Research Article

Characterization of Σ -Semicompleteness via Caristi's Fixed Point Theorem in Semimetric Spaces

Tomonari Suzuki

Department of Basic Sciences, Faculty of Engineering, Kyushu Institute of Technology, Tobata, Kitakyushu 804-8550, Japan

Correspondence should be addressed to Tomonari Suzuki; suzuki-t@mns.kyutech.ac.jp

Received 7 December 2017; Accepted 2 January 2018; Published 1 March 2018

Academic Editor: Satish Shukla

Copyright © 2018 Tomonari Suzuki. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

Introducing the concept of Σ -semicompleteness in semimetric spaces, we extend Caristi's fixed point theorem to Σ -semicomplete semimetric spaces. Via this extension, we characterize Σ -semicompleteness. We also give generalizations of the Banach contraction principle.

1. Introduction

The following, very famous theorem is referred to as Caristi's fixed point theorem. See also [1–10] and references therein.

Theorem 1 (Theorem 1 in [11]). *Let (X, d) be a complete metric space and let T be a mapping on X . Let h be a lower semicontinuous function from X into $[0, \infty)$. Assume $h(Tx) + d(x, Tx) \leq h(x)$ for all $x \in X$. Then T has a fixed point.*

In 1976, Kirk proved that Caristi's fixed point theorem characterizes the metric completeness.

Theorem 2 (Theorem 2 in [12]). *Let (X, d) be a metric space. Then the following are equivalent:*

- (i) X is complete.
- (ii) Every mapping T on X has a fixed point provided there exists a lower semicontinuous function h from X into $[0, \infty)$ satisfying $h(Tx) + d(x, Tx) \leq h(x)$ for all $x \in X$.

Very recently, Theorem 1 was extended to semimetric spaces.

Theorem 3 (see [13]). *Let (X, d) be a $(\Sigma, \neq)$ -complete semimetric space and let T be a mapping on X . Let h be a function from X into $(-\infty, +\infty]$ which is proper, bounded from below, and sequentially lower semicontinuous from above in the sense*

of Definition 6. Assume $h(Tx) + d(x, Tx) \leq h(x)$ for all $x \in X$. Then T has a fixed point.

Remark 4. See Definitions 5 and 6 for the definitions of $(\Sigma, \neq)$ -completeness and others.

It is a very natural question of whether Theorem 3 characterizes $(\Sigma, \neq)$ -completeness of the underlying space.

In this paper, we give a negative answer to this question (see Example 17). Motivated by this fact, we introduce the concept of Σ -semicompleteness and extend Theorem 3 to Σ -semicomplete semimetric spaces (see Corollary 12). And we characterize the Σ -semicompleteness via Corollary 12 (see Theorem 13). Also we give generalizations of the Banach contraction principle (see Section 4).

2. Preliminaries

Throughout this paper, we denote by $\mathbb{N}$, $\mathbb{Q}$, and $\mathbb{R}$ the sets of all positive integers, all rational numbers, and all real numbers, respectively. For an arbitrary set A , we also denote by $\#A$ the cardinal number of A .

In this section, we give some preliminaries.

Definition 5. Let X be a nonempty set and let d be a function from $X \times X$ into $[0, \infty)$. Then (X, d) is said to be a semimetric space if the following hold:

$$(D1) \quad d(x, x) = 0.$$

$$(D2) \ d(x, y) = 0 \Rightarrow x = y.$$

$$(D3) \ d(x, y) = d(y, x) \text{ (symmetry).}$$

Definition 6. Let (X, d) be a semimetric space, let $\{x_n\}$ be a sequence in X , and let $x \in X$. Let $\kappa \in \mathbb{N}$ and let h be a function from X into $(-\infty, +\infty]$.

- (i) $\{x_n\}$ is said to *converge* to x if $\lim_n d(x_n, x) = 0$.
- (ii) $\{x_n\}$ is said to be *Cauchy* if $\lim_n \sup\{d(x_n, x_m) : m > n\} = 0$.
- (iii) $\{x_n\}$ is said to be Σ -*Cauchy* if $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$.
- (iv) $\{x_n\}$ is said to be $(\Sigma, \neq)$ -*Cauchy* if x_n ($n \in \mathbb{N}$) are all different and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$.
- (v) X is said to be *Hausdorff* if $\lim_n d(x_n, x) = 0$ and $\lim_n d(x_n, y) = 0$ imply $x = y$.
- (vi) X is said to be κ -*Hausdorff* if

$$\lim_{n \rightarrow \infty} D(x, u_n^{(1)}, \dots, u_n^{(\kappa)}, y) = 0 \quad (1)$$

implies $x = y$, where

$$\begin{aligned} D(x, u_n^{(1)}, \dots, u_n^{(\kappa)}, y) \\ = d(x, u_n^{(1)}) + d(u_n^{(1)}, u_n^{(2)}) + \dots \\ + d(u_n^{(\kappa-1)}, u_n^{(\kappa)}) + d(u_n^{(\kappa)}, y). \end{aligned} \quad (2)$$

- (vii) X is said to be *complete* if every Cauchy sequence converges.
- (viii) X is said to be Σ -*complete* if every Σ -Cauchy sequence converges.
- (ix) X is said to be $(\Sigma, \neq)$ -*complete* if every $(\Sigma, \neq)$ -Cauchy sequence converges.
- (x) X is said to be *semicomplete* if every Cauchy sequence has a convergent subsequence.
- (xi) X is said to be Σ -*semicomplete* if every Σ -Cauchy sequence has a convergent subsequence.
- (xii) X is said to be $(\Sigma, \neq)$ -*semicomplete* if every $(\Sigma, \neq)$ -Cauchy sequence has a convergent subsequence.
- (xiii) d is said to be *sequentially lower semicontinuous* if $d(x, y) \leq \liminf_n d(x_n, y_n)$ provided $\{x_n\}$ converges to x and $\{y_n\}$ converges to y .
- (xiv) h is said to be *sequentially lower semicontinuous* if $h(x) \leq \liminf_n h(x_n)$ provided $\{x_n\}$ converges to x .
- (xv) h is said to be *sequentially lower semicontinuous from above* if $h(x) \leq \lim_n h(x_n)$ provided $\{x_n\}$ converges to x and $\{h(x_n)\}$ is strictly decreasing.
- (xvi) h is said to be *proper* if $\{x \in X : h(x) \in \mathbb{R}\} \neq \emptyset$.

Remark 7.

- (i) The definitions of κ -Hausdorffness and Σ -semicompleteness are new.
- (ii) It is obvious that X is Hausdorff $\Leftrightarrow X$ is 1-Hausdorff.

- (iii) It is also obvious that X is λ -Hausdorff $\Rightarrow X$ is κ -Hausdorff provided $\kappa < \lambda$.

Proposition 8. Let (X, d) be a semimetric space. Then the following implications hold:

- (I) $(i) \Rightarrow (ii) \Rightarrow (iii) \Leftrightarrow (iv) \Rightarrow (vi)$.
- (II) $(i) \Rightarrow (v) \Rightarrow (vi)$.
- (i) X is Σ -*complete*.
- (ii) X is $(\Sigma, \neq)$ -*complete*.
- (iii) X is Σ -*semicomplete*.
- (iv) X is $(\Sigma, \neq)$ -*semicomplete*.
- (v) X is *complete*.
- (vi) X is *semicomplete*.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iv), (iii) $\Rightarrow$ (iv), and (v) $\Rightarrow$ (vi) obviously hold. We have already proved (i) $\Rightarrow$ (v) in [13].

In order to prove (iv) $\Rightarrow$ (iii), we assume (iv). Let $\{x_n\}$ be a Σ -Cauchy sequence in X . We consider the following two cases:

- (a) There exists $z \in X$ satisfying $\#\{n \in \mathbb{N} : x_n = z\} = \infty$.
- (b) For any $x \in X$, $\#\{n \in \mathbb{N} : x_n = x\} < \infty$.

In the first case, some subsequence of $\{x_n\}$ converges to z . In the second case, we define a subsequence $\{f(n)\}$ of the sequence $\{n\}$ in $\mathbb{N}$ as follows: $f(1) = 1$. We assume that $f(n)$ is defined. Then we define $f(n+1)$ by

$$f(n+1) = \max\{k \in \mathbb{N} : x_k = x_{f(n)}\} + 1. \quad (3)$$

By induction, we have defined $\{f(n)\}$. We note that $x_{f(n)}$ ($n \in \mathbb{N}$) are all different. We also have

$$\sum_{n=1}^{\infty} d(x_{f(n)}, x_{f(n+1)}) \leq \sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty. \quad (4)$$

Thus, $\{x_{f(n)}\}$ is $(\Sigma, \neq)$ -Cauchy. From (iv), there exists a subsequence of $\{x_{f(n)}\}$ which converges. It is obvious that the subsequence is also one of subsequences of $\{x_n\}$. Therefore we have shown (iii).

Let us prove (iii) $\Rightarrow$ (vi). We assume (iii). Let $\{x_n\}$ be a Cauchy sequence in X . Choose a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying

$$\sup\{d(x_\ell, x_m) : m > \ell\} < 2^{-n} \quad (5)$$

for any $\ell, n \in \mathbb{N}$ with $\ell \geq f(n)$. Then $\{x_{f(n)}\}$ satisfies

$$\sum_{n=1}^{\infty} d(x_{f(n)}, x_{f(n+1)}) \leq \sum_{n=1}^{\infty} 2^{-n} = 1 < \infty, \quad (6)$$

thus $\{x_{f(n)}\}$ is Σ -Cauchy. From (iii), there exist $z \in X$ and a subsequence $\{g(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying $\lim_n d(x_{f \circ g(n)}, z) = 0$. Since $\{x_{f \circ g(n)}\}$ is also a subsequence of $\{x_n\}$, we obtain (vi). $\square$

Proposition 9. Let (X, d) be a semimetric space. Assume that d is sequentially lower semicontinuous. Then X is 2-Hausdorff.

Proof. Suppose

$$\lim_{n \rightarrow \infty} (d(x, u_n) + d(u_n, v_n) + d(v_n, y)) = 0. \quad (7)$$

Then $\{u_n\}$ converges to x and $\{v_n\}$ converges to y . So we have

$$d(x, y) \leq \liminf_{n \rightarrow \infty} d(u_n, v_n) = 0. \quad (8)$$

Thus, we obtain $x = y$. $\square$

Proposition 10. Let (X, d) be a Σ -complete semimetric space. Then X is κ -Hausdorff for any $\kappa \in \mathbb{N}$.

Proof. Suppose

$$\lim_{n \rightarrow \infty} D(x, u_n^{(1)}, \dots, u_n^{(\kappa)}, y) = 0. \quad (9)$$

Choose a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ satisfying

$$D(x, u_{f(n)}^{(1)}, \dots, u_{f(n)}^{(\kappa)}, y) < 2^{-n} \quad (10)$$

for $n \in \mathbb{N}$. Define a sequence $\{v_n\}$ in X as follows:

$$\begin{aligned} v_{(2\kappa+2)i+1} &= x, \\ v_{(2\kappa+2)i+j+1} &= u_{f(2i+1)}^{(j)}, \\ v_{(2\kappa+2)i+\kappa+2} &= y, \\ v_{(2\kappa+2)i+2\kappa-j+3} &= u_{f(2i+2)}^{(j)} \end{aligned} \quad (11)$$

for $i \in \mathbb{N} \cup \{0\}$ and $j \in \{1, \dots, \kappa\}$. That is, $\{v_n\}$ is as follows:

$$\begin{aligned} &x, u_{f(1)}^{(1)}, \dots, u_{f(1)}^{(\kappa)}, y, u_{f(2)}^{(1)}, \dots, u_{f(2)}^{(\kappa)}, \\ &x, u_{f(3)}^{(1)}, \dots, u_{f(3)}^{(\kappa)}, y, u_{f(4)}^{(1)}, \dots, u_{f(4)}^{(\kappa)}, x, \dots \end{aligned} \quad (12)$$

We have

$$\begin{aligned} \sum_{i=1}^{\infty} d(v_i, v_{i+1}) &= \sum_{i=0}^{\infty} D(v_{(\kappa+1)i+1}, \dots, v_{(\kappa+1)(i+1)+1}) \\ &< \sum_{i=1}^{\infty} 2^{-i} = 1 < \infty. \end{aligned} \quad (13)$$

Thus, $\{v_n\}$ is Σ -Cauchy. Since X is Σ -complete, $\{v_n\}$ converges to some z . From the definition of $\{v_n\}$, we have $x = z = y$. Thus, X is κ -Hausdorff. $\square$

3. Caristi's Theorem

In this section, we first prove a Kirk-Saliga type fixed point theorem [14] in Σ -semicomplete semimetric spaces. See also [13].

Let α be an ordinal number. We denote by α^+ and α^- the successor and the predecessor of α , respectively. α is said to be *isolated* if α^- exists. On the other hand, α is said to be *limit* if $\alpha \neq 0$ holds and α^- does not exist. For $\kappa \in \mathbb{N}$, we define $\alpha + \kappa$ by

$$\alpha + \kappa = \alpha^{\overbrace{+ \dots +}^{\kappa}}. \quad (14)$$

Theorem 11. Let (X, d) be a Σ -semicomplete semimetric space and let T be a mapping on X . Let h be a function from X into $(-\infty, +\infty]$ which is proper and bounded from below. Assume that h is sequentially lower semicontinuous from above in the sense of Definition 6. Assume also that there exists $\kappa \in \mathbb{N}$ satisfying the following:

- (i) $h(Tx) \leq h(x)$ for all $x \in X$.
- (ii) $h(T^\kappa x) + d(x, Tx) \leq h(x)$ for all $x \in X$.

Then T has a fixed point.

Proof. Define a function H from X into $(-\infty, +\infty]$ by

$$H(x) = \sum_{j=0}^{\kappa-1} h(T^j x), \quad (15)$$

where T^0 is the identity mapping on X . We have from (ii)

$$H(Tx) + d(x, Tx) \leq H(x) \quad \text{for } x \in X. \quad (16)$$

Arguing by contradiction, we assume $Tx \neq x$ for any $x \in X$. Let Ω be the first uncountable ordinal number. Using transfinite induction, we will define a net $\{u_\alpha : \alpha \in \Omega\}$ satisfying the following:

- $(P_1 : \alpha)$ $h(u_\alpha) \leq h(u_\beta)$ and $H(u_\alpha) < H(u_\beta)$ for any $\beta \in \Omega$ with $\beta < \alpha$.
- $(P_2 : \alpha)$ $h(u_\alpha) < h(u_\beta)$ for any $\beta \in \Omega$ with $\beta + \kappa \leq \alpha$.
- $(P_3 : \alpha)$ For any $\varepsilon > 0$ and for any $\beta \in \Omega$ with $\beta < \alpha$, there exists a finite sequence $(\gamma_0, \dots, \gamma_n) \in \Omega^{n+1}$ satisfying

$$\beta = \gamma_0 < \gamma_1 < \dots < \gamma_n = \alpha, \quad (17)$$

$$H(u_\alpha) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) < H(u_\beta) + \varepsilon.$$

Fix $u \in X$ with $h(u) < \infty$. It follows from (i) that $H(u) < \infty$ holds. Put $u_0 = u$. Then $(P_1 : 0) - (P_3 : 0)$ obviously hold.

Fix $\alpha \in \Omega$ with $0 < \alpha$ and assume that $(P_1 : \beta) - (P_3 : \beta)$ hold for $\beta < \alpha$. We consider the following two cases:

- (a) α is isolated.
- (b) α is limit.

In the first case, we put $u_\alpha = Tu_{\alpha^-}$. For any $\beta < \alpha$, since $\beta \leq \alpha^-$ and $u_{\alpha^-} \neq u_\alpha$ hold, we have by $(P_1 : \alpha^-)$, (i) and (16)

$$h(u_\alpha) \leq h(u_{\alpha^-}) \leq h(u_\beta), \quad (18)$$

$$H(u_\alpha) < H(u_\alpha) + d(u_{\alpha^-}, u_\alpha) \leq H(u_{\alpha^-}) \leq H(u_\beta).$$

Thus, we have shown $(P_1 : \alpha)$. For $\beta \in \Omega$ with $\beta + \kappa \leq \alpha$, we have by $(P_1 : \alpha)$ and (ii)

$$\begin{aligned} h(u_\alpha) &\leq h(u_{\beta+\kappa}) = h(T^\kappa u_\beta) \\ &< h(T^\kappa u_\beta) + d(u_\beta, Tu_\beta) \leq h(u_\beta). \end{aligned} \quad (19)$$

Thus, we have shown $(P_2 : \alpha)$. Fix $\varepsilon > 0$ and $\beta \in \Omega$ with $\beta < \alpha$. In the case where $\beta = \alpha^-$, putting $\gamma_0 = \beta$ and $\gamma_1 = \alpha$, we have by (16)

$$\begin{aligned} H(u_\alpha) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) &= H(Tu_\beta) + d(u_\beta, Tu_\beta) \\ &\leq H(u_\beta) < H(u_\beta) + \varepsilon. \end{aligned} \quad (20)$$

In the other case, where $\beta < \alpha^-$, from $(P_3 : \alpha^-)$, there exists a finite sequence $(\gamma_0, \dots, \gamma_n) \in \Omega^{n+1}$ satisfying

$$\begin{aligned} \beta &= \gamma_0 < \gamma_1 < \dots < \gamma_n = \alpha^-, \\ H(u_{\alpha^-}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) &< H(u_\beta) + \varepsilon. \end{aligned} \quad (21)$$

Putting $\gamma_{n+1} = \alpha$, we have by (16)

$$\begin{aligned} H(u_\alpha) + \sum_{j=0}^n d(u_{\gamma_j}, u_{\gamma_{j+1}}) \\ \leq H(u_{\alpha^-}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) < H(u_\beta) + \varepsilon. \end{aligned} \quad (22)$$

Thus, we have shown $(P_3 : \alpha)$. Therefore we have defined u_α satisfying $(P_1 : \alpha)-(P_3 : \alpha)$ in the first case.

In the second case, let $\{\beta_n\}$ be a strictly increasing sequence in Ω converging to α ; that is, the following hold:

(i) $\beta_n < \alpha$ for $n \in \mathbb{N}$.

(ii) For any $\beta < \alpha$, there exists $n \in \mathbb{N}$ satisfying $\beta < \beta_n$.

For any $n \in \mathbb{N}$, from $(P_3 : \beta_{n+1})$, we can choose a finite sequence $(\gamma^{(n)}_0, \dots, \gamma^{(n)}_{\gamma_n}) \in \Omega^{\gamma_n+1}$ satisfying

$$\begin{aligned} \beta_n &= \gamma^{(n)}_0 < \gamma^{(n)}_1 < \dots < \gamma^{(n)}_{\gamma_n} \\ &= \beta_{n+1}, \\ \sum_{j=0}^{\gamma_n-1} d(u_{\gamma^{(n)}_j}, u_{\gamma^{(n)}_{j+1}}) &< H(u_{\beta_n}) - H(u_{\beta_{n+1}}) + 2^{-n}. \end{aligned} \quad (23)$$

Since h is bounded from below, H is also bounded from below. So we have

$$\begin{aligned} \sum_{n=1}^{\infty} \sum_{j=0}^{\gamma_n-1} d(u_{\gamma^{(n)}_j}, u_{\gamma^{(n)}_{j+1}}) &< H(u_{\beta_1}) - \lim_{n \rightarrow \infty} H(u_{\beta_n}) + 1 \\ &< \infty. \end{aligned} \quad (24)$$

Since X is Σ -semicomplete, the sequence

$$\begin{aligned} (\beta_1 =) \gamma^{(1)}_0, \dots, \gamma^{(1)}_{\gamma_1} (= \beta_2 = \gamma^{(2)}_0), \gamma^{(2)}_1, \dots, \\ \gamma^{(2)}_{\gamma_2} (= \beta_3 = \gamma^{(3)}_0), \gamma^{(3)}_1, \dots \end{aligned} \quad (25)$$

has a subsequence $\{\delta_n\}$ such that $\{u_{\delta_n}\}$ converges to some $u_\alpha \in X$. We note that $\{\delta_n\}$ is strictly increasing and it converges to α . Taking a subsequence, without loss of generality, we may assume $\delta_n + \kappa \leq \delta_{n+1}$ for $n \in \mathbb{N}$. We have by $(P_2 : \delta_{n+1})$

$$h(u_{\delta_{n+1}}) < h(u_{\delta_n}) \quad \text{for } n \in \mathbb{N}. \quad (26)$$

Thus, $\{h(u_{\delta_n})\}$ is strictly decreasing. Fix $\varepsilon > 0$ and $\beta \in \Omega$ with $\beta < \alpha$. We can choose $\nu \in \mathbb{N}$ satisfying

$$\begin{aligned} \beta &< \delta_\nu, \\ d(u_{\delta_\nu}, u_\alpha) &< \varepsilon. \end{aligned} \quad (27)$$

Since h is sequentially lower semicontinuous from above, we have from (i)

$$\begin{aligned} h(u_\alpha) &\leq \lim_{n \rightarrow \infty} h(u_{\delta_n}) < h(u_{\delta_\nu}) \leq h(u_\beta), \\ H(u_\alpha) &\leq \kappa h(u_\alpha) \leq \kappa \lim_{n \rightarrow \infty} h(u_{\delta_n}) \\ &= \kappa \inf \{h(u_\gamma) : \gamma < \alpha\} = \lim_{n \rightarrow \infty} H(u_{\delta_n}) \\ &< H(u_{\delta_\nu}) < H(u_\beta). \end{aligned} \quad (28)$$

We have shown $(P_1 : \alpha)$ and $(P_2 : \alpha)$. We can choose a finite sequence $(\gamma_0, \dots, \gamma_n) \in \Omega^{n+1}$ satisfying

$$\begin{aligned} \beta &= \gamma_0 < \gamma_1 < \dots < \gamma_n = \delta_\nu, \\ H(u_{\delta_\nu}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) &< H(u_\beta) + \varepsilon. \end{aligned} \quad (29)$$

Putting $\gamma_{n+1} = \alpha$, we have by $(P_1 : \alpha)$

$$\begin{aligned} H(u_\alpha) + \sum_{j=0}^n d(u_{\gamma_j}, u_{\gamma_{j+1}}) \\ < H(u_{\delta_\nu}) + \sum_{j=0}^{n-1} d(u_{\gamma_j}, u_{\gamma_{j+1}}) + \varepsilon < H(u_\beta) + 2\varepsilon. \end{aligned} \quad (30)$$

Thus we have defined u_α satisfying $(P_1 : \alpha)-(P_3 : \alpha)$ in the second case.

Therefore by transfinite induction, we have defined the net $\{u_\alpha : \alpha \in \Omega\}$ satisfying $(P_1 : \alpha)-(P_3 : \alpha)$ for any $\alpha \in \Omega$. Since the net $\{H(u_\alpha) : \alpha \in \Omega\}$ is strictly decreasing, we obtain

$$\#\mathbb{Q} = \#\mathbb{N} < \#\Omega \leq \#\mathbb{Q}, \quad (31)$$

which implies a contradiction. Therefore there exists a fixed point of T . $\square$

Using Theorem 11, we can generalize Theorem 3.

Corollary 12. Let (X, d) be a Σ -semicomplete semimetric space and let T be a mapping on X . Let h be a function from X into $(-\infty, +\infty]$ which is proper and bounded from below. Assume that h is sequentially lower semicontinuous from above in the sense of Definition 6. Assume also

$$h(Tx) + d(x, Tx) \leq h(x) \quad (32)$$

for all $x \in X$. Then T has a fixed point.

Via Corollary 12, we characterize the Σ -semicompleteness of X .

Theorem 13. *Let (X, d) be a semimetric space. Then the following are equivalent:*

- (i) X is Σ -semicomplete.
- (ii) Every mapping T on X has a fixed point provided there exists a function h from X into $[0, \infty]$ such that h is proper and sequentially lower semicontinuous and (32) holds for all $x \in X$.

Proof. By Corollary 12, we obtain (i) $\Rightarrow$ (ii).

In order to prove (ii) $\Rightarrow$ (i), we will show $\neg$ (i) $\Rightarrow \neg$ (ii). We assume that X is not Σ -semicomplete. Then by Proposition 8, X is not $(\Sigma, \neq)$ -semicomplete. So there exists a sequence $\{x_n\}$ in X such that x_n ($n \in \mathbb{N}$) are all different, $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ holds, and there does not exist a subsequence which converges. Define a mapping T on X and a function h from X into $(0, \infty]$ by

$$Tx = \begin{cases} x_{n+1} & \text{if } x = x_n \text{ for some } n \in \mathbb{N} \\ x_1 & \text{if } x \notin \{x_n : n \in \mathbb{N}\}, \end{cases}$$

$$h(x) = \begin{cases} \sum_{j=n}^{\infty} d(x_j, x_{j+1}) & \text{if } x = x_n \text{ for some } n \in \mathbb{N} \\ \infty & \text{if } x \notin \{x_n : n \in \mathbb{N}\}. \end{cases} \quad (33)$$

We note that T and h are well defined because x_n ($n \in \mathbb{N}$) are all different and $\sum_{n=1}^{\infty} d(x_n, x_{n+1}) < \infty$ holds. Then h is proper, (32) holds for any $x \in X$, and T does not have a fixed point. Let $\{y_n\}$ be a sequence in X converging to some $y \in X$. Arguing by contradiction, we assume

$$h(y) > \liminf_{n \rightarrow \infty} h(y_n). \quad (34)$$

Then from the definition of h , there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $y_{f(n)} \in \{x_m : m > \nu\}$ for any $n \in \mathbb{N}$, where we put $\nu \in \mathbb{N} \cup \{0\}$ by

$$\nu = \begin{cases} n & \text{if } y = x_n \text{ for some } n \in \mathbb{N} \\ 0 & \text{if } y \notin \{x_n : n \in \mathbb{N}\}. \end{cases} \quad (35)$$

Define a function g from $\mathbb{N}$ into $\{m \in \mathbb{N} : m > \nu\}$ by $y_{f(n)} = x_{g(n)}$. We consider the following two cases:

- (a) $\limsup_n g(n) = \infty$.
- (b) $\mu := \limsup_n g(n) < \infty$.

In the first case, $\{x_n\}$ has a subsequence converging to y . This is a contradiction. In the second case, we have

$$\begin{aligned} \infty &= \#\{n \in \mathbb{N} : g(n) = \mu\} = \#\{n \in \mathbb{N} : y_{f(n)} = x_\mu\} \\ &\leq \#\{n \in \mathbb{N} : y_n = x_\mu\}, \end{aligned} \quad (36)$$

which implies $y = x_\mu$. This is also a contradiction. Therefore we obtain $h(y) \leq \liminf_n h(y_n)$. Thus, h is sequentially lower semicontinuous. $\square$

4. Banach's Theorem

The author has extended the Banach contraction principle [15, 16] to semicomplete semimetric spaces. Such a result will be published somewhere else. See also [17]. In this section, we give other generalizations.

Theorem 14. *Let (X, d) be a 2-Hausdorff Σ -semicomplete semimetric space. Let T be a contraction on X . Then T has a unique fixed point z . Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Proof. Let $r \in [0, 1)$ satisfy

$$d(Tx, Ty) \leq rd(x, y) \quad \text{for } x, y \in X. \quad (37)$$

Fix $u \in X$. Then we have

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) \leq \sum_{n=1}^{\infty} r^n d(u, Tu) < \infty. \quad (38)$$

Since X is Σ -semicomplete, there exists a subsequence $\{f(n)\}$ of $\{n\}$ in $\mathbb{N}$ such that $\{T^{f(n)} u\}$ converges to some $z \in X$. We have

$$\lim_{n \rightarrow \infty} d(T^{f(n)+1} u, Tz) \leq \lim_{n \rightarrow \infty} rd(T^{f(n)} u, z) = 0. \quad (39)$$

Thus, $\{T^{f(n)+1} u\}$ converges to Tz . So we have

$$\lim_{n \rightarrow \infty} D(z, T^{f(n)} u, T^{f(n)+1} u, Tz) = 0. \quad (40)$$

From 2-Hausdorffness of X , we obtain $Tz = z$.

For any $x \in X$, we have

$$\begin{aligned} \lim_{n \rightarrow \infty} d(T^n x, z) &= \lim_{n \rightarrow \infty} d(T^n x, T^n z) \leq \lim_{n \rightarrow \infty} r^n d(x, z) \\ &= 0. \end{aligned} \quad (41)$$

The uniqueness of the fixed point follows from (41). $\square$

Theorem 15. *Let (X, d) be a $(\Sigma, \neq)$ -complete Hausdorff semimetric space. Let T be a contraction on X . Then T has a unique fixed point z . Moreover, $\{T^n x\}$ converges to z for all $x \in X$.*

Proof. Let $r \in [0, 1)$ satisfy (37). Fix $u \in X$. We consider the following two cases:

- (a) There exists $\nu \in \mathbb{N}$ satisfying $T^{\nu+1} u = T^\nu u$.
- (b) $T^n u \neq T^{n+1} u$ for any $n \in \mathbb{N}$.

In the first case, $T^\nu u$ is a fixed point of T . In the second case, we have

$$\begin{aligned} d(T^{n+1} u, T^{n+2} u) &\leq rd(T^n u, T^{n+1} u) \\ &< d(T^n u, T^{n+1} u) \end{aligned} \quad (42)$$

for any $n \in \mathbb{N}$. Hence $T^n u$ ($n \in \mathbb{N}$) are all different. We also have

$$\sum_{n=1}^{\infty} d(T^n u, T^{n+1} u) \leq \sum_{n=1}^{\infty} r^n d(u, Tu) < \infty. \quad (43)$$

Since X is $(\sum, \neq)$ -complete, $\{T^n u\}$ converges to some $z \in X$. We have

$$\lim_{n \rightarrow \infty} d(T^{n+1}u, Tz) \leq \lim_{n \rightarrow \infty} rd(T^n u, z) = 0. \quad (44)$$

Thus, $\{T^{n+1}u\}$ converges to Tz . Since X is Hausdorff, we obtain $Tz = z$. Therefore in both cases, there exists a fixed point of T . As in the proof of Theorem 14, we can prove the remainder. $\square$

By Propositions 8–10 and Theorems 14 and 15, we obtain the following corollary.

Corollary 16. *Let (X, d) be a semimetric space. Assume that either of the following holds:*

- (i) X is $\sum$ -complete.
- (ii) X is $\sum$ -semicomplete and d is sequentially lower and semicontinuous.

Let T be a contraction on X . Then T has a unique fixed point z . Moreover, $\{T^n x\}$ converges to z for all $x \in X$.

5. Example

We finally give an example which tells that Theorem 3 does not characterize $(\sum, \neq)$ -completeness of the underlying space.

Example 17. Put $X = \{0\} \cup \{2^{-n} : n \in \mathbb{N}\}$. Define a function d from $X \times X$ into $[0, \infty)$ as follows:

$$\begin{aligned} d(x, x) &= 0, \\ d(2^{-n}, 2^{-(n+1)}) &= d(2^{-(n+1)}, 2^{-n}) = 2^{-(n+1)} \\ &\quad \text{for } n \in \mathbb{N}, \end{aligned} \quad (45)$$

$$d(2^{-2n}, 0) = d(0, 2^{-2n}) = 2^{-2n} \quad \text{for } n \in \mathbb{N},$$

$$d(x, y) = 1 \quad \text{otherwise.}$$

Then the following assertions holds:

- (i) (X, d) is a semimetric space.
- (ii) X is not $(\sum, \neq)$ -semicomplete.
- (iii) X is $\sum$ -semicomplete.
- (iv) Every mapping T on X has a fixed point provided there exists a function h from X into $[0, \infty]$ such that h is proper and sequentially lower semicontinuous and (32) holds for all $x \in X$.

Proof. (i) obviously holds. (iv) follows from (iii) and Corollary 12.

We will show (ii). It is obvious that $\{2^{-n}\}$ is a $(\sum, \neq)$ -Cauchy sequence. However, we have $\limsup_n d(2^{-n}, x) = 1$ for all $x \in X$. Therefore we obtain (ii).

In order to prove (iii), we will show that X is $(\sum, \neq)$ -semicomplete. Let $\{x_n\}$ be a $(\sum, \neq)$ -Cauchy sequence in X . We can choose $\mu, \nu \in \mathbb{N}$ satisfying

$$\begin{aligned} 0 &\notin \{x_n : n \geq \nu\}, \\ x_\nu &= 2^{-\mu}. \end{aligned} \quad (46)$$

From the definition of d , we have $x_{\nu+j} = 2^{-(\mu+j)}$ for $j \in \mathbb{N}$. So $\{x_n\}$ has a subsequence converging to 0. We have shown that X is $(\sum, \neq)$ -semicomplete. By Proposition 8, X is $\sum$ -semicomplete. $\square$

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

The author is supported in part by JSPS KAKENHI Grant no. 16K05207 from Japan Society for the Promotion of Science.

References

- [1] B. Alamri, T. Suzuki, and L. A. Khan, “Caristi’s fixed point theorem and Subrahmanyam’s fixed point theorem in v -generalized metric spaces,” *Journal of Function Spaces*, vol. 2015, Article ID 709391, 2015.
- [2] J. Caristi, “Fixed point theorems for mappings satisfying inwardness conditions,” *Transactions of the American Mathematical Society*, vol. 215, pp. 241–251, 1976.
- [3] D. Downing and W. A. Kirk, “A generalization of Caristi’s theorem with applications to nonlinear mapping theory,” *Pacific Journal of Mathematics*, vol. 69, no. 2, pp. 339–346, 1977.
- [4] I. Ekeland, “On the variational principle,” *Journal of Mathematical Analysis and Applications*, vol. 47, pp. 324–353, 1974.
- [5] I. Ekeland, “Nonconvex minimization problems,” *Bulletin (New Series) of the American Mathematical Society*, vol. 1, no. 3, pp. 443–474, 1979.
- [6] T. Suzuki, “Generalized distance and existence theorems in complete metric spaces,” *Journal of Mathematical Analysis and Applications*, vol. 253, no. 2, pp. 440–458, 2001.
- [7] T. Suzuki, “On Downing-Kirk’s theorem,” *Journal of Mathematical Analysis and Applications*, vol. 286, no. 2, pp. 453–458, 2003.
- [8] T. Suzuki, “Generalized Caristi’s fixed point theorems by Bae and others,” *Journal of Mathematical Analysis and Applications*, vol. 302, no. 2, pp. 502–508, 2005.
- [9] T. Suzuki, “Redefinition of,” *Journal of Function Spaces*, vol. 2017, pp. 1–8, 2017.
- [10] T. Suzuki, “Nadler’s fixed point theorem in v -generalized metric spaces,” *Fixed Point Theory and Applications*, vol. 2017, no. 1, 2017.
- [11] J. Caristi and W. A. Kirk, “Geometric fixed point theory and inwardness conditions,” in *The Geometry of Metric and Linear Spaces*, vol. 490 of *Lecture Notes in Mathematics*, pp. 74–83, Springer, Berlin, Germany, 1975.
- [12] W. A. Kirk, “Caristi’s fixed point theorem and metric convexity,” *Colloquium Mathematicum*, vol. 36, no. 1, pp. 81–86, 1976.
- [13] M. R. Alfuraidan, “Remarks on Caristi’s fixed point theorem in metric spaces with a graph,” *Fixed Point Theory and Applications*, vol. 2014, no. 1, article no. 240, 2014.
- [14] W. A. Kirk and L. M. Saliga, “The Brézis-Browder order principle and extensions of Caristi’s theorem,” *Nonlinear Analysis: Theory, Methods & Applications*, vol. 47, no. 4, pp. 2765–2778, 2001.
- [15] S. Banach, “Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales,” *Fundamenta Mathematicae*, vol. 3, pp. 133–181, 1922.

- [16] R. Caccioppoli, "Un teorema generale sull'esistenza di elementi uniti in una trasformazione funzionale," *Rendiconti dell'Accademia Nazionale dei Lincei*, vol. 11, pp. 794–799, 1930.
- [17] J. Jachymski, J. Matkowski, and T. Świątkowski, "Nonlinear contractions on semimetric spaces," *Journal of Applied Analysis*, vol. 1, no. 2, pp. 125–134, 1995.