Integration in Orlicz-Bochner Spaces

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1. Introduction and Preliminaries

Throughout the paper, $(X, \| \cdot \|_X)$ and $(Y, \| \cdot \|_Y)$ denote real Banach spaces and $X^*$ and $Y^*$ denote their Banach duals, respectively. By $B_X$ and $B_{Y^*}$ we denote the closed unit ball in $X$ and in $Y^*$. Let $\mathcal{S}(X, Y)$ stand for the space of all bounded operators from $X$ and $Y$, equipped with the uniform operator norm $\| \cdot \|$. We assume that $(\Omega, \Sigma, \mu)$ is a complete $\sigma$-finite measure space. Denote by $\Sigma_f(\mu)$ the $\delta$-ring of sets $A \in \Sigma$ with $\mu(A) < \infty$. By $L^0(X)$ we denote the linear space of $\mu$-equivalence classes of all strongly $\Sigma$-measurable functions $f : \Omega \to X$, equipped with the topology $T_0$ of convergence in measure on sets of finite measure.

Now we recall the basic concepts and properties of Orlicz-Bochner spaces (see [1–6] for more details).

By a Young function we mean here a continuous convex mapping $\varphi : [0, \infty) \to [0, \infty]$ that vanishes only at 0 and $\varphi(t)/t \to 0$ as $t \to 0$ and $\varphi(t)/t \to \infty$ as $t \to \infty$. Let $\varphi^*$ stand for the complementary Young function of $\varphi$ in the sense of Young.

Let $L^0(X)$ (resp., $L^p$) denote the Orlicz-Bochner space (resp., Orlicz space) defined by a Young function $\varphi$; that is,

$$L^0(X) := \left\{ f \in L^0(X) : \int_\Omega \varphi(\lambda \| f(\omega) \|_X) \, d\mu < \infty \right\},$$

where $\lambda > 0$. Let $E^0(X)$ denote the Orlicz space defined by $\varphi$; that is,

$$E^0(X) := \left\{ f \in L^0(X) : \int_\Omega \varphi(\lambda \| f(\omega) \|_X) \, d\mu < \infty \right\}.$$

Then $E^0(X)$ is a $\| \cdot \|_\varphi$-closed subspace of $L^0(X)$.
Recall that a subset $H$ of $L^p(X)$ is said to be **solid** whenever $\|f_1\|_X \leq \|f_2\|_X$ $\mu$-a.e. and $f_1, f_2 \in H$ imply $f_1 + H \subseteq H$. A linear topology $\tau$ on $L^p(X)$ is said to be **locally solid** if it has a local basis at 0 consisting of solid sets (see [4]).

According to [7, Definition 2.2] and [6] we have the following definition.

**Definition 1.** A locally solid topology $\tau$ on $L^p(X)$ is said to be a **Lebesgue topology** if for a net $(f_\alpha)$ in $L^p(X)$, $\|f_\alpha\|_X \to 0$ in the Banach lattice $L^p$ if and only if $\|f_\alpha\|_X \to 0$ $\mu$-a.e. and $\|f_\alpha\|_X \leq u(\omega)$ $\mu$-a.e. for some $0 \leq u \in L^p$.

In view of the super Dedekind completeness of $L^p$ one can restrict in the above definition to usual sequences $(f_\alpha)$ in $L^p(X)$ (see [7, Definition 2.2, p. 173]).

Note that, for a sequence $(f_n)$ in $L^p(X)$, $\|f_n\|_X \to 0$ in $L^p$ if and only if $\|f_n\|_X \to 0$ $\mu$-a.e. and $\|f_n\|_X \leq u(\omega)$ $\mu$-a.e. for some $0 \leq u \in L^p$.

For $\varepsilon > 0$ let $U_\varepsilon(f) = \{ f \in L^p(X) : \int f(\omega) d\mu \leq \varepsilon \}$.

Then the family of all sets of the form:

$$\bigcup_{n=1}^{\infty} \left( \sum_{j=1}^{n} U_\varepsilon(e_j) \right),$$

where $(e_j)$ is a sequence of positive numbers and is a local basis at 0 for a linear topology $\tau^\wedge(X)$ on $L^p(X)$ (see [4, 6] for more details). Using [4, Lemma 1.1] one can show that the sets of the form $(\ast)$ are convex and solid, so $\tau^\wedge(X)$ is a locally convex-solid topology.

We now recall terminology and basic facts concerning the spaces of weak-$^*$-measurable functions $g : \Omega \to X^*$ (see [8, 9]). Given a function $g : \Omega \to X^*$ and $x \in X$, let $g_\omega(x) = g(\omega)(x)$ for $\omega \in \Omega$. By $L^0(X^*, X)$ we denote the linear space of the weak-$^*$-equivalence classes of all weak-$^*$-measurable functions $g : \Omega \to X^*$. In view of the super Dedekind completeness of $L^0$, the set $[g_\omega] : x \in B_X$ is order bounded in $L^0$ for each $g \in L^0(X^*, X)$. Thus one can define the so-called **abstract norm** $\theta : L^0(X^*, X) \to L^0$ by

$$\theta(g) = \sup \{ \|g_\omega\|_X : x \in B_X \} \quad (\ast)$$

One can easily check that the following properties of $\theta$ hold:

1. $\theta(g) = 0$ if and only if $g = 0$ and $g \in L^0(X^*, X)$,
2. $\theta(\lambda g) = |\lambda| \theta(g)$ for $\lambda \in \mathbb{R}$ and $g \in L^0(X^*, X)$,
3. $\theta(g_1 + g_2) \leq \theta(g_1) + \theta(g_2)$ for $g_1, g_2 \in L^0(X^*, X)$,
4. $\theta(\chi A g) = \chi \theta(g)$ for $A \subseteq \Omega$ and $g \in L^0(X^*, X)$.

It is known that, for $f \in L^0(X)$, $g \in L^0(X^*, X)$, the function $\langle f, g \rangle : \Omega \to \mathbb{R}$ defined by $\langle f, g \rangle(\omega) = \langle f(\omega), g(\omega) \rangle$ is measurable and

$$\|\langle f, g(\omega) \rangle\|_X \leq \|f(\omega)\|_X \theta(g(\omega)) \quad (\mu\text{-a.e.})$$

Moreover, $\theta(g) = \|g(\cdot)\|_X$ for $g \in L^0(X^*)$. Let

$$L^p(X^*, X) = \{ g \in L^0(X^*, X) : \theta(g) \in L^p \}.$$ Clearly $L^p(X^*) \subset L^p(X^*, X)$. If, in particular, $X^*$ has the Radon-Nikodym property (i.e., $X$ is an Asplund space; see [10, p. 213]), then $L^p(X^*, X) = L^p(X^*)$.

Let $L^p(X)$ stand for the Banach dual of $L^p(X)$, equipped with the conjugate norm $\|\cdot\|_{L^p}^\ast$. Recall that a Young function $\varphi$ satisfies the $\Delta_2$-condition if $\varphi(2\varepsilon) \leq d\varphi(t)$ for some $\varepsilon > 1$ and all $t \geq 0$. We shall say that a Young function $\psi$ is completely weaker than another $\varphi$ (in symbols, $\varphi < \psi$) if for an arbitrary $c > 1$ there exists $d > 1$ such that $\psi(ct) \leq d\varphi(t)$ for all $t \geq 0$. Note that a Young function $\varphi$ satisfies the $\Delta_2$-condition if and only if $\varphi < \psi$. If $\psi < \varphi$, then $L^p \subset L^{p^\ast}$ and it follows that $L^p(X) \subset L^{p^\ast}(X)$.

Now we present basic properties of the topology $\tau^\wedge(X)$ on $L^p(X)$.

**Theorem 2.** Let $\varphi$ be a Young function. Then the following statements hold:

(i) $\tau^\wedge < \tau^\wedge \varphi < \tau^\wedge$ if $\varphi$ satisfies the $\Delta_2$-condition.

(ii) $\tau^\wedge$ is the finest Lebesgue topology on $L^p(X)$.

(iii) $\tau^\wedge$ is generated by the family of norms $\{ \|\cdot\|_{\varphi}^\wedge : \psi < \varphi \}$.

(iv) $(L^p(X), \tau^\wedge)$ is a Banach space.

(v) $(L^p(X), \tau^\wedge)$ is a compact set for the Banach space $(L^p(X), \|\cdot\|_p)$.

(vi) If $X^*$ has the Radon-Nikodym property, then the space $(L^p(X), \tau^\wedge)$ is strongly Mackey; hence $\tau^\wedge$ coincides with the Mackey topology $\tau(L^p(X), L^{p^\ast}(X^*))$.

**Proof.** (i)–(iii) See [4, Theorems 6.1, 6.3 and 6.5].

(iv) In view of [6, Corollary 4.4 and Theorem 1.2], we get $(L^p(X), \tau^\wedge) = (L^p(X^\ast), \tau)$ where $(L^p(X^\ast))$ stands for the order continuous dual of $L^p(X^\ast)$ (see [7, 8, 11] for more details). According to [8, Theorem 4.1] $L^p(X^\ast) = \{ F_g : g \in L^p(X^\ast) \}$.

Using [11, Theorem 1.3] for $g \in L^p(X^\ast, X)$ we have

$$\|F_g\|_{\varphi^\ast} = \sup \left\{ \int_\Omega \langle f, g(\omega) \rangle d\mu : f \in B_L(X^\ast) \right\}$$

$$\|\theta(g)\|_{\varphi^\ast}.$$ (v) See [12, § 3, Theorem 2].

(vi) See [6, Theorem 4.5].
Let \( \gamma_0[\mathcal{S}_q, \mathcal{T}_0] \) (briefly \( \gamma_q \)) denote the natural mixed topology on \( L^p(X) \); that is, \( \gamma_q \) is the finest linear topology that agrees with \( \mathcal{T}_0 \) on \( \| \cdot \|_q \)-bounded sets in \( L^p(X) \) (see [5, 13, 14] for more details). Then \( \gamma_q \) is a locally convex-solid Hausdorff topology (see [14, Theorem 3.2]) and \( \gamma_0 \) and \( \mathcal{S}_q \) have the same bounded sets. This means that \( (L^p(X), \gamma_q) \) is a generalized DF-space (see [15]) and its follows that \( (L^p(X), \gamma_q) \) is quasi-normable (see [15, p. 422]). Moreover, for a sequence \( (f_n) \) in \( L^p(X) \), \( f_n \to 0 \) in \( \gamma_q \) if and only if \( f_n \to 0 \) in \( \mathcal{T}_0 \) and \( \sup \| f_n \|_q < \infty \) (see [14, Theorem 3.1]).

We say that a Young function \( \varphi \) increases essentially more rapidly than another \( \psi \) (in symbols, \( \psi \ll \varphi \)) if for arbitrary \( c > 0, \psi(ct)/\varphi(t) \to 0 \) as \( t \to 0 \) and \( t \to \infty \).

**Theorem 3.** Let \( \varphi \) be a Young function. Then the mixed topology \( \gamma_0 \) on \( L^p(X) \) is generated by the family of norms \( \{ \| \cdot \|_q \}_{q \in (0, \infty)} : X \to [0, \infty) \).

**Proof.** It is known that the mixed topology \( \gamma_0 \) on \( L^p \) is generated by the family of norms \( \{ \| \cdot \|_q \}_{q \in (0, \infty)} 

\) (see [16, Theorem 2.1]). Since \( \| f \|_q = \| f \|_{L^q(X)} \) for \( f \in L^p(X) \), by [14, (5.4), p. 97], the mixed topology \( \gamma_0 \) on \( L^p(X) \) is generated by the family of norms \( \{ \| \cdot \|_q \}_{q \in (0, \infty)} : X \to [0, \infty) \).

Since \( \psi \ll \varphi \) implies \( \psi \ll \varphi \), in view of Theorems 2 and 3, we get

\[ \gamma_0 \subset \mathcal{S}_q. \]  

(10)

The problem of integral representation of bounded linear operators on Banach function spaces of vector-valued functions to Banach spaces in terms of the corresponding operator-valued measures has been the object of much study (see [5, 17–24]). In particular, Dinculeanu (see [19, § 13, Sect. 3], [20], [21, § 8, Sect. B]) studied the problem of integral representation of bounded linear operators from \( L^p(X) \) to a Banach space \( Y \). It is known that if \( 1 \leq p < \infty, \mu(\Omega) < \infty \) and an operator measure \( m : \Sigma \to \mathcal{L}(X, Y) \) vanishes on \( \mu \)-null sets and has the finite \( q \)-seminorm \( \tilde{m}_q(\mu) \) (\( q \leq 1/p + 1/q = 1 \)), then one can define the integral \( \int f \, dm \) for all \( f \in L^p(X) \). Moreover, if \( T : L^p(X) \to Y \) is a bounded linear operator, then the associated operator measure \( m : \Sigma \to \mathcal{L}(X, Y) \) has the finite \( q \)-seminorm \( \tilde{m}_q(\mu) \) and \( T(f) = \int f \, dm \) for all \( f \in L^p(X) \) (see [19, § 13, Theorem 1, p. 239], [20, Theorem 4]). The relationships of the \( q \)-seminorm \( \tilde{m}_q \) to the properties of operators from \( L^p(X) \) to \( Y \) were studied in [22]. Diestel [23] found the integral representation of bounded linear operators from an Orlicz-Boccher space \( L^p(X) \) to a Banach spaces if \( \mu(\Omega) < \infty \) and a Young \( \varphi \) satisfies the \( \Delta_2 \)-condition.

The present paper is a continuation of [5], where we establish integral representation of \( \gamma_q \)-\( \| \cdot \|_q \)-continuous linear operators \( T : L^p(X) \to Y \). We study the problem of integration of functions in \( L^p(X) \) with respect to the representing operator measures of \( \mathcal{S}_q \)-\( \| \cdot \|_q \)-continuous linear operators \( T : L^p(X) \to Y \). An integral representation theorem for \( \mathcal{S}_q \)-\( \| \cdot \|_q \)-continuous linear operators \( T : L^p(X) \to Y \) is established (see Theorem 9 below). We study the relationships between \( \mathcal{S}_q \)-\( \| \cdot \|_q \)-continuous operators \( T : L^p(X) \to Y \) and the properties of their representing measures \( m : \Sigma_f(\mu) \to \mathcal{L}(X, Y) \).

## 2. \( \varphi^* \)-Semivariation of Operator Measures

Assume that \( m : \Sigma_f(\mu) \to \mathcal{L}(X, Y) \) is an additive measure such that \( m(\mu) = 0 \) if \( \mu(\Omega) = 0 \).

Let \( \delta(\Sigma_f(\mu), X) \) denote the space of all \( X \)-valued \( \Sigma_f(\mu) \)-simple functions on \( \Omega \). Then \( s \in \delta(\Sigma_f(\mu), X) \) if \( s = \sum (1_{A_i} \otimes x_i) \), where \( (A_i) \) is a finite pairwise disjoint sequence in \( \Sigma_f(\mu) \) and \( x_i \in X \). For \( s = \sum_{i=1}^n (1_{A_i} \otimes x_i) \in \delta(\Sigma_f(\mu), X) \) and \( A \in \Sigma_f(\mu) \), we can define the integral \( \int_A s \, dm \) by

\[ \int_A s \, dm = \sum_{i=1}^n m(A_i \cap A)(x_i). \]  

(11)

Note that

\[ \int_A s \, dm = \int_{\Omega} 1_A s \, dm. \]  

(12)

For \( y^* \in Y^* \), we define a measure \( m_{y^*} : \Sigma_f(\mu) \to X^* \) by the equality

\[ m_{y^*}(A)(x) = y^*(m(A)(x)) \]  

for \( A \in \Sigma_f(\mu) \), \( x \in X \).

For \( s = \sum_{i=1}^n (1_{A_i} \otimes x_i) \in \delta(\Sigma_f(\mu), X) \) and \( A \in \Sigma_f(\mu) \), we define the integral \( \int_A s \, dm_{y^*} \) by the equality:

\[ \int_A s \, dm_{y^*} = \sum_{i=1}^n m_{y^*}(A_i \cap A)(x_i). \]  

(14)

Then

\[ y^*(\int_A s \, dm) = \int_A s \, dm_{y^*}. \]  

(15)

Following [23], [19, § 13] one can define the \( \varphi^* \)-semivariation \( \tilde{m}_{\varphi^*}(A) \) of \( m \) on \( A \in \Sigma_f(\mu) \) by

\[ \tilde{m}_{\varphi^*}(A) = \sup \left\{ \sum_{i=1}^n m(A \cap A_i)(x_i) \right\}_Y, \]  

(16)

where the supremum is taken over all finite pairwise disjoint sets \( \{A_1, \ldots, A_n\} \) in \( \Sigma_f(\mu) \) and \( x_i \in X \) for \( i = 1, \ldots, n \) such that \( \| \sum_{i=1}^n (1_{A_i} \otimes x_i) \|_\varphi \leq 1 \).

One can observe that

\[ \tilde{m}_{\varphi^*}(A) = \sup \left\{ \int_A s \, dm : s \in \delta(\Sigma_f(\mu), X), \| s \|_\varphi \leq 1 \right\}. \]  

(17)

Note that

\[ \tilde{m}_{\varphi^*}(A \cup B) \leq \tilde{m}_{\varphi^*}(A) + \tilde{m}_{\varphi^*}(B) \]  

for \( A, B \in \Sigma_f(\mu) \).
Let \((m_{\mu,\nu})_\nu(A)\) stand for the \(\varphi^*\)-semivariation of \(m_\mu\) on \(A \in \Sigma\); that is,
\[
(m_{\mu,\nu})_\nu(A) = \sup \left\{ \int_A s \, dm_{\mu,\nu} : s \in \delta(S_f(\mu),X), \|s\|_\varphi \leq 1 \right\}.
\]

The following lemma will be useful.

**Lemma 4.** Let \(\varphi\) be a Young function and \(m : \Sigma_f(\mu) \to \mathcal{L}(X,Y)\) be a measure with \(m \ll \mu\) and \(m_{\mu,\nu}(\Omega) < \infty\). Then the following statements hold:

(i) If \(f \in E^\varphi(X)\), then there exists a \(\|\cdot\|_\varphi\)-Cauchy sequence \((s_n)\) in \(\delta(S_f(\mu),X)\) such that \(\|s_n(\omega) - f(\omega)\|_X \to 0\) \(\mu\)-a.e.

(ii) If \((s_n)\) is a \(\|\cdot\|_\varphi\)-Cauchy sequence in \(\delta(S_f(\mu),X)\), then for \(A \in \Sigma\), \(\int_A s_n dm\) is a Cauchy sequence in a Banach space \(Y\) and for every \(y^* \in Y^*\), \((\int_A s_n dm_{y^*})\) is a Cauchy sequence in \(\mathbb{R}\).

(iii) If \(f \in E^\varphi(X)\) and \((s'_n)\) and \((s''_n)\) are \(\|\cdot\|_\varphi\)-Cauchy sequence in \(\delta(S_f(\mu),X)\) such that \(\|s'_n(\omega) - f(\omega)\|_X \to 0\) \(\mu\)-a.e. and \(\|s''_n(\omega) - f(\omega)\|_X \to 0\) \(\mu\)-a.e., then for \(A \in \Sigma\), one has
\[
\lim \int_A s'_n dm = \lim \int_A s''_n dm,
\]
and for every \(y^* \in Y^*\), one has
\[
\lim \int_A s'_n dm_{y^*} = \lim \int_A s''_n dm_{y^*}.
\]

**Proof.**

(i) Let \(f \in E^\varphi(X)\). Then there exists a sequence \((s_n)\) in \(\delta(S_f(\mu),X)\) such that \(\|s_n(\omega) - f(\omega)\|_X \to 0\) \(\mu\)-a.e. and \(\|s_n(\omega)\|_X \leq \|f(\omega)\|_X\) \(\mu\)-a.e. for all \(n \in \mathbb{N}\) (see [21, Theorem 6, p. 4]). Using the Lebesgue dominated convergence theorem, we obtain that \(\int_A \varphi(\lambda(\|s_n(\omega) - f(\omega)\|_X))d\mu \to 0\) for all \(\lambda > 0\), so \(s_n - f\|_\varphi \to 0\). Hence \((s_n)\) is a \(\|\cdot\|_\varphi\)-Cauchy sequence.

(ii) Assume that \((s_n)\) is a \(\|\cdot\|_\varphi\)-Cauchy sequence in \(\delta(S_f(\mu),X)\). Hence for \(n, k \in \mathbb{N}\), we have
\[
\left\| \int_A s_n dm - \int_A s_k dm \right\|_Y = \left\| \int (s_n - s_k) dm \right\|_Y \\
\leq \|s_n - s_k\|_\varphi \overline{m}_\varphi(A) \leq \|s_n - s_k\|_\varphi \overline{m}_\varphi(\Omega).
\]

It follows that \((\int_A s_k dm)\) is a Cauchy sequence in \(Y\). Hence in view of (15), for \(y^* \in Y^*\), \((\int_A s_n dm_{y^*})\) is a Cauchy sequence in \(\mathbb{R}\).

(iii) Note that \((s'_n - s''_n)\) is a \(\|\cdot\|_\varphi\)-Cauchy sequence and \(\|s'_n(\omega) - s''_n(\omega)\|_X \to 0\) \(\mu\)-a.e. Hence there exists \(h \in E^\varphi(X)\) such that \(\|s'_n - s''_n - h\|_\varphi \to 0\). Note that \(\mathcal{T}_{\mu,\nu}(X,Y) \subset \mathcal{T}_{\varphi}(E^\varphi(X))\).

3. Integral Representation of Continuous Operators on Orlicz-Bochner Spaces

For a bounded linear operator \(T : L^\varphi(X) \to Y\) let
\[
\|T\|_\varphi = \sup \left\{ \|T(f)\|_Y : f \in B_{L^\varphi(X)} \right\}.
\]

**Proposition 6.** Let \(T : L^\varphi(X) \to Y\) be a bounded linear operator
and
\[
m(A,\mu) = T(\mathbb{1}_A \otimes x) \quad \text{for} \quad A \in \Sigma_f(\mu), \quad x \in X.
\]

Then the following statements hold:

(i) If \(A \in \Sigma_f(\mu)\) and \(m(A) \leq \|T\|_\varphi \cdot \|1_A\|_\varphi\).

(ii) If \(m \ll \mu\).

(iii) \(\|m(A,\mu)\| \to 0\) if \(A_n \downarrow \emptyset\) with \(A_n \in \Sigma_f(\mu)\).

(iv) \(m : \Sigma_f(\mu) \to \mathcal{L}(X,Y)\) is countably additive; that is,
\[
m(\bigcup_{n=1}^\infty B_n) = \sum_{n=1}^\infty m(B_n) \quad \text{if} \quad (B_n) \text{ is a pairwise disjoint sequence in } \Sigma_f(\mu) \text{ with } \bigcup_{n=1}^\infty B_n \in \Sigma_f(\mu).
\]

(v) \(\overline{m}_\varphi(\Omega) \leq \|T\|_\varphi\).
Proof. (i) Let \( A \in \Sigma_f(\mu) \). Then for \( x \in B_X \), we have \( \|1_A \otimes x\|_p \leq \|1_A\|_p \) and hence
\[
\|m(A)(x)\|_Y = \|T(1_A \otimes x)\|_Y \leq \|T\|_p \cdot \|1_A \otimes x\|_p \\
\leq \|T\|_p \cdot \|1_A\|_p,
\]
so \( \|m(A)\| \leq \|T\|_p \cdot \|1_A\|_p \).

(ii) This follows from (i) because \( \|1_A\|_p = 0 \) if \( \mu(A) = 0 \).

(iii) Assume that \( A_n \downarrow 0 \) with \( A_n \in \Sigma_f(\mu) \). Then \( 1_{A_n}(\omega) \geq 1_{A_n}(\omega) \downarrow 0 \) for \( \omega \in \Omega \). By the Lebesgue dominated convergence theorem, we obtain that \( \int_{\Omega} \phi(A_n(\omega))d\mu \to 0 \) for every \( \lambda > 0 \). This means that \( \|1_{A_n}\|_p \to 0 \) and by (i), \( \|m(A_n)\| \to 0 \).

(iv) Assume that \( (B_n) \) is a pairwise disjoint sequence in \( \Sigma_f(\mu) \) with \( B = \bigcup_{n=1}^{\infty} B_n \in \Sigma_f(\mu) \). Let \( A_n = B \setminus \bigcup_{i=1}^{n} B_i \) for \( n \in \mathbb{N} \). Then \( A_n \in \Sigma_f(\mu) \) and \( A_n \downarrow 0 \). Hence by (iii) \( \|m(B) - \sum_{i=1}^{n} m(B_i)\| = \|m(B) - m(\bigcup_{i=1}^{n} B_i)\| = \|m(A_n)\| \to 0 \).

Statement (v) is obvious. \( \square \)

Definition 7. Let \( T : L^q(X) \to Y \) be a bounded linear operator and
\[
m(A)(x) = T(1_A \otimes x) \quad \text{for} \ A \in \Sigma_f(\mu), \ x \in X.
\]
Then the measure \( m : \Sigma_f(\mu) \to \mathcal{L}(X,Y) \) will be called a representing measure of \( T \).

Proposition 8. Let \( T : L^q(X) \to Y \) be a \((\mathcal{T}_q^\wedge, \|\cdot\|_Y)\)-continuous linear operator and \( m : \Sigma_f(\mu) \to \mathcal{L}(X,Y) \) be its representing measure. Then there exists a Young function \( \psi \) such that \( \psi \preccurlyeq \varphi \) and \( m_q(\psi) < \infty \).

Proof. According to Theorem 2 there exist a finite set \( \{\psi_i : i = 1, \ldots, n\} \) of Young functions with \( \psi_i \preccurlyeq \varphi \) for \( i = 1, \ldots, n \) and \( a > 0 \) such that
\[
\|T(f)\|_Y \leq a \max_{1 \leq i \leq n} \|f\|_{\psi_i} \quad \forall f \in L^q(X).
\]
Let \( \psi(t) = \max_{1 \leq i \leq n} \psi_i(t) \) for \( t \geq 0 \). Then \( \psi \) is a Young function with \( \psi \preccurlyeq \varphi \) and
\[
\|T(f)\|_Y \leq a \|f\|_{\psi} \quad \forall f \in L^q(X).
\]
Hence
\[
\overline{m}_q(\psi)(\Omega) = \sup \{\|T(s)\|_Y : s \in \mathcal{S}(\Sigma_f(\mu), X), \|s\|_{\psi} \leq 1\}
\leq a < \infty.
\]
\( \square \)

For a linear operator \( T : L^q(X) \to Y \) and \( A \in \Sigma \), let
\[
T_A(f) = T(1_A f) \quad \text{for} \ f \in L^q(X).
\]

Now we can state our main result that extends the classical results concerning the integral representation of operators on Lebesgue-Bochner spaces \( L^p(X) \) (1 \( \leq p < \infty \) (see [19, § 13, Theorem 1, pp. 259–261]) to operators on Orlicz-Bochner spaces \( L^q(X) \).

Theorem 9. Let \( T : L^q(X) \to Y \) be a \((\mathcal{T}_q^\wedge, \|\cdot\|_Y)\)-continuous linear operator and \( m : \Sigma_f(\mu) \to \mathcal{L}(X,Y) \) be its representing measure. Then for \( A \in \Sigma \) the following statements hold:

(i) \( T_A : L^q(X) \to Y \) is a \((\mathcal{T}_q^\wedge, \|\cdot\|_Y)\)-continuous linear operator.

(ii) For \( f \in L^q(X) \), one has
\[
T_A(f) = \int_A f \ dm
\]
and for \( y^* \in Y^* \), one has
\[
y^*(T_A(f)) = \int_A f \ dm_{y^*}.
\]

(iii) For \( f \in L^q(X) \), the measure \( m_f : \Sigma \to Y \) defined by the equality
\[
m_f(A) = \int_A f \ dm \quad \text{for} \ A \in \Sigma
\]
is countably additive.

(iv) \( \|T_A\|_p = \overline{m}_q(A) \)

and for \( y^* \in Y^* \), \( \|y^* \circ T_A\|_{\psi} = \|y^* \circ T\|_{\psi} \cdot \overline{m}_q(\psi)(A) \).

(v) \( \overline{m}_q(A) = \sup \{\|\overline{m}_{\psi_i}(\psi_i)(A) : \psi_i \in \mathcal{S}(\Sigma_f(\mu), X)\} \).

(vi) For \( f \in L^q(X) \), one has
\[
\left\| \int_A f \ dm \right\|_{\psi} \leq \overline{m}_q(A) \|f\|_{\psi}
\]
and for \( y^* \in Y^* \), one has
\[
\left\| \int_A f \ dm_{y^*} \right\|_{\psi} \leq \left(\overline{m}_q(\psi)\right)^*_{\psi}(A) \|f\|_{\psi}.
\]
Since $T_A(s_n) = \int_A s_n dm$ and by (i), $T_A$ is $(\mathcal{S}_\varphi^\wedge, \| \cdot \|_Y)$-continuous, we get
\[ T_A(f) = \lim_{n \to \infty} \int_A s_n dm. \tag{43} \]
Hence
\[ T_A(f) = \int_A f dm \tag{44} \]
and for $y^* \in Y^*$, we have
\[ y^*(T_A(f)) = \lim_{n \to \infty} y^*(\int_A s_n dm) = \lim_{n \to \infty} \int_A s_n dm. \tag{45} \]

(iii) Let $f \in L^p_\mathcal{S}(X)$ and $(A_n)$ be a sequence in $\Sigma$ such that $A_n \downarrow A$. Then $1_{A_n}(\omega) \downarrow 0$ for $\omega \in \Omega$, and hence $\| 1_{A_n}(\omega) f(\omega) \| X \to 0$ $\mu$-a.e. and $\| 1_{A_n}(\omega) f(\omega) \| X \leq \| f(\omega) \| X$ $\mu$-a.e. Hence $1_{A_n} f \to 0$ in $\mathcal{S}_\varphi^\wedge$ because $\mathcal{S}_\varphi^\wedge$ is a Lebesgue topology, and by (i) we get
\[ \| m_f(A_n) \|_Y = \int_{A_n} f dm \to 0. \tag{46} \]

(iv) Note that $\bar{m}_\varphi(A) \leq \| T_A \|_\varphi$. To show that $\| T_A \|_\varphi \leq \bar{m}_\varphi(A)$, assume that $f \in L^\varphi(X)$, Choose a sequence $(s_n)$ in $\mathcal{S}(\varphi)$ such that $\| s_n(\omega) - f(\omega) \| X \to 0$ $\mu$-a.e. and $\| s_n(\omega) \| X \leq \| f(\omega) \| X$ $\mu$-a.e. for all $n \in \mathbb{N}$. Since $\mathcal{S}_\varphi^\wedge$ is a Lebesgue topology, we have $s_n \to f$ in $\mathcal{S}_\varphi^\wedge$ and hence $\| T_A(s_n) - T_A(f) \|_Y \to 0$. Note that $T_A(s_n) = \int_A s_n dm$.

Let $\varepsilon > 0$ be given. Choose $n_0 \in \mathbb{N}$ such that $\| T_A(f) - \int_A s_{n_0} dm \|_Y < \varepsilon$. Then
\[ \| T_A(f) \|_Y \leq \| T_A(f) - \int_A s_{n_0} dm \|_Y + \| \int_A s_{n_0} dm \|_Y \tag{47} \]
\[ \leq \varepsilon + \bar{m}_\varphi(A). \]

It follows that $\| T_A \|_\varphi \leq \bar{m}_\varphi(A)$, so $\bar{m}_\varphi(A) = \| T_A \|_\varphi$. Hence for $y^* \in Y^*$, we easily get
\[ \| (y^* \circ T_A) \|_\varphi = \| y^* \circ T_A \|_\varphi = \left( \bar{m}_\varphi \right)_\varphi(A). \tag{48} \]

(v) Using (iv) we have
\[ \bar{m}_\varphi(A) = \| T_A \|_\varphi \]
\[ = \sup \{ \| T_A(f) \|_Y : f \in L^\varphi(X), \| f \|_Y \leq 1 \} \]
\[ = \sup_{y^* \in \mathcal{E}_\varphi^\wedge} \{ \| y^* \circ T_A(f) \|_Y : f \in L^\varphi(X), \| f \|_Y \leq 1 \} \tag{49} \]
\[ = \sup_{y^* \in \mathcal{E}_\varphi^\wedge} \| y^* \circ T_A \|_\varphi = \sup_{y^* \in \mathcal{E}_\varphi^\wedge} \left( \bar{m}_\varphi \right)_\varphi(A). \]

(vi) This follows from (ii) and (iv).

For a sequence $(A_n)$ in $\Sigma$, we will write $A_n \searrow 0$ if $A_n \downarrow 0$ and $\mu(A_n \cap A) \to 0$ for every $A \in \mathcal{F}_\varphi(\mu)$.

**Definition 10.** A measure $m : \Sigma_f(\mu) \to \mathcal{L}(X,Y)$ with $m \ll \mu$ and $\bar{m}_\varphi(\Omega) < \infty$ is said to be $\varphi^*$-semivariationally $\mu$-continuous if $\bar{m}_\varphi(A_n) \to 0$ whenever $A_n \searrow A \in \Sigma_f(\mu)$.

Using a standard argument we can show the following.

**Proposition 11.** Let $m : \Sigma_f(\mu) \to \mathcal{L}(X,Y)$ be an additive measure such that $m \ll \mu$ and $\bar{m}_\varphi(\Omega) < \infty$. Then the following statements are equivalent:

(i) $m$ is $\varphi^*$-semivariationally $\mu$-continuous.

(ii) The following two conditions hold simultaneously:

(a) For every $\varepsilon > 0$ there exists $\delta > 0$ such that $\bar{m}_\varphi(A) \leq \varepsilon$ whenever $\mu(A) \leq \delta$, $A \in \Sigma_f(\mu)$.

(b) For every $\varepsilon > 0$ there exists $A_0 \in \Sigma_f(\mu)$ such that $\bar{m}_\varphi(\Omega \setminus A_0) \leq \varepsilon$.

The following theorem characterizes $\varphi^*$-semivariationally $\mu$-continuous representing measures.

**Theorem 12.** Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{S}_\varphi^\wedge, \| \cdot \|_Y)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathcal{L}(X,Y)$ be its representing measure. Then the following statements are equivalent:

(i) $m$ is $\varphi^*$-semivariationally $\mu$-continuous.

(ii) $T$ is $\left( \mathcal{S}_\varphi^\wedge, \| \cdot \|_Y \right)$-continuous.

(iii) $\| T(f_n) \|_Y \to 0$ if $f_n \to 0$ in $\mathcal{S}_\varphi(\mu)$ and $\sup_n \| f_n \|_\varphi < \infty$.

(iv) $\| T_{A_n} \|_Y \to 0$ if $A_n \searrow A \in \Sigma_f(\mu)$.

**Proof.** (i) $\iff$ (ii) $\iff$ (iii) See [5, Corollary 2.8 and Proposition 11].

(i) $\iff$ (iv) This follows from Theorem 9.

Now assume that $\Omega$ is a completely regular Hausdorff space. Let $\mathcal{B}_\varphi$ denote the $\sigma$-algebra of Baire sets in $\Omega$, which is the $\sigma$-algebra generated by the class $\mathcal{E}$ of all zero sets of bounded continuous positive functions on $\omega$. By $\mathcal{P}$ we denote the family of all cozero (=positive) in $\Omega$ (see [25, p. 108]).

Let $\mu : \mathcal{B}_\varphi \to [0,\infty)$ be a countably additive measure. Then $\mu$ is zero-set regular; that is, for every $A \in \mathcal{B}_\varphi$ and $\varepsilon > 0$ there exists $Z \in \mathcal{E}$ with $Z \subset A$ such that $\mu(A \setminus Z) \leq \varepsilon$ (see [25, p. 118]). It follows that for every $A \in \mathcal{B}_\varphi$ and $\varepsilon > 0$ there exist $U \in \mathcal{P}, U \supset A$ such that $\mu(U \setminus A) \leq \varepsilon$.

We can assume that $\mu$ to be complete (if necessary we can take the completion $(\Omega, \mathcal{B}_\varphi, \overline{\mu})$ of the measure space $(\Omega, \mathcal{B}_\varphi, \mu)$).

**Proposition 13.** Assume that $\Omega$ is a completely regular Hausdorff space and $(\Omega, \mathcal{B}_\varphi, \mu)$ is a complete finite measure space. Let $T : L^\varphi(X) \to Y$ be a $(\mathcal{S}_\varphi^\wedge, \| \cdot \|_Y)$-continuous linear operator and $m : \mathcal{B}_\varphi \to \mathcal{L}(X,Y)$ be its representing measure. Then the following statements are equivalent:

(i) $m$ is $\varphi^*$-semivariationally $\mu$-continuous.
Choose $\mathcal{A}$ such that $A_n \downarrow$ and $\mu(A_{n}) \to 0$. Then there exists a sequence $(U_{n})$ in $\mathcal{P}$ with $A_{n} \subset U_{n}$ such that $\overline{m}_{\varphi}(U_{n}) \to 0$. 

(iii) For every sequence $(A_{n})$ in $\mathcal{B}$ such that $A_n \downarrow$ and $\mu(A_{n}) \to 0$ there exists a sequence $(U_{n})$ in $\mathcal{P}$ with $A_{n} \subset U_{n}$ such that 

$$\sup \{\|T(f)\|_{Y} : f \in B_{L^{\varphi}(X)}, \text{ supp } f \subset U_{n} \} \to 0. \quad (50)$$

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds and $(A_{n})$ is a sequence in $\mathcal{B}$ such that $A_{n} \downarrow$ and $\mu(A_{n}) \to 0$. Then there exists a sequence $(U_{n})$ in $\mathcal{P}$ such that $A_{n} \subset U_{n}$ and $\mu(U_{n} \setminus A_{n}) \leq 1/n$ for $n \in \mathbb{N}$.

Let $\varepsilon > 0$ be given. Then in view of Proposition 11 there exists $\delta > 0$ such that $\overline{m}_{\varphi}(U_{n}) \leq \varepsilon/2$ if $\mu(A_{n}) \leq \delta$ with $A_{n} \in \mathcal{B}$. Choose $n_{1} \in \mathbb{N}$ such that $\mu(U_{n} \setminus A_{n}) \leq \delta$ for $n \geq n_{2}$. Then $\overline{m}_{\varphi}(U_{n}) \leq \varepsilon/2$ for $n \geq n_{2}$. Therefore, for $n \geq n_{0} = \max(n_{1}, n_{2})$, we get 

$$\overline{m}_{\varphi}(U_{n}) \leq \overline{m}_{\varphi}(U_{n} \setminus A_{n}) + \overline{m}_{\varphi}(A_{n}) \leq \varepsilon; \quad (51)$$

that is, (ii) holds.

(ii) $\Rightarrow$ (iii) Assume that (ii) holds and $(A_{n})$ is a sequence in $\mathcal{B}$ such that $A_{n} \downarrow$ and $\mu(A_{n}) \to 0$. Then there exists a sequence $(U_{n})$ in $\mathcal{P}$ with $A_{n} \subset U_{n}$ such that $\overline{m}_{\varphi}(U_{n}) \to 0$. Note that, for $f \in B_{L^{\varphi}(X)}$ with $\text{ supp } f \subset U_{n}$ for $n \in \mathbb{N}$, by Theorem 9 we have 

$$\|T(f)\|_{Y} = \int_{\Omega} f \, dm \leq \int_{U_{n}} f \, dm \leq \overline{m}_{\varphi}(U_{n}). \quad (52)$$

It follows that (iii) holds.

(iii) $\Rightarrow$ (i) Assume that (iii) holds and $A_{n} \downarrow$ with $\mu(A_{n}) \to 0$. Then there exists a sequence $(U_{n})$ in $\mathcal{P}$ with $A_{n} \subset U_{n}$ such that 

$$\sup \{\|T(f)\|_{Y} : f \in B_{L^{\varphi}(X)}, \text{ supp } f \subset U_{n} \} \to 0. \quad (53)$$

Assume on the contrary that (i) fails to hold. Then without loss of generality we can assume that 

$$\overline{m}_{\varphi}(A_{n}) > \varepsilon_{0} \quad \text{for some } \varepsilon_{0} > 0, \quad \text{all } n \in \mathbb{N}. \quad (54)$$

Choose $n_{0} \in \mathbb{N}$ such that 

$$\sup \{\|T(f)\|_{Y} : f \in B_{L^{\varphi}(X)}, \text{ supp } f \subset U_{n_{0}} \} < \varepsilon_{0}/2. \quad (55)$$

In view of (54) there exists a pairwise disjoint set $\{B_{1}, \ldots, B_{k}\}$ in $\mathcal{B}$ and $x_{i} \in X$ for $i = 1, \ldots, k$ and $y^{*} \in B_{Y^{*}}$, such that 

$$\| \sum_{i=1}^{k} \|1_{B_{i}} \otimes x_{i}\|_{\varphi} \leq 1 \quad \text{and} \quad \left| y^{*} \left( \sum_{i=1}^{k} m(A_{n_{0}} \cap B_{i}) (x_{i}) \right) \right| \geq \varepsilon_{0}. \quad (56)$$

Let $s_{0} = \sum_{i=1}^{k} \|1_{A_{n_{0}} \cap B_{i}} \otimes x_{i}\|_{\varphi}$. Then $\|s_{0}\|_{\varphi} \leq 1$ and $\text{ supp } s_{0} \subset A_{n_{0}} \subset U_{n_{0}}$. Then by (55) we get $\|T(s_{0})\|_{Y} < \varepsilon_{0}/2$. On the other hand, in view of (56) we have $\|T(s_{0})\|_{Y} \geq \varepsilon_{0}$.

This contradiction establishes that (i) holds.

**Corollary 14.** Assume that $\Omega$ is a completely regular Hausdorff space and $(\Omega, \mathcal{B}, \mu)$ is a complete finite measure space. Let $T : L^{\varphi}(X) \to Y$ be a $(\varphi, \|\cdot\|_{Y})$-continuous linear operator and $m : \mathcal{B} \to \mathcal{L}(X, Y)$ be its representing measure. Then $m_{\varphi}$ is regular; that is, for every $A \in \mathcal{B}$ and $\varepsilon > 0$ there exists $Z \in \mathcal{L}$ and $U \in \mathcal{P}$ with $Z \subset A \subset U$ such that $\overline{m}_{\varphi}(U \setminus Z) \leq \varepsilon$. 

Proof. In view of Theorem 12 $m$ is $\varphi^{*}$-semivariationally $\mu$-continuous. Let $A \in \mathcal{B}$ and $\varepsilon > 0$ be given. Then by Proposition 11 there exists $\delta > 0$ such that $\overline{m}_{\varphi}(B) \leq \varepsilon$ whenever $B \in \mathcal{B}$ and $\mu(B) \leq \delta$. By the regularity of $\mu$ one can choose $Z \in \mathcal{L}$ and $U \in \mathcal{P}$ with $Z \subset A \subset U$ such that $\mu(U \setminus Z) \leq \delta$. Hence $\overline{m}_{\varphi}(U \setminus Z) \leq \varepsilon$, as desired. 

**4. Compact Operators on Orlicz-Bochner Spaces**

The following theorem presents necessary conditions for a $(\varphi, \|\cdot\|_{Y})$-continuous operator $T : L^{\varphi}(X) \to Y$ to be compact.

**Theorem 15.** Assume that a Young function $\varphi$ such that $\varphi^{*}$ satisfies the $\Delta_{2}$-condition. Let $T : L^{\varphi}(X) \to Y$ be a $(\varphi, \|\cdot\|_{Y})$-continuous linear operator and $m : \Sigma(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. If $T$ is compact, then $m$ is $\varphi^{*}$-semivariationally $\mu$-continuous.

Proof. Assume that $T$ is compact and $m$ fails to be $\varphi^{*}$-semivariationally $\mu$-continuous. Then there exist $\varepsilon > 0$ and a sequence $(A_{n})$ in $\Sigma$ with $A_{n} \subset \Omega_{0}$ such that $\|T_{A_{n}}\|_{\varphi} = \overline{m}_{\varphi}(A_{n}) > \varepsilon$ for $n \in \mathbb{N}$ (see Theorem 9). Hence one can choose a sequence $(y^{*}_{n})$ in $B_{Y^{*}}$ such that 

$$\|y^{*}_{n} \circ T_{A_{n}}\|_{\varphi} \geq \varepsilon \quad \forall n \in \mathbb{N}. \quad (57)$$

By Schauder's theorem the conjugate mapping $T^{*} : Y^{*} \to L^{\varphi}(X)$ is compact. Note that $T^{*}(y^{*}_{n}) = y^{*}_{n} \circ T \in L^{\varphi}(X)_{n}$ for all $n \in \mathbb{N}$, where $L^{\varphi}(X)_{n}$ is a closed subspace of the Banach space $(L^{\varphi}(X)^{*}, \|\cdot\|_{\varphi}^{*})$ (see Theorem 2). Then for every $n \in \mathbb{N}$ there exists $g_{n} \in L^{\varphi}(X^{*}, X)$ such that 

$$\langle y^{*}_{n} \circ T(f) \rangle = \int_{\Omega} \langle f(\omega), g_{n}(\omega) \rangle \, d\mu$$

$$\text{ for } f \in L^{\varphi}(X), \quad (58)$$

$$\|y^{*}_{n} \circ T\|_{\varphi}^{*}$$

$$= \sup \left\{ \int_{\Omega} \|f(\omega)\|_{X} \theta(g_{n}(\omega)) \, d\mu : f \in B_{L^{\varphi}(X)} \right\}$$

$$= \|\theta(g_{n})\|_{\varphi^{*}}. \quad (59)$$

Hence we obtain that, for each $n \in \mathbb{N}$,

$$\|y^{*}_{n} \circ T_{A_{n}}\|_{\varphi}^{*} = \|1_{A_{n}} \circ \theta(g_{n})\|_{\varphi^{*}} = \|\theta(1_{A_{n}}g_{n})\|_{\varphi^{*}}. \quad (59)$$
Since $T^*(B_{Y^*})$ is a relatively sequentially compact subset of $(L^p(X), \| \cdot \|_p)$, there exist a subsequence $(g_{k_n})$ of $(g_n)$ and $g \in L^p(X^*, X)$ such that
\[ \left\| F_{g_n} - F_g \right\|_p = \left\| \theta(g_{k_n} - g) \right\|_p \to 0. \]
(60)
Choose $n_\varepsilon \in \mathbb{N}$ such that $\left\| \theta(g_{k_n} - g) \right\|_p \leq \varepsilon / 2$ for $n \geq n_\varepsilon$.

Hence for $n \geq n_\varepsilon$,
\[ \left\| \theta \left( 1_A \theta(g_{k_n} - g) \right) \right\|_p^* \leq \left\| \theta(g_{k_n} - g) \right\|_p \leq \frac{\varepsilon}{2}. \]
(61)
Using (57) and (61), for $n \geq n_\varepsilon$, we get
\[ \varepsilon \leq \left\| \theta^* T_{A \theta}(g_{k_n}) \right\|_p^* = \left\| \theta \left( 1_A \theta(g_{k_n}) \right) \right\|_p \leq \frac{\varepsilon}{2} + \left\| \theta(1_A g) \right\|_p. \]
(62)
and hence
\[ \left\| 1_A \theta(g) \right\|_p = \left\| \theta(1_A g) \right\|_p \geq \frac{\varepsilon}{2}. \]
(63)

On the other hand, since $\varphi^*$ is supposed to satisfy the $\Delta_2$-condition, we have that $\left\| 1_A \theta(g) \right\|_p \to 0$ (see [26, Theorem 3, pp. 58-59]). This contradiction establishes that $m$ is $\varphi^*$-semivariationally $\mu$-continuous. \hfill \Box

**Corollary 16.** Assume that $\varphi$ is a Young function such that $\varphi^*$ satisfies the $\Delta_2$-condition. Let $T : L^p(X) \to Y$ be a $(\mathcal{F}_\varphi^*, \| \cdot \|_\varphi)$-continuous linear operator. Then the following statements are equivalent:

(i) $T$ is compact.

(ii) $T$ is $(y_{p*}, \| \cdot \|_\varphi)$-compact; that is, there exists a $y_{p*}$-neighborhood $V$ of $0$ in $L^p(X)$ such that $T(V)$ is a relatively compact set in $Y$.

(iii) There exists a Young function $\psi$ with $\varphi \ll \psi$ such that
\[ \left\{ \int f \, dm : f \in L^p(X), \| f \|_p \leq 1 \right\} \text{ is a relatively compact set in } Y. \]

Proof. (i) $\Rightarrow$ (ii) Assume that (i) holds. Then by Theorems 12 and 15 $T$ is $(y_{p*}, \| \cdot \|_\varphi)$-continuous. Since the space $(L^p(X), y_{p*})$ is quasinormable, by Grothendieck's classical result (see [15, p. 429]), we obtain that $T$ is $(y_{p*}, \| \cdot \|_\varphi)$-compact.

(ii) $\Rightarrow$ (i) The implication is obvious.

(iii) $\Rightarrow$ (ii) This follows from Theorem 3. \hfill \Box

## 5. Topology Associated with the $\varphi^*$-Semivariation of a Representing Measure

Assume that $T : L^p(X) \to Y$ be a $(\mathcal{F}_\varphi^*, \| \cdot \|_\varphi)$-continuous linear operator. Let $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. Let us put
\[ p_m(y^*) = (m_{y^*})^*_\varphi, (\Omega) \text{ for } y^* \in Y^*. \]
(64)
Note that $p_m$ is a seminorm on $Y^*$. Following [22, 27] let $\delta_{m,p}$ stand for the topology on $B_{Y^*}$ defined by the seminorm $p_m$ restricted to $B_{Y^*}$.

The following theorem characterizes $(\mathcal{F}_\varphi^*, \| \cdot \|_\varphi)$-continuous compact operators $T : L^p(X) \to Y$ in terms of the topological properties of the space $(B_{Y^*}, \delta_{m,p})$ (see [22, Theorem 3]).

**Theorem 17.** Let $T : L^p(X) \to Y$ be a $(\mathcal{F}_\varphi^*, \| \cdot \|_\varphi)$-continuous linear operator and $m : \Sigma_f(\mu) \to \mathcal{L}(X, Y)$ be its representing measure. Then the following statements are equivalent:

(i) The space $(B_{Y^*}, \delta_{m,p})$ is compact.

(ii) $T$ is compact.

Proof. (i) $\Rightarrow$ (ii) Assume that $(B_{Y^*}, \delta_{m,p})$ is compact. Let $(y_{p*}^n)$ be a sequence in $B_{Y^*}$. Without loss of generality we can assume that $y_{p*}^n \to y_0^*$ in $\delta_{m,p}$ for some $y^* \in B_{Y^*}$. Then using Theorem 9 for $f \in L^p(X)$, we have
\[ \left| (T^*(y_{p*}^n) - T^*(y_0^*)) (f) \right| = \left| (y_{p*}^n - y_0^*) (T (f)) \right| \]
\[ = \left\{ \int f \, dm \big| y_{p*}^n - y_0^* \big( m, \varphi \beta \big) \leq \left( \frac{m_{y_{p*}^n - y_0^*}}{\varphi^*_\beta}, (\Omega) \right) \right\} \to 0. \]
(65)
It follows that $\left\| T^*(y_{p*}^n) - T^*(y_0^*) \right\|_\varphi \leq (m_{y_{p*}^n - y_0^*})^*_\varphi, (\Omega)$, where $\delta_{m,p}$ is a seminorm on $(\mathcal{F}_\varphi^*, \| \cdot \|_\varphi)$. This means that $T^*$ is compact and hence $T$ is compact.

(ii) $\Rightarrow$ (i) Assume that $T$ is compact and $(y_{p*}^n)$ is a net in $B_{Y^*}$. Since $B_{Y^*}$ is $\sigma(Y^*, Y)$-compact, without loss of generality we can assume that $y_{p*}^n \to y_0^*$ in $\sigma(Y^*, Y)$ for some $y^* \in B_{Y^*}$.

In view of the compactness of the conjugate operator $T^* : Y^* \to L^p(X)^*$, there exists a subset $(y_{p*})$ of $(y_{p*}^n)$ and $\Phi_0 \in L^p(X)^*$ such that $\left\| T^* (y_{p*}) - \Phi_0 \right\|_p \to 0$. On the other hand, since $T^*$ is $\sigma(Y^*, Y), \sigma(L^p(X)^*, L^p(X))$-continuous, we get $T^*(y_{p*}^n) \to T^*(y_0^*)$ in $\sigma(L^p(X)^*, L^p(X))$. Hence $\Phi_0 = T^*(y_0^*)$; that is, $\left\| T^* (y_{p*}^n) - T^*(y_0^*) \right\|_\varphi \to 0$.

Let $\varepsilon > 0$ be given. Then there exist a pairwise disjoint set $A_1, \ldots, A_n$ in $\Sigma_f(\mu)$ and $x_i \in X$ for $i = 1, \ldots, n$ such that $\left\| \sum_{i=1}^n (1_A \otimes x_i) \right\|_p \leq 1$ and
\[ \left( m_{y_{p*}^n - y_0^*} ight)^*_\varphi, (\Omega) \leq \left( \sum_{i=1}^n (y_{p*}^n - y_0^*) (m (A_i) (x_i)) \right) + \varepsilon. \]
(66)
Hence
\[ \left( m_{y_{p*}^n - y_0^*} ight)^*_\varphi, (\Omega) \leq \left( \sum_{i=1}^n (y_{p*}^n - y_0^*) (T (1_A \otimes x_i)) \right) + \varepsilon \]
\[ \leq \left( y_{p*}^n - y_0^* \right) T \left( \sum_{i=1}^n (1_A \otimes x_i) \right) + \varepsilon + \varepsilon \]
Assume that Corollary 18.

Hence \( p \Delta \) satisfies the space \( \Delta_2 \)-condition. Let \( T \colon L^p(X) \to Y \) be a \( (\mathcal{F}^n, \|\cdot\|_Y) \)-continuous linear operator and \( m : \Sigma_1(\mu) \to \mathcal{L}(X, Y) \) be its representing measure. If the space \( (B_{Y^*}, \delta_{m, \rho^*}) \) is compact, then \( m \) is \( \rho^* \)-seminormally \( \mu \)-continuous.

Corollary 18. Assume that \( \varphi \) is a Young function such that \( \varphi^* \) satisfies the \( \Delta_2 \)-condition. Let \( T : L^p(X) \to Y \) be a \( (\mathcal{F}^n, \|\cdot\|_Y) \)-continuous linear operator and \( m : \Sigma_1(\mu) \to \mathcal{L}(X, Y) \) be its representing measure. If the space \( (B_{Y^*}, \delta_{m, \rho^*}) \) is compact, then \( m \) is \( \rho^* \)-semivariationally \( \mu \)-continuous.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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