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Research Article

Existence of Equilibria for Discontinuous Games in General Topological Spaces with Binary Relations

Ji-Cheng Hou

School of Science, Beijing Information Science and Technology University, Beijing 100192, China

Correspondence should be addressed to Ji-Cheng Hou; houjc163@163.com

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We provide several results on the existence of equilibria for discontinuous games in general topological spaces without any convexity structure. All of the theorems yielding existence of equilibria here are stated in terms of the player's preference relations over joint strategies.

1. Introduction

Nash equilibrium is a fundamental concept in the theory of games and the most widely used method of predicting the outcome of a strategic interaction in almost all areas of economics as well as in business and other social sciences.

Following Reny [1] and Tian [2], a game G is simply a family of ordered tuples $(X_i, \succeq_i)_{i\in I}$, where I is a finite or infinite (countable or uncountable) set of players, and, for each $i \in I$, X_i is the set of strategies of player i, and $\succeq_i$ is a binary relation on X.

When $\succeq_i$ can be represented by a payoff function $u_i : X \to \boldsymbol{R}$, the game $G = (X_i, u_i)_{i\in I}$ introduced by Nash in [3] is a special case of $G = (X_i, \succeq_i)_{i\in I}$.

A strategy profile $x \in X$ is a pure strategy Nash equilibrium of a game G if $x \succeq_i (y_i, x_i)$ for all $i \in I$ and $y_i \in X_i$.

Nash [3] proved that an (Nash) equilibrium of the game exists if the set X_i of pure strategies of player i is a compact convex subset of an Euclidean space, and if payoff function u_i of player i is continuous and (quasi-)concave in x_i, for each $i \in I$. However, it is known that many important games frequently exhibit discontinuities or non-quasi-concavity in payoffs, such as those in [4, 5]. Also, many economic models do not have convex strategy spaces, so payoff functions under consideration do not have any form of quasi-concavity.

Accordingly, many economists continually strive to seek to weaken the continuity and quasi-concavity of payoff functions. Dasgupta and Maskin [5], Reny [6], Nessah [7], Nessah and Tian [8], and others established the existence of pure strategy Nash equilibrium for discontinuous, compact, and quasi-concave games. Baye et al. [4], Yu [9], Tan et al. [10], Zhang [11], Lignola [12], Nessah and Tian [13, 14], Kim and Lee [15], Hou [16], Chang [17], and Tian [10] and others investigated the existence of pure strategy Nash equilibrium for discontinuous and/or non-quasi-concave games with finite or countable players by using the approach to consider a mapping of individual payoffs into an aggregator function (the aggregator function $U : X \times X \to \boldsymbol{R}$ is defined by $U(y,x) = \sum_{i\in I} u_i(y_i, x_{-i})$ for each $(x,y) \in X \times X$.), which is pioneered by Nikaido and Isoda [18]. To use these results, one must analyze the aggregator function. Such an analysis involves a high dimension and is hard to check in a particular game. Also, it was already indicated in [4, 19] that the quasi-concavity of individual payoffs is not sufficient to establish these concavities that appeared in [4, 9–12, 15–17] for the aggregator function. In addition, to use the method of [18], the countability of amount of players in the game considered is needed.

In this paper, we firstly establish a new existence result of Nash equilibria for discontinuous games in general topological spaces with binary relations. Then, we give some results on the existence of symmetric Nash equilibria and dominant strategy equilibria in general topological spaces without any convexity structure (geometrical or abstract). All of the theorems yielding existence of equilibria here are stated in terms of the players preference relations over joint strategies. It should be emphasized that the method we use is different in essence from those methods given in all results mentioned above.

The paper proceeds as follows. Section 2 provides some notations. Section 3 provides a new notion called generalized convex game and our main result, Theorem 8, as well as an example which holds our assumptions, but the old ones do not hold. Section 4 provides a theorem which is a generalization of Proposition 5.2 of Reny [1] to general topological spaces. Section 5 provides a new notion called generalized uniformly quasi-concavity which is a natural extension of the uniformly transfer quasi-concavity introduced by Bay et al. [4] to topological spaces, and a characterization of dominant strategy equilibrium for games in general topological spaces.

2. Preliminaries

Throughout this work, all topological spaces are assumed to be Hausdorff. Let A be a subset of a topological space X. We denote by $\overline{A}$ the closure of A in X. If A is a subset of a vector space, we denote by co$A$ the convex hull of A. We use $\boldsymbol{R}$ to denote the set of all real numbers, $\boldsymbol{R}^{n+1}$ to denote the $n+1$ dimensional Euclidean space, Δ_n to denote the standard n-dimensional simplex in R^{n+1}, and e_i for $i = 0, 1, \dots, n$ to denote the standard base in Δ_n. Let I be the set of players that is either finite or infinite (even uncountable). Each player i's strategy space X_i is a general topological space without any convexity (geometric or abstract). Denote by X the Cartesian product of all X_i's equipped with the product topology, which is the set of strategy profiles. For each player $i \in I$, denote by x_{-i} all other players rather than player i. Also denote by $X_{-I'} = \prod_{j\in I\setminus I'} X_j$ the Cartesian product of the sets of strategies of players j with $j \notin I'$, and we sample write X_{-i} for the set $\{i\}$ consisting of a single point i. Let $\succ_i$ denote the asymmetric part of $\succeq_i$, i.e., $y \succ_i x$ if and only if $y \succeq_i x$ but not $x \succeq_i y$.

3. Existence of Nash Equilibrium for Generalized Convex Game with Single Player Deviation Property

In this section, we introduce the notion called generalized convex game which is a natural extension of the convex game of Reny [1] to topological spaces and is unrelated to the diagonal transfer quasi-concavity of Baye et al. [4], the $\mathscr{C}$−concavity of Kim and Lee [15], the $\mathscr{C}$−quasi-concavity of Hou [16], the 0-pair-concavity of Chang [17], and the 0-diagonal quasi-concavity that appeared in [5, 10–12] and establish an existence result of a pure strategy Nash equilibrium for noncooperative games in topological spaces.

Definition 1 (see [1]). Let X_i be a convex subset of a topological vector space for each $\in I$. If for each $i \in I$ and each $x \in X$, $\{x_i' : (x_i', x_{-i}) \succeq_i x\}$ is a convex set, then the game $G = (X_i, \succeq_i)_{i\in I}$ is said to be convex.

Definition 2. If for each $i \in I$ and each finite subset $\{x_{i0}, x_{i1}, \dots, x_{in_i}\}$ of X_i there exists a continuous mapping $\phi_{n_i} : \Delta_{n_i} \to X_i$ such that for any $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n_i}) \in \Delta_{n_i}$,

$$\left(\phi_{n_i}(\lambda), x_{-i}\right) \succeq_i \min\left\{\left(x_{ij}, x_{-i}\right) \mid j \in J(\lambda)\right\} \quad \text{for each } x \in X, \tag{1}$$

where $J(\lambda) = \{j : \lambda_j \neq 0\}$, then one says that the game $(X_i, \succeq_i)_{i\in I}$ *is generalized convex.*

For the generalized convexity, we have the following proposition which shows that the generalized convexity is a natural extension of Reny's convexity to topological spaces without any convexity structure.

Proposition 3. *For each $i \in I$, let X_i be a convex subset of a topological vector space and $\succeq_i$ be complete and transitive. If the game $(X_i, \succeq_i)_{i\in I}$ is convex, then it is generalized convex.*

Proof. Let $i \in I$ and $\{x_{i0}, x_{i1}, \dots, x_{in_i}\}$ be a finite subset of X_i. Define a mapping $\phi_{n_i} : \Delta_{n_i} \to X_i$ by $\phi_{n_i}(\lambda) = \sum_{j=0}^{n_i} \lambda_j x_{ij}$ for each $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n_i}) \in \Delta_{n_i}$. Obviously, ϕ_{n_i} is continuous. For any $\lambda \in \Delta_{n_i}$, let $J(\lambda) = \{j : \lambda_j \neq 0\}$. Let $x \in X$ and $\lambda \in \Delta_{n_i}$. Since $\succeq_i$ is complete and transitive, there exists $j_0 \in J(\lambda)$ such that $(x_{ij}, x_{-i}) \succeq_i (x_{ij_0}, x_{-i})$ for all $j \in J(\lambda)$. Since $(X_i, \succeq_i)_{i\in I}$ is convex, one has that $(\sum_{j\in J(\lambda)} \lambda_j x_{ij}, x_{-i}) \succeq_i (x_{ij_0}, x_{-i}) = \min\{(x_{ij}, x_{-i}) \mid j \in J(\lambda)\}$. Therefore, for any $\lambda \in \Delta_{n_i}$, one has

$$\begin{aligned} \left(\phi_{n_i}(\lambda), x_{-i}\right) &= \left(\sum_{j\in J(\lambda)} \lambda_j x_{ij}, x_{-i}\right) \\ &\succeq \min\left\{\left(x_{ij}, x_{-i}\right) \mid j \in J(\lambda)\right\} \end{aligned} \tag{2}$$

for all $x \in X$. □

Motivated by the proof of Corollary 2.2 of Guillerme [20], we have the following Lemma which, albeit simple, provides seemingly a new approach on the investigation for the existence of Nash equilibria.

Lemma 4. *Let $G = (X_i, \succeq)_{i\in I}$ be a game. Then G has a Nash equilibrium if and only if $\bigcap_{i\in I}\bigcap_{x_i\in X_i} K_i(x_i) \neq \emptyset$, where $K_i(x_i) = \{y \in X \mid y \succeq_i (x_i, y_{-i})\}$.*

Proof.

Sufficiency. Since $\bigcap_{i\in I}\bigcap_{x_i\in X_i} K_i(x_i) \neq \emptyset$, we pick up an element $\overline{x} \in \bigcap_{i\in I}\bigcap_{x_i\in X_i} K_i(x_i)$. Then, for each $i \in I$ and $x_i \in X_i$, we have $\overline{x} \in K_i(x_i)$, and thus $\overline{x} \succeq_i (x_i, \overline{x}_{-i})$.

Necessity. Suppose that $\overline{x}$ is a Nash equilibrium point of G. Then for each $i \in I$ and $x_i \in X_i$, we have $\overline{x} \succeq_i (x_i, \overline{x}_{-i})$, and thus $\overline{x} \in K_i(x_i)$. □

Definition 5. A game $G = (X_i, \succeq_i)_{i\in I}$ is said to have D-single player deviation property if, whenever $x^* \in D$ is not a Nash equilibrium, there exist player i, $y_i \in X_i$, and a neighborhood

$N(x^*)$ of x^* such that $(y_i, x'_{-i}) \succ_i x'$ for each $x' \in N(x^*) \cap (X \setminus E_G)$, where D is a subset of X.

Remark 6. If $D = X$, then the D-single player deviation property is the single player deviation property due to Prokopovych [21, pp. 387] (see also Remark 4 of Reny [1] and Definition 3.2 of Nessah and Tian [13] where it was called weak transfer continuity). The single player deviation property holds in a large class of discontinuous games and is often quite simple to check in a particular game.

Lemma 7 (see [22]). *Let $I_0 = \{1, 2, \ldots, n\}$ and n_i be a natural number for each $i \in I_0$. For each $i \in I_0$, let $F_i : \{e^i_j : j = 0, 1, 2, \ldots, n_i\} \to \prod_{i\in I_0} \Delta_{n_i}$ be a closed set-valued map, where $\{e^i_j : j = 0, 1, 2, \ldots, n_i\}$ denotes the standard basis of Δ_{n_i}. If, for any $i \in I_0$ and any finite subset S_i of $\{0, 1, 2, \ldots, n_i\}$, one has $\prod_{i\in I_0} \mathrm{co}\{e^i_j \mid j \in S_i\} \subseteq \bigcap_{i\in I_0} \bigcup_{j\in S_i} F_i(e^i_j)$, then $\bigcap_{i\in I_0} \bigcap_{j=0}^{n_i} F_i(e^i_j) \neq \emptyset$.*

Theorem 8. *Let $G = (X_i, \succeq_i)_{i\in I}$ have D-single player deviation property such that*

$$D = \bigcap_{i\in I^*} \bigcap_{a_i\in A_i} \overline{\{y \in X \mid y \succeq_i (a_i, y_{-i})\}} \tag{3}$$

is compact, where I^* is a nonempty finite subset of I and A_i is a nonempty finite subset of X_i for each $i \in I^*$. Suppose that, for each $i \in I$, $\succeq_i$ is complete and transitive. If G is generalized convex, then G has a Nash equilibrium.

Proof. Assume, by way of contradiction, that G has no Nash equilibrium in pure strategies. For each $i \in I$ and each $x_i \in X_i$, we use $K_i(x_i)$ to denote the set

$$\{y \in X \mid y \succeq_i (x_i, y_{-i})\}. \tag{4}$$

We show that $\bigcap_{i\in I} \bigcap_{x_i\in X_i} K_i(x_i) \neq \emptyset$ which means that G has Nash equilibrium by Lemma 4, a contradiction.

Since

$$D = \bigcap_{i\in I^*} \bigcap_{a_i\in A_i} \overline{\{y \in X \mid y \succeq_i (a_i, y_{-i})\}} \tag{5}$$

is compact, it is immediate to verify that $\bigcap_{i\in I^*} \bigcap_{a_i\in A_i} \overline{K_i(a_i)}$ is compact.

We first show that

$$\bigcap_{i\in I} \bigcap_{x_i\in X_i} \overline{K_i(x_i)} = \bigcap_{i\in I} \bigcap_{x_i\in X_i} K_i(x_i). \tag{6}$$

Obviously,

$$\bigcap_{i\in I} \bigcap_{x_i\in X_i} K_i(\mathrm{x}_i) \subseteq \bigcap_{i\in I} \bigcap_{x_i\in X_i} \overline{K_i(x_i)}. \tag{7}$$

Now we show that

$$\bigcap_{i\in I} \bigcap_{x_i\in X_i} \overline{K_i(x_i)} \subseteq \bigcap_{i\in I} \bigcap_{x_i\in X_i} K_i(x_i). \tag{8}$$

If not, then there exist a

$$y \in \bigcap_{i\in I} \bigcap_{x_i\in X_i} \overline{K_i(x_i)}, \tag{9}$$

an $i \in I$, and an $x_i \in X_i$ such that $y \notin K_i(x_i)$. By Lemma 4, it follows that y is not a Nash equilibrium. Clearly,

$$y \in \bigcap_{i\in I} \bigcap_{a_i\in A_i} \overline{K_i(a_i)}. \tag{10}$$

Since G has D-single player deviation property, there exist a $j \in I$, an $x_j \in X_j$, and a neighborhood $N(y)$ of y such that $(x_j, z_{-j}) \succ_i z$ for each $z \in N(y)$. It follows that

$$N(y) \cap K_j(x_j) = \emptyset, \quad \text{i.e., } y \notin \overline{K_j(x_j)}. \tag{11}$$

This contradicts (9), and so we have that

$$\bigcap_{i\in I} \bigcap_{x_i\in X_i} \overline{K_i(x_i)} = \bigcap_{i\in I} \bigcap_{x_i\in X_i} K_i(x_i). \tag{12}$$

In order to complete the proof, we only need to show that $\bigcap_{i\in I} \bigcap_{x_i\in X_i} \overline{K_i(x_i)} \neq \emptyset$.

Since $\bigcap_{i\in I^*} \bigcap_{a_i\in A_i} \overline{K_i(a_i)}$ is compact, we only need to show that the family

$$\left\{\overline{K_i(x_i)} \cap D \mid i \in I,\ x_i \in X_i\right\} \tag{13}$$

has the finite intersection property. Toward this end, let I' be an arbitrary finite subset of I, $I_0 = I^* \cup I'$, and $B_i = \{b_{i1}, b_{i2}, \ldots, b_{in_i}\}$ be a finite subset of X_i for each $i \in I_0$.

By the generalized convexity condition, for each $i \in I_0$, there exists a continuous mapping $\phi_{n_i} : \Delta_{n_i} \to X_i$ such that, for any $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_{n_i}) \in \Delta_{n_i}$, one has that

$$\left(\phi_{n_i}(\lambda), x_{-i}\right) \succeq_i \min\left\{\left(b_{ij}, x_{-i}\right) \mid j \in J(\lambda)\right\} \tag{14}$$

for each $x \in X$, where $J(\lambda) = \{j \in \{0, 1, 2, \ldots, n_i\} \mid \lambda_j \neq 0\}$.

Take an arbitrary point $x^0_{-I_0} \in X_{-I_0}$. Define a mapping $\phi : \prod_{i\in I_0} \Delta_{n_i} \to X$ as follows:

$$\phi\left((\Lambda_i)_{i\in I_0}\right) = \left(\left(\phi_{n_i}(\Lambda_i)\right)_{i\in I_0}, x^0_{-I_0}\right), \tag{15}$$

for each $(\Lambda_i)_{i\in I_0} \in \prod_{i\in I_0} \Delta_{n_i}$.

Obviously, ϕ is a continuous mapping from $\prod_{i\in I_0} \Delta_{n_i}$ into X.

For each $i \in I_0$, we take an arbitrary finite subset S_i of $\{0, 1, \ldots, n_i\}$.

We show that

$$\prod_{i\in I_0} \mathrm{co}\left\{e^i_j : j \in S_i\right\} \subseteq \bigcap_{i\in I_0} \bigcup_{j\in S_i} \phi^{-1}\left(\overline{K_i(b_{ij})}\right). \tag{16}$$

Indeed, if $(\Lambda_i)_{i\in I_0} \notin \bigcap_{i\in I_0} \bigcup_{j\in S_i} \phi^{-1}(\overline{K_i(b_{ij})})$, then there exists an $i^* \in I_0$ such that $(\Lambda_i)_{i\in I_0} \notin \phi^{-1}(\overline{K_{i^*}(b_{i^*j})})$ for each $j \in$

S_{i^*}. Thus $\phi((\Lambda_i)_{i\in I_0}) \notin \overline{K_{i^*}(b_{i^*j})}$ for each $j \in S_{i^*}$. Particularly, $\phi((\Lambda_i)_{i\in I_0}) \notin K_{i^*}(b_{i^*j})$ for each $j \in S_{i^*}$.

Let $x^* = \phi((\Lambda_i)_{i\in I_0})$. Then $x_i^* = \phi_{n_i}(\Lambda_i)$ for each $i \in I_0$, and $x_{-I_0}^* = x_{-I_0}^0$, by the definition of ϕ.

Since $x^* \notin K_{i^*}(b_{i^*j})$ for each $j \in S_{i^*}$, one has that

$$\left(b_{i^*j}, x_{-i^*}\right) \succ_{i^*} x^* \tag{17}$$

for each $j \in S_{i^*}$. We show that $\Lambda_{i^*} \notin \text{co}\{e_j^{i^*} : j \in S_{i^*}\}$. If not, then $\Lambda_{i^*} = \sum_{j\in S_{i^*}} \lambda_j e_j^{i^*}$, where $\lambda_j \geq 0$ for each $j \in S_{i^*}$ and $\sum_{j\in S_{i^*}} \lambda_j = 1$. By (14),

$$\begin{aligned} x^* &= (x_{i^*}^*, x_{-i^*}^*) = \left(\phi_{n_{i^*}}\left(\Lambda_{i^*}\right), x_{-i^*}^*\right) \\ &\succeq_{i^*} \min\left\{\left(b_{i^*j}, x_{-i^*}^*\right) \mid j \in J\left(\Lambda_{i^*}\right)\right\}, \end{aligned} \tag{18}$$

where $J(\Lambda_{i^*}) = \{j \in S_{i^*} \mid \lambda_j \neq 0\}$.

Obviously, $J(\Lambda_{i^*}) \subseteq S_{i^*}$. By (17),

$$\begin{aligned} x^* &= (x_{i^*}^*, x_{-i^*}^*) = \left(\phi_{n_{i^*}}\left(\Lambda_{i^*}\right), x_{-i^*}^*\right) \\ &\succeq_{i^*} \min\left\{\left(b_{i^*j}, x_{-i^*}^*\right) \mid j \in J\left(\Lambda_{i^*}\right)\right\} \succ_{i^*} x^* \\ &\qquad (\text{because } \succeq_{i^*} \text{ is complete and transitive}). \end{aligned} \tag{19}$$

This is impossible. Therefore, for any $i \in I_0$ and any subset S_i of $\{0, 1, \ldots, n_i\}$, we have that

$$\prod_{i\in I_0} \text{co}\left\{e_j^i : j \in S_i\right\} \subseteq \bigcap_{i\in I_0} \bigcup_{j\in S_i} \phi^{-1}\left(\overline{K_i\left(b_{ij}\right)}\right). \tag{20}$$

By using Lemma 7 with $F_i(e_j^i) = \phi^{-1}(\overline{K_i(b_{ij})})$ for each $i \in I_0$ and $j = 0, 1, \ldots, n_i$, we have that

$$\bigcap_{i\in I_0} \bigcap_{j=0}^{n_i} \phi^{-1}\left(\overline{K_i\left(b_{ij}\right)}\right) \neq \emptyset. \tag{21}$$

Pick up an element

$$\left(\overline{\Lambda}_i\right)_{i\in I_0} \in \bigcap_{i\in I_0} \bigcap_{j=0}^{n_i} \phi^{-1}\left(\overline{K_i\left(b_{ij}\right)}\right). \tag{22}$$

Then

$$\phi\left(\left(\overline{\Lambda}_i\right)_{i\in I_0}\right) \in \bigcap_{i\in I_0} \bigcap_{j=0}^{n_i} \overline{K_i\left(b_{ij}\right)} = \bigcap_{i\in I'} \bigcap_{j=0}^{n_i} \overline{K_i\left(b_{ij}\right)} \cap D. \tag{23}$$

This completes the proof of the theorem. □

From Theorem 8, we obtain immediately the following corollary which improves and generalizes Theorem 3.2 of [13], Corollary 2.2 of [20], and Corollary 3.1 of [7].

Corollary 9. *Let $G = (X_i, \succeq_i)_{i\in I}$ be compact and have the single player deviation property such that $\succeq_i$ is complete and transitive. If G is generalized convex, then G has a Nash equilibrium.*

Now, we give an example of problem of existence of pure strategy Nash equilibrium for discontinuous games, which holds our assumptions, but the old ones do not hold.

Example 10. Consider the game $(X_i, \succeq_i)_{i\in\{1,2\}}$ that consists of two players where X_1 and X_2 are the closed intervals $[-1, 1]$ and $[0, 2]$, respectively, and the player i's preference relations $\succeq_i$ are defined as follows:

$$\begin{aligned} &(x_1, x_2) \succeq_1 (y_1, y_2) \\ &\qquad\qquad \text{if and only if } x_1^2 + x_2^2 \geq y_1^2 + y_2^2, \\ &(x_1, x_2) \succeq_2 (y_1, y_2) \\ &\qquad\qquad \text{if and only if } 2\left|x_1\right| \geq x_2 \text{ or } 2\left|y_1\right| < y_2 \end{aligned} \tag{24}$$

for each $(x_1, x_2), (y_1, y_2) \in X = X_1 \times X_2$. We firstly show the single player deviation property of the game. If $\overline{x} \in [-1, 1] \times [0, 2]$ is not a Nash equilibrium, then there is an i and an $x_i \in X_i$ such that $(x_i, \overline{x}_{-i}) \succ_i \overline{x}$. If $i = 1$, then it is obvious that there is a neighborhood $N(\overline{x})$ of $\overline{x}$ such that $(x_1, x_2') \succ_1 x'$ for each $x' \in N(\overline{x})$. If $i = 2$, then $2|\overline{x}_1| < \overline{x}_2$. Let $N(\overline{x}) = (-1, 1) \times [0, 2]$. Then $N(\overline{x})$ is a neighborhood of $\overline{x}$, and thus $x_1^2 + x_2^2 < 1 + x_2^2$ for each $(x_1, x_2) \in N(\overline{x})$; it implies that $(1, x_2) \succ_1 (x_1, x_2)$ for all $(x_1, x_2) \in N(\overline{x})$.

Now we check the generalized convexity of the game. Let $\{x_{10}, x_{11}, \ldots, x_{1n}\}$ be a finite subset of $[-1, 1]$. Define a mapping $\phi_n : \Delta_n \to [-1, 1]$ by $\phi_n(\lambda) = \sum_{i=0}^n \lambda_i |x_{1i}|$ for each $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_n) \in \Delta_n$. Clearly, ϕ_n is continuous, and $(\phi_n(\lambda))^2 + x_2^2 = (\sum_{i=0}^n \lambda_i |x_{1i}|)^2 + x_2^2 = (\sum_{i\in J(\lambda)} \lambda_i |x_{1i}|)^2 + x_2^2 \geq (\min\{|x_{1i}| \mid i \in J(\lambda)\})^2 + x_2^2 = |x_{1i}|^2 + x_2^2 = x_{1i}^2 + x_2^2$, and thus $(\phi_n(\lambda), x_2) \succeq_1 (x_{1i}, x_2)$, for some $i \in J(\lambda)$ and all $x_2 \in X_2$, where $J(\lambda) = \{i \mid \lambda_i \neq 0\}$.

Let $\{x_{20}, x_{21}, \ldots, x_{2m}\}$ be a finite subset of $[0, 2]$. Define a mapping $\psi_m : \Delta_m \to [0, 2]$ by $\psi_m(\lambda) = \sum_{i=0}^m \lambda_i x_{2i}$ for each $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m) \in \Delta_m$. Clearly, ψ_m is continuous. Let $\lambda = (\lambda_0, \lambda_1, \ldots, \lambda_m) \in \Delta_m$ and $x_1 \in [-1, 1]$. Without loss the generality, we assume that $\lambda_i \neq 0$ for each $i = 0, 1, \ldots, m$. If $x_{2i} \leq 2|x_1|$ for all $i = 0, 1, \ldots, m$, then $\sum_{i=0}^m \lambda_i x_{2i} \leq 2|x_1|$, and thus

$$\left(x_1, \phi_m\left(\lambda\right)\right) = \left(x_1, \sum_{i=0}^m \lambda_i x_{2i}\right) \succeq_2 \left(x_1, x_{2i}\right). \tag{25}$$

If there exists a $j \in \{0, 1, \ldots, m\}$ such that $x_{2j} > 2|x_1|$, then $(x_1, \phi_m(\lambda)) \succeq_2 (x_1, x_{2j})$. Therefore, the game is generalized convex. By Theorem 8, the game has a Nash equilibrium. In fact, the strategy profile $(1, 2)$ is a Nash equilibrium of the game.

On the other hand, we show that the game is not convex. Indeed, if we pick up a point $y_0 \in (0, 2]$, and take $x_{11} = 1$ and $x_{12} = -1$, then $x_{11}^2 + y_0^2 = x_{12}^2 + y_0^2 = 1 + y_0^2 > y_0^2$. Obviously, $0 \in \text{co}\{x_{11}, x_{12}\}$ and $y_0^2 < 1 + y_0^2$. Therefore, the set $\{x_1 \in X_1 \mid (x_1, y_0) \succeq_1 (0, y_0)\}$ is not convex. So the game is not convex.

4. Existence of Symmetric Pure Strategy Nash Equilibria

Throughout this section, we assume that the strategy spaces for all players are the same. As such, let $X_i = Z$ for each $i \in I$, and let $(x; y)_i$ denote the strategy profile in which player i chooses x and every other player chooses y. If, in addition, for every pair of players i and j, $(x; y)_i \succeq_i (y; y)_i$ if and only if $(x; y)_j \succeq_j (y; y)_j$, then we say that $G = (X_i = Z, \succeq_i)_{i\in I}$ is a quasi-symmetric game. A Nash equilibrium $\overline{x} \in X$ of a game $G = (X_i = Z, \succeq_i)_{i\in I}$ is said to be symmetric iff $\overline{x}_i = \overline{x}_j$ for each $i, j \in I$. For each $z \in Z$, we use $[z]$ to denote the strategy profiles $x \in X$ in which $x_i = z$ for all $i \in I$.

The following notion of a diagonally point secure game was introduced in Reny [1, Definition 5.1]. Let I be a finite set of players. A quasi-symmetric convex game $G = (X_i = Z, \succeq_i)_{i\in I}$ is diagonally point secure if whenever $[z]$ is not a Nash equilibrium, there is a point $\widehat{z} \in Z$ and a neighborhood U of z such that for every $w \in U$, $(\widehat{z}, z', \dots, z') \succ (w, w, \dots, w)$ for every $z' \in U$. To show that $(\widehat{z}, z', \dots, z') \succ (w, w, \dots, w)$ for every $z' \in U$ is excessive, we introduce the following definition.

Definition 11. A quasi-symmetric game $G = (X_i = Z, \succeq_i)_{i\in I}$ is said to have the D-diagonal deviation property if, whenever $[z] \in D$ is not a Nash equilibrium, there is a point $\widehat{z} \in Z$ and a neighborhood U of $[z]$ such that, for every $[w] \in U$, $(\widehat{z}; w) \succ [w]$, where D is a subset of X.

Theorem 12. *Let $G = (X_i = Z, \succeq_i)_{i\in I}$ be a quasi-symmetric game and have D-diagonal deviation property such that*

$$D = \bigcap_{a\in A} \overline{\{[y] \in X \mid [y] \succeq (a; y)\}} \tag{26}$$

is compact for some nonempty finite subset A of Z and $\succeq$ is complete and transitive. If G is generalized convex, then G has a symmetric Nash equilibrium.

Proof. For each $z \in Z$, we use $K(z)$ to denote the set $\{[y] \in X \mid [y] \succeq (z; y)\}$. We show that $\bigcap_{z\in Z} K(z) \neq \emptyset$ which means that G has a symmetric Nash equilibrium.

Since $D = \bigcap_{a\in A} \overline{\{[y] \in X \mid [y] \succeq (a; y)\}}$ is compact, it is obvious that $\bigcap_{a\in A} \overline{K(a)}$ is compact.

We first show that $\bigcap_{z\in Z} \overline{K(z)} = \bigcap_{z\in Z} K(z)$. Obviously, $\bigcap_{z\in Z} K(z) \subseteq \bigcap_{z\in Z} \overline{K(z)}$. Now we show that $\bigcap_{z\in Z} \overline{K(z)} \subseteq \bigcap_{z\in Z} K(z)$. If not, then there exists a

$$y \in \bigcap_{z\in Z} \overline{K(z)} \tag{27}$$

such that $y \notin \bigcap_{z\in Z} K(z)$. We show that $y \in \{[z] \mid z \in Z\}$. If not, then there exist $i, j \in I$ such that $y_i \neq y_j$. Since Z is Hausdorff, there exist a neighborhood $U(y_i)$ of y_i and a neighborhood $U(y_j)$ of y_j such that $U(y_i) \cap U(y_j) = \emptyset$. Obviously, $U(y_i) \times U(y_j) \times \prod_{k\in I\setminus\{i,j\}} X_k$ is a neighborhood of y with $U(y_i) \times U(y_j) \times \prod_{k\in I\setminus\{i,j\}} X_k \cap \{[z] \mid z \in Z\} = \emptyset$, which contradicts (27). It is easy to see that y is not a Nash equilibrium and y belongs to D. By the D-diagonal deviation property, there exist an $x \in Z$ and a neighborhood $N(y)$ of y such that $(x; z) \succ [z]$ for each $[z] \in N(y)$. It follows that $N(y) \cap K(x) = \emptyset$, i.e., $y \notin \overline{K(x)}$. This contradicts (27), and so $\bigcap_{z\in Z} \overline{K(z)} = \bigcap_{z\in Z} K(z)$.

In order to complete the proof, we only need to show that $\bigcap_{z\in Z} \overline{K(z)} \neq \emptyset$.

Since $\bigcap_{a\in A} \overline{K(a)}$ is compact, we only need to show that the family $\{\overline{K(x)} \mid x \in Z\}$ has the finite intersection property. Toward this end, let $B = \{b_0, b_1, b_2, \dots, b_n\}$ be a finite subset of Z.

By the generalized convexity condition, there exists a continuous mapping $\phi_n : \Delta_n \to Z$ such that, for any $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_n) \in \Delta_n$, one has the fact that

$$(\phi_n(\lambda); z) \succeq \min\left\{(b_j; z) \mid j \in J(\lambda)\right\} \tag{28}$$

for each $z \in Z$, where $J(\lambda) = \{j \in \{0, 1, 2, \dots, n\} \mid \lambda_j \neq 0\}$.

Define a mapping $\phi : \Delta_n \to X$ as follows: $\phi(\lambda) = [\phi_n(\lambda)]$, for each $\lambda \in \Delta_n$.

Obviously, ϕ is a continuous mapping from Δ_n into X.

We take an arbitrary finite subset S of $\{0, 1, \dots, n\}$.

We show that $\mathrm{co}\{e_j : j \in S\} \subseteq \bigcup_{j\in S} \phi^{-1}(\overline{K(b_j)})$.

Indeed, if $\lambda \notin \bigcup_{j\in S} \phi^{-1}(\overline{K(b_j)})$, then $\phi(\lambda) \notin \overline{K(b_j)}$ for each $j \in S$. Particularly, $\phi(\lambda) \notin K(b_j)$ for each $j \in S$, and thus

$$\left(b_j; \phi_n(\lambda)\right) \succ [\phi_n(\lambda)] \tag{29}$$

For each $j \in S$. We show that $\lambda \notin \mathrm{co}\{e_j : j \in S\}$. If not, then $\lambda = \sum_{j\in S} \lambda_j e_j$, where $\lambda_j \geq 0$ for each $j \in S$ and $\sum_{j\in S} \lambda_j = 1$. By (28), $[\phi_n(\lambda)] \succeq \min\{(b_j; \phi_n(\lambda)) \mid j \in J(\lambda)\}$, where $J(\lambda) = \{j \in S \mid \lambda_j \neq 0\}$.

Obviously, $J(\lambda) \subseteq S$. Since $\succeq$ is complete and transitive, we have

$$\begin{aligned}[\phi_n(\lambda)] &\succeq \min\left\{\left(b_j; \phi_n(\lambda)\right) \mid j \in J(\lambda)\right\} \\ &\succ [\phi_n(\lambda)] \quad (\text{by } (29)).\end{aligned} \tag{30}$$

This is impossible. Therefore, for any subset S of $\{0, 1, \dots, n\}$, we have that $\mathrm{co}\{e_j : j \in S\} \subseteq \bigcup_{j\in S} \phi^{-1}(\overline{K(b_j)})$. By the classic KKM theorem, we have that $\bigcap_{j=0}^{n} \phi^{-1}(\overline{K(b_j)}) \neq \emptyset$.

Pick up an element $\lambda \in \bigcap_{j=0}^{n} \phi^{-1}(\overline{K(b_j)})$. Then $\phi(\lambda) \in \bigcap_{j=0}^{n} \overline{K(b_j)}$. This completes the proof of the theorem. □

From Theorem 12, we obtain immediately the following corollary which improves and generalizes Proposition 5.2 of Reny [1].

Corollary 13. *Let $G = (X_i = Z, \succeq_i)_{i\in I}$ be a quasi-symmetric compact game and have the D-diagonal deviation property with $D = X$. Suppose that $\succeq$ is complete and transitive. If G is generalized convex, then G has a symmetric Nash equilibrium.*

5. Existence of Dominant Strategy Equilibria

Bay et al. [4] gave a complete characterization for the existence of dominant strategy equilibrium in games with the set X_i of pure strategies of player i being a compact convex subset of a topological vector space, by introducing a concavity notion called uniformly transfer quasi-concavity.

Let $G = (X_i, \succeq_i)_{i\in I}$ be a game. A point $x^* \in X$ is said to be a *dominant strategy equilibrium* if, for all $i \in I$, $(x_i^*, x_{-i}) \succeq_i x$ for all $x = (x_i, x_{-i}) \in X$.

Definition 14. A game $G = (X_i, \succeq_i)_{i\in I}$ is said to be *generalized uniformly quasi-concave* if, for any $i \in I$ and any finite subset $\{x^0, x^1, \dots, x^n\}$ of X, there exists a continuous mapping $\phi_{n_i} : \Delta_{n_i} \to X_i$ such that, for any $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n_i}) \in \Delta_{n_i}$, $(\phi_{n_i}(\lambda), x_{-i}^j) \succeq_i \min\{x^j \mid j \in J(\lambda)\}$ for all $j \in J(\lambda)$, where $J(\lambda) = \{i : \lambda_i \neq 0\}$.

Definition 15. A game $G = (X_i, \succeq_i)_{i\in I}$ is said to be *transfer upper semicontinuous* if, for every $i \in I$, $y_i \in X_i$, and $x \in X$, $x \succ_i (y_i, x_{-i})$ implies that there exists a point $x' \in X$ and a neighborhood $N(y_i)$ of y_i such that $x \succ_i (y_i', x_{-i})$ for all $y_i' \in N(y_i)$.

Remark 16. When X_i is a convex subset of a topological vector space and $\succeq_i$ can be represented by a payoff function $u_i : X \to \boldsymbol{R}$, Definition 15 is due to Bay et al. [4].

Theorem 17. *Let $G = (X_i, \succeq_i)_{i\in I}$ be a transfer upper semicontinuous game with $\succeq_i$ being complete and transitive. Suppose for each $i \in I$ there exists a finite subset $\{z^1, z^2, \dots, z^{m_i}\}$ of X such that $\bigcap_{k=1}^{m_i} \overline{F_i(z^k)}$ is compact where $F_i(z^k) = \{y_i \in X_i \mid (y_i, z_{-i}^k) \succeq_i z^k\}$. Then G has a dominant strategy equilibrium if and only if G is generalized uniformly quasi-concave.*

Proof.

Necessity. Suppose that the game G has a dominant strategy equilibrium $x^* \in X$. We want to show that G is generalized uniformly quasi-concave. Let $i \in I$ and $\{x^0, x^1, \dots, x^{n_i}\}$ be an arbitrary finite subset of X. Define a mapping $\phi_{n_i} : \Delta_{n_i} \to X_i$ as follows:

$$\phi_{n_i}(\lambda) = x_i^*, \tag{31}$$

for each $\lambda \in \Delta_{n_i}$. Obviously, ϕ_{n_i} is continuous.

For any $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n_i}) \in \Delta_{n_i}$,

$$\begin{aligned} \left(\phi_{n_i}(\lambda), x_{-i}^j\right) &= \left(x_i^*, x_{-i}^j\right) \\ &\succeq_i x^j \quad \text{for all } j = 0, 1, \dots, n_i \\ &\succeq_i \min\left\{x^j : j \in J(\lambda)\right\} \end{aligned} \tag{32}$$

For all $j \in J(\lambda)$, where $J(\lambda) = \{j : \lambda_j \neq 0\}$.

Sufficiency. Let $i \in I$. Define a mapping $F_i : X \to X_i$ as follows:

$$F_i(x) = \{y_i \in X_i \mid (y_i, x_{-i}) \succeq_i x\}, \tag{33}$$

for each $x \in X$.

We show firstly that $\bigcap_{x\in X} F_i(x) = \bigcap_{x\in X} \overline{F_i(x)}$. To this end, we only need to show $\bigcap_{x\in X} F_i(x) \supseteq \bigcap_{x\in X} \overline{F_i(x)}$. Indeed, if $y_i \notin \bigcap_{x\in X} F_i(x)$, then there exists an $x \in X$ such that $y_i \notin F_i(x)$. By the definition of F_i, one has that $x \succ_i (y_i, x_{-i})$. By the transfer upper semicontinuity, there exists an $x' \in X$ and a neighborhood $N(y_i)$ of y_i such that $x' \succ_i (y_i', x_{-i}')$ for all $y_i' \in N(y_i)$. So, $N(y_i) \cap F_i(x') = \emptyset$. Therefore, $y_i \notin \overline{F_i(x')}$. Thus

$$\bigcap_{x\in X} F_i(x) = \bigcap_{x\in X} \overline{F_i(x)} \tag{34}$$

Now we show that $\bigcap_{x\in X} \overline{F_i(x)} \neq \emptyset$. Toward this end, we firstly show that the family $\{\overline{F_i(x)} \mid x \in X\}$ has the finite intersection property.

Let $\{x^0, x^1, \dots, x^{n_i}\}$ be an arbitrary finite subset of X. By the generalized uniform quasi-concavity, there exists a continuous mapping $\phi_{n_i} : \Delta_{n_i} \to X_i$ such that, for any $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n_i}) \in \Delta_{n_i}$, one has that

$$\left(\phi_{n_i}(\lambda), x_{-i}^j\right) \succeq_i \min\left\{x^j \mid j \in J(\lambda)\right\} \tag{35}$$

for all $j \in J(\lambda)$, where $J(\lambda) = \{i \in \{0, 1, 2, \dots, n_i\} \mid \lambda_i \neq 0\}$.

Let S be an arbitrary subset of $\{0, 1, 2, \dots, n_i\}$. We show that

$$\operatorname{co}\left\{e_j : j \in S\right\} \subseteq \bigcup_{j\in S} \phi_{n_i}^{-1}\left(\overline{F_i(x^j)}\right). \tag{36}$$

Indeed, if $\lambda = (\lambda_0, \lambda_1, \dots, \lambda_{n_i}) \notin \bigcup_{j\in S} \phi_{n_i}^{-1}(\overline{F_i(x^j)})$, then $\phi_{n_i}(\lambda) \notin \bigcup_{j\in S} \overline{F_i(x^j)}$. So, $\phi_{n_i}(\lambda) \notin \bigcup_{j\in S} F_i(x^j)$, i.e., for each $j \in S$, one has that $\phi_{n_i}(\lambda) \notin F_i(x^j)$. By the definition of F_i, we have that

$$x^j \succ_i \left(\phi_{n_i}(\lambda), x_{-i}^j\right) \tag{37}$$

for all $j \in S$. Now we demonstrate that $\lambda \notin \operatorname{co}\{e_j : j \in S\}$. If not, then $J(\lambda) \subseteq S$. By (37), one has that $x^j \succ_i (\phi_{n_i}(\lambda), x_{-i}^j)$ for all $j \in J(\lambda)$. This contradicts (35). Therefore, $\operatorname{co}\{e_j : j \in S\} \subseteq \bigcup_{j\in S} \phi_{n_i}^{-1}(\overline{F_i(x^j)})$. By the classic KKM theorem,

$$\bigcap_{j=0}^{n_i} \phi_n^{-1}\left(\overline{F_i(x^j)}\right) \neq \emptyset. \tag{38}$$

Pick up an element $\lambda^* \in \bigcap_{j=0}^{n_i} \phi_{n_i}^{-1}(\overline{F_i(x^j)})$. Then $\phi_{n_i}(\lambda^*) \in \bigcap_{j=0}^{n_i} \overline{F_i(x^j)}$. Therefore, $\bigcap_{j=0}^{n_i} \overline{F_i(x^j)} \neq \emptyset$. Therefore, the family $\{\overline{F_i(x)} \mid x \in X\}$ has the finite intersection property. Particularly, the family

$$\left\{\overline{F_i(x)} \cap \left(\bigcap_{k=1}^{m_i} \overline{F_i(z^k)}\right) \mid x \in X\right\} \tag{39}$$

has the finite intersection property. By the supposition, the set $\bigcap_{k=1}^{m_i} \overline{F_i(z^k)}$ is compact. So,

$$\bigcap_{x\in X} \left(\overline{F_i(x)} \cap \left(\bigcap_{k=1}^{m_i} \overline{F_i(z^k)}\right)\right) \neq \emptyset. \tag{40}$$

By (34), we have that $\bigcap_{x\in X} F_i(x) \neq \emptyset$. Take an element $x_i^* \in \bigcap_{x\in X} F_i(x)$. It is easy to verify that $x^* = (x_i^*)_{i\in I}$ is a dominant strategy equilibrium of G. □

Remark 18. In Theorem 17, the compactness of X_i implies that $\bigcap_{k=1}^{m} \overline{F_i(z^k)}$ is compact. So, Theorem 17 extends Theorems 4 and 5 of Bay et al. [4] to general topological spaces without any convexity (geometrical or abstract) structure.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that they have no conflicts of interest.

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