Research Article

Spectral Functions for the Vector-Valued Fourier Transform

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A scalar distribution function $\sigma(s)$ is called a spectral function for the Fourier transform $\widehat{\varphi}(s) = \int_{\mathbb{R}} e^{its} \varphi(t) dt$ (with respect to an interval $\mathcal{J} \subset \mathbb{R}$) if for each function $\varphi \in L^2(\mathbb{R})$ with support in $\mathcal{J}$ the Parseval identity $\int_{\mathbb{R}} |\widehat{\varphi}(s)|^2 d\sigma(s) = \int_{\mathbb{R}} |\varphi(t)|^2 dt$ holds. We show that in the case $\mathcal{J} = \mathbb{R}$ there exists a unique spectral function $\sigma(s) = (1/2\pi)s$, in which case the above Parseval identity turns into the classical one. On the contrary, in the case of a finite interval $\mathcal{J} = (0, b)$, there exist infinitely many spectral functions (with respect to $\mathcal{J}$). We introduce also the concept of the matrix-valued spectral function $\sigma(s)$ (with respect to a system of intervals $\{\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n\}$) for the vector-valued Fourier transform of a vector-function $\varphi(t) = \{\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)\} \in L^2(\mathcal{J}, \mathbb{C}^n)$, such that support of φ_j lies in $\mathcal{J}_j$. The main result is a parametrization of all matrix (in particular scalar) spectral functions $\sigma(s)$ for various systems of intervals $\{\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n\}$.

1. Introduction

Recall that according to the Plancherel theorem (see, e.g., [1, Ch.3.37]) the Fourier transform

$$\widehat{\varphi}(s) = \int_{\mathbb{R}} e^{its} \varphi(t) dt \quad (1)$$

of a function $\varphi \in L^2(\mathbb{R})$ satisfies the Parseval identity

$$\frac{1}{2\pi} \int_{\mathbb{R}} |\widehat{\varphi}(s)|^2 ds = \int_{\mathbb{R}} |\varphi(t)|^2 dt \quad (2)$$

and the inverse Fourier transform is

$$\varphi(t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-its} \widehat{\varphi}(s) ds. \quad (3)$$

As is known a number of classical problems (moment problems, eigenfunctions expansion of differential operators, and their various modifications) are reduced to searching for the distribution functions $\sigma(\cdot)$ ("spectral functions") such that the Stieltjes integral $I(\lambda) = \int_{\mathbb{R}} g_{\lambda}(s) d\sigma(s)$ with a given family of functions $g_{\lambda}(s)$ satisfies certain preassigned conditions. The problem of such a kind for the Fourier transform (1) can be formulated as follows. Assume that $\mathcal{J} = (a, b)$, $-\infty \leq a <$

$b \leq \infty$, is an interval in $\mathbb{R}$ and let $L^2(\mathbb{R}; \mathcal{J})$ be the set of all functions $\varphi \in L^2(\mathbb{R})$ with support in $\mathcal{J}$. It is necessary to describe all distribution functions $\sigma(\cdot)$ such that instead of (2) the following Parseval identity holds:

$$\int_{\mathbb{R}} |\widehat{\varphi}(s)|^2 d\sigma(s) = \int_{\mathbb{R}} |\varphi(t)|^2 dt, \quad \varphi \in L^2(\mathbb{R}; \mathcal{J}). \quad (4)$$

A function $\sigma(\cdot)$ satisfying (4) will be called a spectral function for the Fourier transform (1) (with respect to $\mathcal{J}$); the set of all such spectral functions will be denoted by $SF(\mathcal{J})$. If $\sigma(\cdot) \in SF(\mathcal{J})$, then for each $\varphi \in L^2(\mathbb{R}; \mathcal{J})$ the inverse Fourier transform is

$$\varphi(t) = \chi_{\mathcal{J}}(t) \int_{\mathbb{R}} e^{-its} \widehat{\varphi}(s) d\sigma(s), \quad (5)$$

where $\chi_{\mathcal{J}}$ is the indicator of $\mathcal{J}$. Clearly formulas (1), (4), and (5) give a natural generalization of the classical Plancherel theorem. We have not seen such a generalization in the literature.

A parametrization of the set $SF(\mathcal{J})$ in the case of a bounded interval $\mathcal{J}$ is given by the following theorem, which directly follows from Corollary 18 of the paper.

Theorem 1. Let $\mathcal{F} = (0, b)$, $b < \infty$, be a bounded interval in $\mathbb{R}$. Then the equalities

$$m_K(\lambda) = \frac{i}{2} \cdot \frac{e^{-i\lambda b} + K(\lambda)}{e^{-i\lambda b} - K(\lambda)}, \quad \lambda \in \mathbb{C}_+ \quad (6)$$

$$\sigma_K(s) = \lim_{\delta \rightarrow +0} \lim_{y \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} m_K(x + iy) dx \quad (7)$$

establish a bijective correspondence $\sigma(s) = \sigma_K(s)$ between all holomorphic functions $K(\lambda)$, $\lambda \in \mathbb{C}_+$, with $|K(\lambda)| \leq 1$ and all (scalar) spectral functions $\sigma(\cdot) \in SF((0, b))$.

In the case $K(\lambda) \equiv 1$ the function $m_K(\lambda)$ is meromorphic with poles on $\mathbb{R}$, $\sigma_K(s)$ is a jump function, and equalities (1) and (5) give an expansion of a function $\varphi \in L^2(\mathbb{R}; (0, b))$ into the Fourier series on $(0, b)$. In the case $K(\lambda) \equiv 0$ one has $\sigma_K(s) = (1/2\pi)s$ and equality (5) turns into the classical inverse Fourier–Plancherel transform (3) of a function $\varphi \in L^2(\mathbb{R})$ with support on $(0, b)$. Moreover, according to Theorem 1 there exist infinitely many spectral functions $\sigma(\cdot) \in SF((0, b))$. At the same time we show that in the case $\mathcal{F} = \mathbb{R}$ the set $SF(\mathbb{R})$ consists of the unique spectral function $\sigma(s) = (1/2\pi)s$, for which equality (5) turns into the inverse Fourier–Plancherel transform (3) of a function $\varphi \in L^2(\mathbb{R})$. Similar situation takes place in the case $\mathcal{F} = (0, \infty)$ (see Corollary 22).

Actually we study a more general object—a matrix-valued spectral function for the vector-valued Fourier transform. Namely, assume that $\widetilde{\mathcal{F}} = \{\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n\}$ is a system of intervals $\mathcal{F}_j = (a_j, b_j)$, $-\infty \leq a_j < b_j \leq \infty$. Denote by $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ the set of all vector-functions

$$\varphi(t) = \{\varphi_1(t), \varphi_2(t), \dots, \varphi_n(t)\} \in \mathbb{C}^n, \quad t \in \mathbb{R}, \quad (8)$$

such that $\int_{\mathbb{R}} \|\varphi(t)\|^2 dt < \infty$ and support of a coordinate function φ_j lies in $\mathcal{F}_j$. For each function $\varphi \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ equality (1) defines the vector-valued Fourier transform $\widehat{\varphi} : \mathbb{R} \rightarrow \mathbb{C}^n$ of φ . A matrix-valued distribution function $\sigma : \mathbb{R} \rightarrow \mathbb{C}^{n \times n}$ will be called a spectral function for the vector-valued Fourier transform (1) (with respect to $\widetilde{\mathcal{F}}$) if the following Parseval identity holds:

$$\int_{\mathbb{R}} (d\sigma(s) \widehat{\varphi}(s), \widehat{\varphi}(s)) = \int_{\mathbb{R}} \|\varphi(t)\|^2 dt, \quad (9)$$

$$\varphi \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}}).$$

We denote the set of all such spectral functions by $SF_n(\widetilde{\mathcal{F}}) = SF_n(\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_n)$. If $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$, then for each $\varphi \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ the inverse Fourier transform is

$$\varphi(t) = \chi_{\widetilde{\mathcal{F}}}(t) \int_{\mathbb{R}} e^{-its} d\sigma(s) \widehat{\varphi}(s), \quad (10)$$

where $\chi_{\widetilde{\mathcal{F}}}(t) = \operatorname{diag}(\chi_{\mathcal{F}_1}(t), \chi_{\mathcal{F}_2}(t), \dots, \chi_{\mathcal{F}_n}(t))$ (see Proposition 12).

In the paper we parameterize spectral functions $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$ for various classes of $\widetilde{\mathcal{F}}$. To illustrate the obtained

results we specify below two theorems, which give a parametrization of the classes $SF_2((0, b), (-c, 0))$ ($b, c \in (0, \infty)$) and $SF_2((0, \infty), (-\infty, 0))$. These theorems are immediate from Corollary 19 and Theorem 21 of the paper.

Theorem 2. Assume that $b, c \in (0, \infty)$ and let

$$\mathcal{Q}_1(\lambda) = \begin{pmatrix} 0 & \sqrt{2}e^{i\lambda c} \\ -\sqrt{2} & 0 \end{pmatrix},$$

$$\mathcal{Q}_2(\lambda) = \begin{pmatrix} -\sqrt{2}e^{-i\lambda b} & 0 \\ 0 & \sqrt{2} \end{pmatrix}, \quad (11)$$

$$\lambda \in \mathbb{C}_+,$$

where $\mathbb{C}_+$ is the upper half-plane of the complex plane. Then the equality

$$m_K(\lambda) = -\frac{i}{2} (K(\lambda) \mathcal{Q}_1(\lambda) + \mathcal{Q}_2(\lambda))^{-1} \cdot (K(\lambda) \mathcal{Q}_1(\lambda) - \mathcal{Q}_2(\lambda)), \quad \lambda \in \mathbb{C}_+ \quad (12)$$

together with formula (7) gives a bijective correspondence $\sigma(\cdot) = \sigma_K(\cdot)$ between all holomorphic contractive matrix functions $K(\lambda) \in \mathbb{C}^{2 \times 2}$, $\lambda \in \mathbb{C}_+$, and all spectral functions $\sigma(\cdot) \in SF_2((0, b), (-c, 0))$.

Theorem 3. Let $\mathcal{S}_{\mathbb{R}}$ be the set of all complex-valued functions F on $\mathbb{R}$ admitting the representation

$$F(x) := \lim_{y \rightarrow +0} K(x + iy) \quad (\text{a.e. on } \mathbb{R}) \quad (13)$$

with a holomorphic function $K(\cdot)$ defined on $\mathbb{C}_+$ and satisfying $|K(\lambda)| \leq 1$, $\lambda \in \mathbb{C}_+$. Then the equalities

$$\Sigma_F(x) = \frac{1}{2\pi} \begin{pmatrix} 1 & \overline{F(x)} \\ F(x) & 1 \end{pmatrix}, \quad x \in \mathbb{R}, \text{ and} \quad (14)$$

$$\sigma_F(s) = \int_0^s \Sigma_F(x) dx$$

give a bijective correspondence $\sigma(\cdot) = \sigma_F(\cdot)$ between all functions $F \in \mathcal{S}_{\mathbb{R}}$ and all spectral functions $\sigma(\cdot) \in SF_2((0, \infty), (-\infty, 0))$.

These theorems show that the sets of spectral functions differ significantly for systems $\widetilde{\mathcal{F}}$ formed by finite or infinite intervals. In particular, by Theorem 3 each spectral function $\sigma(\cdot) \in SF_2((0, \infty), (-\infty, 0))$ is absolutely continuous with the matrix density $\Sigma_F(x)$ defined by the first equality in (14).

In conclusion note that the results of the paper on spectral functions for the classical Fourier transform are obtained with the aid of the results from [2–4] concerning spectral functions for canonical differential systems.

2. Preliminaries

2.1. Notations and Definitions. In the following $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_- = (-\infty, 0]$ are semiaxis of the real axis, $\mathbb{C}_+$ is an upper

half-plane of the complex plane, $\overline{m, n} := \{k \in \mathbb{N} \cup \{0\} : m \leq k \leq n\}$ ($m, n \in \mathbb{N} \cup \{0\}$ and $m \leq n$), δ_{jk} is the Kronecker delta (i.e., $\delta_{jk} = 1$ if $j = k$ and $\delta_{jk} = 0$ if $j \neq k$), and $\mathbf{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is the set of all linear bounded operators from the Hilbert space $\mathfrak{H}_1$ to the Hilbert space $\mathfrak{H}_2$, $\mathbf{B}(\mathfrak{H}) = \mathbf{B}(\mathfrak{H}, \mathfrak{H})$.

We denote by $\mathcal{S}(\mathbb{C}^k, \mathbb{C}^n)$ the set of all holomorphic operator-functions $K : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^k, \mathbb{C}^n)$ such that $\|K(\lambda)\| \leq 1$, $\lambda \in \mathbb{C}_+$ (the latter means that the function K is contractive). Moreover, we put $\mathcal{S}(\mathbb{C}^n) := \mathcal{S}(\mathbb{C}^n, \mathbb{C}^n)$ and $\mathcal{S} := \mathcal{S}(\mathbb{C})$ (that is, $\mathcal{S}$ is the set of all scalar contractive holomorphic functions on $\mathbb{C}_+$). If $K \in \mathcal{S}(\mathbb{C}^n)$, then according to the Fatou's theorem there exists the limit

$$F(x) := \lim_{y \rightarrow +0} K(x + iy) \quad (\text{a.e. on } \mathbb{R}). \quad (15)$$

We denote by $\mathcal{S}_{\mathbb{R}}(\mathbb{C}^n)$ the set of all functions $F : \mathbb{R} \rightarrow \mathbf{B}(\mathbb{C}^n)$ admitting the representation (15) with some function $K \in \mathcal{S}(\mathbb{C}^n)$. Clearly, each function $K \in \mathcal{S}_{\mathbb{R}}(\mathbb{C}^n)$ is contractive and Borel measurable.

Let $\mathcal{I} = (a, b)$, $-\infty \leq a < b \leq \infty$, be an open interval in $\mathbb{R}$. We denote by $L^2(\mathcal{I}, \mathbb{C}^n)$ the Hilbert space of Borel measurable functions $f : \mathcal{I} \rightarrow \mathbb{C}^n$ such that $\int_{\mathcal{I}} \|f(t)\|^2 dt < \infty$.

Recall that a nondecreasing left-continuous operator-function $\sigma : \mathbb{R} \rightarrow \mathbf{B}(\mathbb{C}^n)$ with $\sigma(0) = 0$ is called a distribution function. For each distribution function σ there exists a Borel measure μ on $\mathbb{R}$ and an operator-function $\Psi : \mathbb{R} \rightarrow \mathbf{B}(\mathbb{C}^n)$ such that $\Psi(s) \geq 0$ (a.e. on $\mathbb{R}$) and $\sigma(\beta) - \sigma(\alpha) = \int_{[\alpha, \beta)} \Psi(s) d\mu$, $-\infty < \alpha < \beta < \infty$. Moreover, a linear space $L^2(\sigma, \mathbb{C}^n)$ of all Borel measurable functions $g : \mathcal{I} \rightarrow \mathbb{C}^n$ satisfying $\int_{\mathbb{R}} (d\sigma(s)g(s), g(s)) := \int_{\mathbb{R}} (\Psi(s)g(s), g(s)) d\mu < \infty$ is a Hilbert space with respect to the inner product

$$\begin{aligned} (g_1, g_2)_{L^2(\sigma, \mathbb{C}^n)} &= \int_{\mathbb{R}} (d\sigma(s)g_1(s), g_2(s)) \\ &:= \int_{\mathbb{R}} (\Psi(s)g_1(s), g_2(s)) d\mu \end{aligned} \quad (16)$$

defined for all $g_1, g_2 \in L^2(\sigma, \mathbb{C}^n)$ (see, e.g., [5, Ch.3.15]).

A distribution function σ is called absolutely continuous if there exists a locally integrable operator-function $\Sigma : \mathbb{R} \rightarrow \mathbf{B}(\mathbb{C}^n)$ (the density of σ) such that $\sigma(s) = \int_0^s \Sigma(x) dx$. As is known, in this case $\sigma'(s) = \Sigma(s)$ (a.e. on $\mathbb{R}$).

Remark 4. Let $\mathbb{C}^{n \times k}$ be the set of all $n \times k$ -matrices with complex entries. As is known each operator $T \in \mathbf{B}(\mathbb{C}^k, \mathbb{C}^n)$ can be identified with its matrix $T = (T_{ij}) \in \mathbb{C}^{n \times k}$ in the standard bases of $\mathbb{C}^k$ and $\mathbb{C}^n$. Hence a (holomorphic) operator-function with values in $\mathbf{B}(\mathbb{C}^k, \mathbb{C}^n)$ can be identified with a (holomorphic) matrix function with values in $\mathbb{C}^{n \times k}$, which implies that the class $\mathcal{S}(\mathbb{C}^k, \mathbb{C}^n)$ is in fact the well-known Schur class $\mathcal{S}^{n \times k}$ of holomorphic in $\mathbb{C}_+$ contractive matrix functions with values in $\mathbb{C}^{n \times k}$. Nevertheless it is more convenient to use the operator terminology in our

considerations (of course, all final results of the paper can be readily translated into the matrix language).

Let A be a closed densely defined symmetric operator in $\mathfrak{H}$ with the domain $\text{dom } A$ and let $\mathfrak{N}_{\lambda}(A) = \ker(A^* - \lambda)$, $\lambda \in \mathbb{C}_{\pm}$, be the defect subspace of A . We will need the following definitions.

Definition 5 (see [6]). A collection $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ consisting of a Hilbert space $\mathcal{H}$ and linear mappings $\Gamma_j : \text{dom } A^* \rightarrow \mathcal{H}$, $j \in \{0, 1\}$, is called a boundary triplet for A^* if the mapping $\Gamma = (\Gamma_0, \Gamma_1)^T$ is surjective and the following Green's identity holds:

$$\begin{aligned} (A^* f, g) - (f, A^* g) &= (\Gamma_1 f, \Gamma_0 g) - (\Gamma_0 f, \Gamma_1 g), \\ f, g &\in \text{dom } A^*. \end{aligned} \quad (17)$$

Definition 6 (see [7, 8]). The operator-function $M : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathcal{H})$ defined by

$$\Gamma_1 \upharpoonright \mathfrak{N}_{\lambda}(A) = M(\lambda) \Gamma_0 \upharpoonright \mathfrak{N}_{\lambda}(A), \quad \lambda \in \mathbb{C}_+ \quad (18)$$

is called the Weyl function of the boundary triplet $\Pi = \{\mathcal{H}, \Gamma_0, \Gamma_1\}$ for A^* .

2.2. Nevanlinna Functions and Nevanlinna Pairs. As is known a holomorphic operator-function $\Phi : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^n)$ is called a Nevanlinna function (and is referred to the class $R[\mathbb{C}^n]$) if $\text{Im } \Phi(\lambda) \geq 0$, $\lambda \in \mathbb{C}_+$. Recall also (see, e.g., [9]) that a pair $C_0(\cdot), C_1(\cdot)$ of holomorphic operator-functions $C_j(\cdot) : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^n)$, $j \in \{0, 1\}$, is called a Nevanlinna pair if $\text{Im}(C_1(\lambda)C_0^*(\lambda)) \geq 0$ and $\text{ran}(C_0(\lambda), C_1(\lambda)) = \mathbb{C}^n$, $\lambda \in \mathbb{C}_+$. Two Nevanlinna pairs $C_0(\cdot), C_1(\cdot)$ and $\tilde{C}_0(\cdot), \tilde{C}_1(\cdot)$ are said to be equivalent if $\tilde{C}_j(\lambda) = \varphi(\lambda)C_j(\lambda)$, $\lambda \in \mathbb{C}_+$, $j \in \{0, 1\}$, with some holomorphic operator-function $\varphi : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^n)$, such that the operator $\varphi(\lambda)$ is invertible for all $\lambda \in \mathbb{C}_+$. Clearly, the set of all Nevanlinna pairs falls into nonintersecting classes of equivalent pairs (the equivalence classes). In the following we do not distinguish equivalent Nevanlinna pairs. Moreover, we denote by $\tau = \{C_0(\cdot), C_1(\cdot)\}$ the equivalence class generating by the Nevanlinna pair $C_0(\cdot), C_1(\cdot)$ and by $\tilde{R}(\mathbb{C}^n)$ the set of all such equivalence classes. Identifying of a function $\Phi \in R[\mathbb{C}^n]$ with a pair $\tau = \{\Phi(\cdot), -I\} \in \tilde{R}(\mathbb{C}^n)$ enables one to consider the class $R[\mathbb{C}^n]$ as a subclass of $\tilde{R}(\mathbb{C}^n)$.

Definition 7 (see [3]). Let $\mathcal{H}$ be a subspace in $\mathbb{C}^n$ and let $\mathcal{H}' = \mathbb{C}^n \ominus \mathcal{H}$. A pair $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \tilde{R}(\mathbb{C}^n)$ is referred to the class $\tilde{R}_{\mathcal{H}}(\mathbb{C}^n)$ if $C_1(\lambda) \upharpoonright \mathcal{H}' = 2iC_0(\lambda) \upharpoonright \mathcal{H}'$, $\lambda \in \mathbb{C}_+$.

Clearly, $\tilde{R}_{\mathbb{C}^n}(\mathbb{C}^n) = \tilde{R}(\mathbb{C}^n)$.

A parametrization of the class $\tilde{R}_{\mathcal{H}}(\mathbb{C}^n)$ in terms of contractive operator-functions is given by the following proposition.

Proposition 8 (see [3]). Let $\mathcal{H}$ be a subspace in $\mathbb{C}^n$ and let $\mathcal{H}' = \mathbb{C}^n \ominus \mathcal{H}$. Then the equalities

$$\begin{aligned} C_0(\lambda) &= \begin{pmatrix} \frac{1}{\sqrt{2}} I_{\mathcal{H}'} & -K_1(\lambda) \\ 0 & K_2(\lambda) + iI_{\mathcal{H}} \end{pmatrix} : \mathcal{H}' \oplus \mathcal{H} \longrightarrow \\ &\quad \mathcal{H}' \oplus \mathcal{H} \\ C_1(\lambda) &= \begin{pmatrix} i\sqrt{2} I_{\mathcal{H}'} & iK_1(\lambda) \\ 0 & -iK_2(\lambda) - I_{\mathcal{H}} \end{pmatrix} : \mathcal{H}' \oplus \mathcal{H} \longrightarrow \\ &\quad \mathcal{H}' \oplus \mathcal{H}, \quad \lambda \in \mathbb{C}_+ \end{aligned} \quad (19)$$

give a bijective correspondence between all operator-functions $K(\cdot) \in \mathcal{S}(\mathcal{H}, \mathbb{C}^n)$ with the block representation $K(\lambda) = (K_1(\lambda), K_2(\lambda))^T : \mathcal{H} \longrightarrow \mathcal{H} \oplus \mathcal{H}'$ and all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \tilde{\mathcal{R}}_{\mathcal{H}}(\mathbb{C}^n)$.

2.3. Canonical Systems. As is known (see, e.g., [10, 11]) a canonical differential system of an order n on the semiaxis $\mathbb{R}_+ = [0, \infty)$ is of the form

$$Jy' = \lambda H(t)y, \quad t \in \mathbb{R}_+, \quad \lambda \in \mathbb{C}, \quad (20)$$

where $J \in \mathbf{B}(\mathbb{C}^n)$ is an operator satisfying $J^* = J^{-1} = -J$ and $H(\cdot) : \mathbb{R}_+ \longrightarrow \mathbf{B}(\mathbb{C}^n)$ is a Borel measurable operator-function integrable on each bounded interval $(0, \beta) \subset \mathbb{R}_+$ and satisfying $H(t) \geq 0$ (a.e. on $\mathbb{R}_+$). In the following we assume also that $H(t)$ is invertible a.e. on $\mathcal{I}$.

Denote by $L_H^2(\mathbb{R}_+, \mathbb{C}^n)$ the Hilbert space of all Borel measurable functions $f : \mathbb{R}_+ \longrightarrow \mathbb{C}^n$ satisfying $\int_{\mathbb{R}_+} (H(t)f(t), f(t))dt < \infty$ and by $\dot{L}_H^2(\mathbb{R}_+, \mathbb{C}^n)$ the set of all functions $f \in L_H^2(\mathbb{R}_+, \mathbb{C}^n)$ such that $H(t)f(t) = 0$ a.e. on $\mathbb{R}_+ \setminus [0, \beta_f]$, where $\beta_f \in (0, \infty)$ depends on f . Since $H(t)$ is invertible a.e. on $\mathbb{R}_+$, system (20) gives rise to the maximal operator $T_{\max}$ and the minimal operator $T_{\min}$ in $L_H^2(\mathbb{R}_+, \mathbb{C}^n)$ defined as follows. Let $AC(\mathbb{R}_+, \mathbb{C}^n)$ be the set of all functions $f : \mathbb{R}_+ \longrightarrow \mathbb{C}^n$, which are absolutely continuous on each compact subinterval $[0, \beta] \subset \mathbb{R}_+$. Then the domain $\text{dom } T_{\max}$ of $T_{\max}$ is $\text{dom } T_{\max} = \{y \in AC(\mathbb{R}_+, \mathbb{C}^n) \cap L_H^2(\mathbb{R}_+, \mathbb{C}^n) : H^{-1}(t)Jy'(t) \in L_H^2(\mathbb{R}_+, \mathbb{C}^n)\}$ and $T_{\max}y = H^{-1}(t)Jy'(t)$, $y \in \text{dom } T_{\max}$. The minimal operator $T_{\min}$ is the closure of the operator $T'_{\min}$ defined as a restriction of $T_{\max}$ onto the set of all functions $y \in \text{dom } T_{\max}$ with compact support. It turns out that $T'_{\min}$ is a closed densely defined symmetric operator, $T_{\min}^* = T_{\max}$, and there exists the limit

$$[y, z]_{\infty} := \lim_{t \rightarrow +\infty} (Jy(t), z(t)), \quad y, z \in \text{dom } T_{\max} \quad (21)$$

(see, e.g., [12]). Clearly, equality (21) defines a skew-Hermitian bilinear form $[\cdot, \cdot]_{\infty}$ on $\text{dom } T_{\max}$.

For each $\lambda \in \mathbb{C}$ there exists a unique locally absolutely continuous operator-function $Y(\cdot, \lambda) : \mathbb{R}_+ \longrightarrow \mathbf{B}(\mathbb{C}^n)$ such that $JY'_t(t, \lambda) = \lambda H(t)Y(t, \lambda)$ (a.e. on $\mathbb{R}_+$) and $Y(0, \lambda) = I_n$. The operator-function $Y(\cdot, \lambda)$ is called the matricant of system (20).

In the following we denote by $N_{\pm}$ the deficiency indices of the system (20), i.e., deficiency indices of $T_{\min}$. Clearly, $N_{\pm} =$

$\dim \mathcal{N}_{\lambda}$, $\lambda \in \mathbb{C}_{\pm}$, where $\mathcal{N}_{\lambda} \subset \text{dom } T_{\max}$ is the linear space of all vector solutions of (20) belonging to $L_H^2(\mathbb{R}_+, \mathbb{C}^n)$. Hence $N_{\pm} \leq n$. Moreover, if $N_+ = n$, then $Y(\cdot, \lambda)x \in \text{dom } T_{\max}$ ($x \in \mathbb{C}^n$, $\lambda \in \mathbb{C}_+$) and hence for any $\lambda \in \mathbb{C}_+$ the equality

$$(Y(\lambda)x)(t) = Y(t, \lambda)x, \quad x \in \mathbb{C}^n \quad (22)$$

defines a linear operator $Y(\lambda) : \mathbb{C}^n \longrightarrow \text{dom } T_{\max}$.

With each function $f \in \dot{L}_H^2(\mathbb{R}_+, \mathbb{C}^n)$ one associates the function $\hat{f} : \mathbb{R} \longrightarrow \mathbb{C}^n$ (the generalized Fourier transform of f) given by

$$\hat{f}(s) = \int_{\mathbb{R}_+} Y^*(t, s) H(t) f(t) dt, \quad s \in \mathbb{R}. \quad (23)$$

Definition 9. A distribution function $\hat{\sigma}(\cdot) : \mathbb{R} \longrightarrow \mathbf{B}(\mathbb{C}^n)$ is called a spectral function of system (20) if the following Parseval identity holds:

$$\begin{aligned} \int_{\mathbb{R}} (d\hat{\sigma}(s) \hat{f}(s), \hat{f}(s)) &= \int_{\mathbb{R}_+} (H(t) f(t), f(t)) dt, \\ f &\in \dot{L}_H^2(\mathbb{R}_+, \mathbb{C}^n). \end{aligned} \quad (24)$$

Below within this subsection we assume that $n = 2q + \nu$ (hence $\mathbb{C}^n = \mathbb{C}^q \oplus \mathbb{C}^{\nu} \oplus \mathbb{C}^q$) and the operator J in (20) is

$$\begin{aligned} J &= \begin{pmatrix} 0 & 0 & -I_q \\ 0 & iI_{\nu} & 0 \\ I_q & 0 & 0 \end{pmatrix} : \mathbb{C}^q \oplus \mathbb{C}^{\nu} \oplus \mathbb{C}^q \longrightarrow \\ &\quad \mathbb{C}^q \oplus \mathbb{C}^{\nu} \oplus \mathbb{C}^q. \end{aligned} \quad (25)$$

Assume that system (20) has the maximal deficiency index $N_+ = n$ and let $m_- = n - N_-$. Then $m_- \leq q$, and by [3, Lemma 4.1] there exists an operator

$$\begin{aligned} \Gamma_{\infty} &= (\Gamma_{0\infty}, \hat{\Gamma}_{\infty}, \Gamma_{1\infty})^T : \text{dom } T_{\max} \longrightarrow \\ &\quad \mathbb{C}^q \oplus \mathbb{C}^{\nu} \oplus \mathbb{C}^q \end{aligned} \quad (26)$$

such that

$$\begin{aligned} \text{ran } \Gamma_{\infty} &= \left\{ x \oplus \hat{x} \oplus x' \in \mathbb{C}^q \oplus \mathbb{C}^{\nu} \oplus \mathbb{C}^q : Px' = -\frac{i}{2}Px \right\} \\ [y, z]_{\infty} &= (\Gamma_{0\infty}y, \Gamma_{1\infty}z) - (\Gamma_{1\infty}y, \Gamma_{0\infty}z) + i(\hat{\Gamma}_{\infty}y, \hat{\Gamma}_{\infty}z), \end{aligned} \quad (27)$$

$$y, z \in \text{dom } T_{\max}$$

(here P is the orthoprojector in $\mathbb{C}^q$ onto $\mathbb{C}^{m_-}$).

A description of all spectral functions of system (20) is given by the following theorem, which directly follows from Theorem 4.17 and formulas (4.68)–(4.71) in [3].

Theorem 10. Let system (20), (25) satisfies $N_+ = n$, let $m_- = n - N_-$, let $\mathcal{H} = \mathbb{C}^{q-m_-} \oplus \mathbb{C}^{\nu} \oplus \mathbb{C}^q \subset \mathbb{C}^n$, let Γ_{∞} be the operator

(26) satisfying (27), (28), and let $Y(\lambda)$ be the operator (22). Moreover, let $B : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^n)$ be an operator-function (the monodromy matrix) defined by $B(\lambda) = \Gamma_\infty Y(\lambda)$, $\lambda \in \mathbb{C}_+$, let

$$B(\lambda) = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) \\ B_{21}(\lambda) & B_{22}(\lambda) & B_{23}(\lambda) \\ B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda) \end{pmatrix} : \mathbb{C}^q \oplus \mathbb{C}^v \oplus \mathbb{C}^q \rightarrow \mathbb{C}^q \oplus \mathbb{C}^v \oplus \mathbb{C}^q \quad (29)$$

be the block representation of $B(\lambda)$, and let

$$w_1(\lambda) = \begin{pmatrix} B_{11}(\lambda) & B_{12}(\lambda) & B_{13}(\lambda) \\ -iB_{21}(\lambda) & -i(B_{22}(\lambda) - I_v) & -iB_{23}(\lambda) \\ 0 & 0 & -I_q \end{pmatrix} \quad (30)$$

$$w_2(\lambda) = \begin{pmatrix} \frac{1}{2}B_{13}(\lambda) & \frac{i}{2}B_{12}(\lambda) & -\frac{1}{2}B_{11}(\lambda) \\ -\frac{i}{2}B_{23}(\lambda) & \frac{1}{2}(B_{22}(\lambda) + I_v) & \frac{i}{2}B_{21}(\lambda) \\ \frac{1}{2}I_q & 0 & 0 \end{pmatrix} \quad (31)$$

$$w_3(\lambda) = \begin{pmatrix} B_{31}(\lambda) & B_{32}(\lambda) & B_{33}(\lambda) \\ -\frac{1}{2}B_{21}(\lambda) & -\frac{1}{2}(B_{22}(\lambda) + I_v) & -\frac{1}{2}B_{23}(\lambda) \\ -I_q & 0 & 0 \end{pmatrix} \quad (32)$$

$$w_4(\lambda) = \begin{pmatrix} \frac{1}{2}B_{33}(\lambda) & \frac{i}{2}B_{32}(\lambda) & -\frac{1}{2}B_{31}(\lambda) \\ -\frac{1}{4}B_{23}(\lambda) & -\frac{i}{4}(B_{22}(\lambda) - I_v) & \frac{1}{4}B_{21}(\lambda) \\ 0 & 0 & -\frac{1}{2}I_q \end{pmatrix}. \quad (33)$$

Then the equalities

$$\widehat{m}_\tau(\lambda) = (C_0(\lambda) w_1(\lambda) + C_1(\lambda) w_3(\lambda))^{-1} \cdot (C_0(\lambda) w_2(\lambda) + C_1(\lambda) w_4(\lambda)), \quad \lambda \in \mathbb{C}_+ \quad (34)$$

$$\widehat{\sigma}_\tau(s) = \lim_{\delta \rightarrow +0} \lim_{\gamma \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} \widehat{m}_\tau(x + iy) dx \quad (35)$$

establish a bijective correspondence $\widehat{\sigma}(s) = \widehat{\sigma}_\tau(s)$ between all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \widetilde{R}_{\mathcal{H}}(\mathbb{C}^n)$ and all spectral functions $\widehat{\sigma}(\cdot)$ of the system.

3. Spectral Functions for the Fourier Transform

3.1. Definitions and Auxiliary Results. Let

$$\widetilde{\mathcal{F}} = \{\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n\} \quad (36)$$

be a system of intervals $\mathcal{J}_j = (a_j, b_j)$, $-\infty \leq a_j < b_j \leq \infty$, $j \in \overline{1, n}$. Denote by $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ the set of all functions $\varphi(\cdot) = \{\varphi_j(\cdot)\}_{j=1}^n \in L^2(\mathbb{R}, \mathbb{C}^n)$ with $\varphi_j(t) = 0$ a.e. on $\mathbb{R} \setminus \mathcal{J}_j$ and by $\dot{L}^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ the set of all functions $\varphi(\cdot) = \{\varphi_j(\cdot)\}_{j=1}^n \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ with the following property: if $\mathcal{J}_j \in \widetilde{\mathcal{F}}$ is an infinite interval, then there exists a finite interval $(\alpha_j, \beta_j) \subset \mathcal{J}_j$ (depending on φ) such that $\varphi_j(t) = 0$ a.e. on $\mathcal{J}_j \setminus (\alpha_j, \beta_j)$. Since $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ is unitarily equivalent to the Hilbert space $L^2(\mathcal{J}_1) \oplus L^2(\mathcal{J}_2) \oplus \dots \oplus L^2(\mathcal{J}_n)$, it follows that $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ is a closed subspace in $L^2(\mathbb{R}, \mathbb{C}^n)$. Moreover, $\dot{L}^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ is a linear manifold dense in $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$.

With each function $\varphi \in \dot{L}^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ one associates the function $\widehat{\varphi} : \mathbb{R} \rightarrow \mathbb{C}^n$ (the classical vector-valued Fourier transform of φ) given by

$$\widehat{\varphi}(s) = \int_{\mathbb{R}} e^{its} \varphi(t) dt, \quad s \in \mathbb{R}. \quad (37)$$

Assume that $\sigma(\cdot) : \mathbb{R} \rightarrow \mathbf{B}(\mathbb{C}^n)$ is an operator-valued distribution function such that for each $\varphi \in \dot{L}^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ the following Parseval identity holds:

$$\begin{aligned} (\|\widehat{\varphi}\|_{L^2(\sigma; \mathbb{C}^n)} &:=) \int_{\mathbb{R}} (d\sigma(s) \widehat{\varphi}(s), \widehat{\varphi}(s)) \\ &= \int_{\mathbb{R}} \|\varphi(t)\|^2 dt (= \|\varphi\|_{L^2(\mathbb{R}, \mathbb{C}^n)}^2). \end{aligned} \quad (38)$$

Then for each $\varphi \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ the integral in (37) converges in the norm of $L^2(\sigma, \mathbb{C}^n)$ and the equality $V_\sigma \varphi = \widehat{\varphi}$, $\varphi \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$, defines an isometry V_σ from $L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$ to $L^2(\sigma; \mathbb{C}^n)$.

Definition 11. A $\mathbf{B}(\mathbb{C}^n)$ -valued distribution function $\sigma(\cdot)$ such that (38) holds will be called a spectral function for the classical Fourier transform (37) (with respect to the system of intervals $\widetilde{\mathcal{F}}$).

In what follows the set of spectral functions in the sense of Definition 11 will be denoted by $SF_n(\widetilde{\mathcal{F}}) = SF_n(\mathcal{J}_1, \mathcal{J}_2, \dots, \mathcal{J}_n)$. Moreover, in the case when $n = 1$ and the system $\widetilde{\mathcal{F}}$ consists of a unique interval $\mathcal{J} = (a, b)$, we let $SF(a, b) := SF_1(\widetilde{\mathcal{F}})$.

Proposition 12. Let $\widetilde{\mathcal{F}}$ be system of intervals (36), let $\chi_{\mathcal{J}_j}(\cdot)$ be the indicator of an interval $\mathcal{J}_j$, let $\chi_{\widetilde{\mathcal{F}}}(t) = \operatorname{diag}(\chi_{\mathcal{J}_1}(t), \chi_{\mathcal{J}_2}(t), \dots, \chi_{\mathcal{J}_n}(t))$, and let $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$. Then the inverse to (37) transform is

$$\varphi(t) = \int_{\mathbb{R}} e^{-its} \chi_{\widetilde{\mathcal{F}}}(t) d\sigma(s) \widehat{\varphi}(s), \quad (39)$$

where the integral converges in the norm of $L^2(\mathbb{R}, \mathbb{C}^n)$. If in addition $\sigma(\cdot)$ is absolutely continuous and $\Sigma(s) = \sigma'(s)$ is

the density of $\sigma(\cdot)$, then Parseval identity (38) and the inverse transform (39) can be written as

$$\int_{\mathbb{R}} (\Sigma(s) \widehat{\varphi}(s), \widehat{\varphi}(s)) ds = \int_{\mathbb{R}} \|\varphi(t)\|^2 dt, \quad (40)$$

$$\varphi \in \dot{L}^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$$

$$\varphi(t) = \int_{\mathbb{R}} e^{-its} \chi_{\widetilde{\mathcal{F}}}(t) \Sigma(s) \widehat{\varphi}(s) ds. \quad (41)$$

Proof. Since V_{σ} is an isometry, to prove (39) it is sufficient to show that

$$(V_{\sigma}^* g)(t) = \int_{\mathbb{R}} e^{-its} \chi_{\widetilde{\mathcal{F}}}(t) d\sigma(s) g(s), \quad (42)$$

$$g \in \dot{L}^2(\sigma, \mathbb{C}^n),$$

where $\dot{L}^2(\sigma, \mathbb{C}^n)$ is the set of all functions $g \in L^2(\sigma, \mathbb{C}^n)$ with compact support.

Let $g \in \dot{L}^2(\sigma, \mathbb{C}^n)$, let $f_g(t) = \{f_{g,j}(t)\}_{j=1}^n$ be given by

$$f_g(t) = \int_{\mathbb{R}} e^{-its} \chi_{\widetilde{\mathcal{F}}}(t) d\sigma(s) g(s) \quad (43)$$

$$:= \chi_{\widetilde{\mathcal{F}}}(t) \int_{\mathbb{R}} e^{-its} \Psi(s) g(s) d\mu$$

(for Ψ and μ see assertions before (16)), and let $f_*(t) = \{f_{j*}(t)\}_{j=1}^n := (V_{\sigma}^* g)(t) \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$. Let us show that for each finite interval $\mathcal{I}'_j \subset \mathcal{I}_j$

$$\int_{\mathbb{R}} \chi_{\mathcal{I}'_j}(t) \overline{f_{g,j}(t)} dt = \int_{\mathbb{R}} \chi_{\mathcal{I}'_j}(t) \overline{f_{j*}(t)} dt, \quad j \in \overline{1, n}. \quad (44)$$

Indeed, let $\alpha_j(t) = \{\delta_{kj} \chi_{\mathcal{I}'_j}(t)\}_{k=1}^n \in L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})$. Then by using the Fubini theorem and (16) one gets

$$\begin{aligned} \int_{\mathbb{R}} \chi_{\mathcal{I}'_j}(t) \overline{f_{g,j}(t)} dt &= \int_{\mathbb{R}} (\alpha_j(t), f_g(t)) dt \\ &= \int_{\mathbb{R}} \left(\alpha_j(t), \int_{\mathbb{R}} e^{-its} \Psi(s) g(s) d\mu \right) dt \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\alpha_j(t), e^{-its} \Psi(s) g(s)) d\mu \right) dt \\ &= \int_{\mathbb{R}} \left(\int_{\mathbb{R}} (\alpha_j(t), e^{-its} \Psi(s) g(s)) dt \right) d\mu \quad (45) \\ &= \int_{\mathbb{R}} \left(\Psi(s) \int_{\mathbb{R}} e^{its} \alpha_j(t) dt, g(s) \right) d\mu \\ &= (V_{\sigma} \alpha_j, g)_{L^2(\sigma; \mathbb{C}^n)} = (\alpha_j, V_{\sigma}^* g)_{L^2(\mathbb{R}, \mathbb{C}^n; \widetilde{\mathcal{F}})} \\ &= \int_{\mathbb{R}} (\alpha_j(t), f_*(t)) dt = \int_{\mathbb{R}} \chi_{\mathcal{I}'_j}(t) \overline{f_{j*}(t)} dt, \end{aligned}$$

which proves (44). Since $f_{g,j}(t) = f_{j*}(t) = 0$ (a.e. on $\mathbb{R} \setminus \mathcal{I}_j$), it follows from (44) that $f_g(t) = f_*(t)$ (a.e. on $\mathbb{R}$) and hence (42) holds.

The last statement of the proposition is obvious. $\square$

Remark 13. If $n = 1$ and system $\widetilde{\mathcal{F}}$ consists of a unique interval $\mathcal{I} = (a, b)$, then equality (37) defines the classical $\mathbb{C}$ -valued Fourier transform $\widehat{\varphi}$ of a scalar function $\varphi \in \dot{L}^2(\mathbb{R})$ with compact support belonging to (a, b) (we denote the set of such functions by $\dot{L}^2(\mathbb{R}; (a, b))$). A spectral function $\sigma(\cdot) \in SF(a, b)$ for this transform is a scalar distribution function such that the Parseval identity

$$\int_{\mathbb{R}} |\widehat{\varphi}(s)|^2 d\sigma(s) = \int_{\mathbb{R}} |\varphi(t)|^2 dt, \quad \varphi \in \dot{L}^2(\mathbb{R}; (a, b)) \quad (46)$$

holds; moreover, the inverse Fourier transform is

$$\varphi(t) = \int_{\mathbb{R}} e^{-its} \chi_{\mathcal{I}}(t) \widehat{\varphi}(s) d\sigma(s). \quad (47)$$

Assume that $r_{\pm}, m_{\pm} \in \mathbb{N} \cup \{0\}$, $p := r_+ + m_+$, $q := r_- + m_-$, $n := p + q$, and $\widetilde{\mathcal{F}}$ is a system of intervals given by

$$\widetilde{\mathcal{F}} = \left\{ \overbrace{(0, b_1), \dots, (0, b_{r_+})}^p, \underbrace{\mathbb{R}_+, \dots, \mathbb{R}_+}_{m_+ \text{ times}}, \right. \quad (48)$$

$$\left. \overbrace{(-c_1, 0), \dots, (-c_{r_-}, 0)}^q, \underbrace{\mathbb{R}_-, \dots, \mathbb{R}_-}_{m_- \text{ times}} \right\}$$

(here $b_j, c_j \in (0, \infty)$). With the system $\widetilde{\mathcal{F}}$ one associates canonical differential system

$$\widetilde{J} y' = \lambda \widetilde{H}(t) y, \quad t \in \mathbb{R}_+, \quad (49)$$

where

$$\begin{aligned} \widetilde{J} &= i \operatorname{diag}(I_p, -I_q), \\ \widetilde{H}(t) &= \operatorname{diag}(h_{1+}(t), \dots, h_{p+}(t), h_{1-}(t), \dots, h_{q-}(t)). \end{aligned} \quad (50)$$

In (50) $h_{j+}, j \in \overline{1, p}$, and $h_{k-}, k \in \overline{1, q}$, are continuous functions on $\mathbb{R}_+$ such that $h_{j+}(t) > 0$, $h_{k-}(t) > 0$, $t \in \mathbb{R}_+$, and

$$\int_{\mathbb{R}_+} h_{j+}(t) dt = b_j, \quad j \in \overline{1, r_+}; \quad (51)$$

$$\int_{\mathbb{R}_+} h_{j+}(t) dt = +\infty, \quad j \in \overline{r_+ + 1, p}$$

$$\int_{\mathbb{R}_+} h_{k-}(t) dt = c_k, \quad k \in \overline{1, r_-}; \quad (52)$$

$$\int_{\mathbb{R}_+} h_{k-}(t) dt = +\infty, \quad k \in \overline{r_- + 1, q}.$$

One can easily verify that the matrizant $\widetilde{Y}(\cdot, \lambda)$ of system (49) is

$$\begin{aligned} \widetilde{Y}(t, \lambda) &= \operatorname{diag}(e^{-i\lambda\Phi_{1+}(t)}, \dots, e^{-i\lambda\Phi_{p+}(t)}, e^{i\lambda\Phi_{1-}(t)}, \dots, \\ &\quad e^{i\lambda\Phi_{q-}(t)}), \end{aligned} \quad (53)$$

where

$$\begin{aligned}\Phi_{j+}(t) &= \int_{[0,t]} h_{j+}(u) du, \quad j \in \overline{1, p}; \\ \Phi_{k-}(t) &= \int_{[0,t]} h_{k-}(u) du, \quad k \in \overline{1, q}.\end{aligned}\quad (54)$$

Assume that $\varepsilon_j = \{\delta_{kj}\}_{k=1}^n \in \mathbb{C}^n$, $j \in \overline{1, n}$, is the standard basis in $\mathbb{C}^n$ and let

$$\begin{aligned}\tilde{y}_{j+}(t, \lambda) &= e^{-i\lambda\Phi_{j+}(t)} \varepsilon_j, \quad j \in \overline{1, p}; \\ \tilde{y}_{k-}(t, \lambda) &= e^{i\lambda\Phi_{k-}(t)} \varepsilon_{p+k}, \quad k \in \overline{1, q}.\end{aligned}\quad (55)$$

Since by (53) $(\tilde{y}_{j+}(t, \lambda))^T$ and $(\tilde{y}_{k-}(t, \lambda))^T$ are columns of the matrizant $\tilde{Y}(t, \lambda)$, the system $\{\tilde{y}_{j+}(\cdot, \lambda)\}_{j=1}^p \cup \{\tilde{y}_{k-}(\cdot, \lambda)\}_{k=1}^q$ is a basis in the space of all solutions of system (49). Moreover, the immediate checking shows that

$$\tilde{y}_{j+}(\cdot, \lambda) \in L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n), \quad j \in \overline{1, r_+}; \quad (56)$$

$$\tilde{y}_{j+}(\cdot, \lambda) \notin L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n), \quad j \in \overline{r_+ + 1, p}$$

$$\tilde{y}_{k-}(\cdot, \lambda) \in L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n), \quad k \in \overline{1, q} \quad (57)$$

for $\lambda \in \mathbb{C}_+$ and

$$\tilde{y}_{j+}(\cdot, \lambda) \in L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n), \quad j \in \overline{1, p}; \quad (58)$$

$$\tilde{y}_{k-}(\cdot, \lambda) \in L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n), \quad k \in \overline{1, r_-}$$

$$\tilde{y}_{k-}(\cdot, \lambda) \notin L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n), \quad k \in \overline{r_- + 1, q} \quad (59)$$

for $\lambda \in \mathbb{C}_-$. Therefore deficiency indices $\tilde{N}_{\pm}$ of system (49) are

$$\begin{aligned}\tilde{N}_+ &= q + r_+, \\ \tilde{N}_- &= p + r_-.\end{aligned}\quad (60)$$

Now assume that $\tilde{\mathcal{F}}$ is system of intervals (48) with $p \geq q$ and let $\nu := p - q$, so that

$$\begin{aligned}p &= q + \nu = r_- + m_- + \nu, \\ n &= p + q = 2q + \nu.\end{aligned}\quad (61)$$

Moreover, let X and X^* be mutually adjoint unitary operators in $\mathbb{C}^n$ defined by

$$\begin{aligned}X &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_q & 0 & iI_q \\ 0 & \sqrt{2}I_\nu & 0 \\ I_q & 0 & -iI_q \end{pmatrix}, \\ X^* &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_q & 0 & I_q \\ 0 & \sqrt{2}I_\nu & 0 \\ -iI_q & 0 & iI_q \end{pmatrix}.\end{aligned}\quad (62)$$

Together with canonical system (49), (50) we associate with system of intervals $\tilde{\mathcal{F}}$ canonical system (20), where J is of the form (25) and

$$H(t) = X^* \tilde{H}(t) X, \quad t \in \mathbb{R}_+. \quad (63)$$

Clearly, $\tilde{J}$ admits the representation $\tilde{J} = \text{diag}(iI_q, iI_\nu, -iI_q)$ and the immediate checking shows that

$$J = X^* \tilde{J} X, \quad \tilde{J} = X J X^*. \quad (64)$$

In the following we denote by $T_{\max}$ and $\tilde{T}_{\max}$ the maximal operators induced by systems (20) and (49), respectively. It follows from (63) and (64) that the equality

$$(U\tilde{f})(t) = X^* \tilde{f}(t), \quad \tilde{f} \in L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n) \quad (65)$$

defines a unitary operator U from $L_{\tilde{H}}^2(\mathbb{R}_+, \mathbb{C}^n)$ onto $L_H^2(\mathbb{R}_+, \mathbb{C}^n)$ such that $U \text{ dom } \tilde{T}_{\max} = \text{dom } T_{\max}$ and

$$\begin{aligned}(J(U\tilde{y})(t), (U\tilde{z})(t)) &= (\tilde{J}\tilde{y}(t), \tilde{z}(t)), \\ [U\tilde{y}, U\tilde{z}]_\infty &= [\tilde{y}, \tilde{z}]_\infty, \\ \tilde{y}, \tilde{z} &\in \text{dom } \tilde{T}_{\max}.\end{aligned}\quad (66)$$

Moreover, the matrizant $Y(\cdot, \lambda)$ of system (20) is

$$Y(t, \lambda) = X^* \tilde{Y}(t, \lambda) X, \quad (67)$$

where $\tilde{Y}(t, \lambda)$ is the matrizant (53) of system (49). Observe also that systems (20) and (49) have the same deficiency indices and by (60) deficiency indices $N_{\pm}$ of the system (20) are

$$\begin{aligned}N_+ &= q + r_+, \\ N_- &= p + r_-.\end{aligned}\quad (68)$$

Proposition 14. *Let $\tilde{\mathcal{F}}$ be system of intervals (48). Then the set $SF_n(\tilde{\mathcal{F}})$ coincides with the set of spectral functions of system (49), (50). If in addition $p \geq q$, then the equality*

$$\sigma(s) = X \hat{\sigma}(s) X^* \quad (69)$$

gives a bijective correspondence between all spectral functions $\hat{\sigma}(\cdot)$ of system (20), (63) and all spectral functions $\sigma(\cdot) \in SF_n(\tilde{\mathcal{F}})$.

Proof. Let $\mathcal{J}_{j+} = [0, b_j)$, $j \in \overline{1, r_+}$, and $\mathcal{J}_{j+} = \mathbb{R}_+$, $j \in \overline{r_+ + 1, p}$. Moreover, let $\mathcal{J}_{k-} = (-c_k, 0]$, $k \in \overline{1, r_-}$, and $\mathcal{J}_{k-} = \mathbb{R}_-$, $k \in \overline{r_- + 1, q}$. Clearly, $\Phi_{j+}(\cdot) (\Phi_{k-}(\cdot))$ is a strictly increasing absolutely continuous function mapping bijectively $\mathbb{R}_+$ onto $\mathcal{J}_{j+}$ (resp. $-\mathcal{J}_{k-} = \{t \in \mathbb{R} : -t \in \mathcal{J}_{k-}\}$). With each vector-function

$$f(\cdot) = \{f_{1+}(\cdot), \dots, f_{p+}(\cdot), f_{1-}(\cdot), \dots, f_{q-}(\cdot)\} : \mathbb{R}_+ \rightarrow \mathbb{C}^n \quad (70)$$

we associate the vector-function

$$\varphi_f(\cdot) = \{\varphi_{1+}(\cdot), \dots, \varphi_{p+}(\cdot), \varphi_{1-}(\cdot), \dots, \varphi_{q-}(\cdot)\} : \mathbb{R} \longrightarrow \mathbb{C}^n \quad (71)$$

given by

$$\begin{aligned} \varphi_{j+}(u) &= \begin{cases} f_{j+}(\Phi_{j+}^{-1}(u)), & u \in \mathcal{J}_{j+} \\ 0, & u \notin \mathcal{J}_{j+}, \end{cases} \\ \varphi_{k-}(u) &= \begin{cases} f_{k-}(\Phi_{k-}^{-1}(-u)), & u \in \mathcal{J}_{k-} \\ 0, & u \notin \mathcal{J}_{k-} \end{cases} \end{aligned} \quad (72)$$

(here $j \in \overline{1, p}$ and $k \in \overline{1, q}$). Since $\Phi'_{j+}(t) = h_{j+}(t)$ and $\Phi'_{k-}(t) = h_{k-}(t)$, it follows that

$$\begin{aligned} \int_{\mathbb{R}_+} h_{j+}(t) |f_{j+}(t)|^2 dt &= \int_{\mathcal{J}_{j+}} |f_{j+}(\Phi_{j+}^{-1}(u))|^2 du \\ &= \int_{\mathbb{R}} |\varphi_{j+}(u)|^2 du, \quad j \in \overline{1, p} \\ \int_{\mathbb{R}_+} h_{k-}(t) |f_{k-}(t)|^2 dt &= \int_{\mathcal{J}_{k-}} |f_{k-}(\Phi_{k-}^{-1}(-u))|^2 du \\ &= \int_{\mathbb{R}} |\varphi_{k-}(u)|^2 du, \quad k \in \overline{1, q} \end{aligned} \quad (73)$$

and hence

$$\begin{aligned} \int_{\mathbb{R}_+} (\bar{H}(t) f(t), f(t)) dt &= \sum_{j=1}^p \int_{\mathbb{R}_+} h_{j+}(t) |f_{j+}(t)|^2 dt \\ &\quad + \sum_{k=1}^q \int_{\mathbb{R}_+} h_{k-}(t) |f_{k-}(t)|^2 dt \\ &= \sum_{j=1}^p \int_{\mathbb{R}} |\varphi_{j+}(u)|^2 du + \sum_{k=1}^q \int_{\mathbb{R}} |\varphi_{k-}(u)|^2 du \\ &= \int_{\mathbb{R}} \|\varphi_f(u)\|^2 du. \end{aligned} \quad (74)$$

Thus, the equality $\varphi(t) = \varphi_f(t)$ gives a bijective correspondence between all functions $f \in \dot{L}^2_{\bar{H}}(\mathbb{R}_+, \mathbb{C}^n)$ and all functions $\varphi \in \dot{L}^2_2(\mathbb{R}, \mathbb{C}^n; \mathcal{F})$. Moreover, $\|f\|_{\dot{L}^2_{\bar{H}}(\mathbb{R}_+, \mathbb{C}^n)} = \|\varphi_f\|_{\dot{L}^2_2(\mathbb{R}, \mathbb{C}^n; \mathcal{F})}$.

Next assume that $f \in \dot{L}^2_{\bar{H}}(\mathbb{R}_+, \mathbb{C}^n)$ and $\varphi_f \in \dot{L}^2_2(\mathbb{R}, \mathbb{C}^n; \mathcal{F})$ are vector-functions (70) and (71) and let

$$\begin{aligned} \hat{f} &= \{\hat{f}_{1+}, \dots, \hat{f}_{p+}, \hat{f}_{1-}, \dots, \hat{f}_{q-}\}, \\ \hat{\varphi}_f &= \{\hat{\varphi}_{1+}, \dots, \hat{\varphi}_{p+}, \hat{\varphi}_{1-}, \dots, \hat{\varphi}_{q-}\} \end{aligned} \quad (75)$$

be the corresponding Fourier transforms (23) and (37) (in (23) $H(t)$ and $Y(t, \lambda)$ should be replaced with $\bar{H}(t)$ and $\bar{Y}(t, \lambda)$, respectively). Then by (53) and (50)

$$\begin{aligned} \hat{f}_{j+}(s) &= \int_{\mathbb{R}_+} e^{is\Phi_{j+}(t)} h_{j+}(t) f_{j+}(t) dt \\ &= \int_{\mathcal{J}_{j+}} e^{isu} f_{j+}(\Phi_{j+}^{-1}(u)) du \\ &= \int_{\mathbb{R}} e^{isu} \varphi_{j+}(u) du = \hat{\varphi}_{j+}(s) \\ \hat{f}_{k-}(s) &= \int_{\mathbb{R}_+} e^{-is\Phi_{k-}(t)} h_{k-}(t) f_{k-}(t) dt \\ &= \int_{\mathcal{J}_{k-}} e^{isu} f_{k-}(\Phi_{k-}^{-1}(-u)) du \\ &= \int_{\mathbb{R}} e^{isu} \varphi_{k-}(u) du = \hat{\varphi}_{k-}(s) \end{aligned} \quad (76)$$

and, consequently, $\hat{f}(s) = \hat{\varphi}_f(s)$. This yields the first statement of the proposition.

If $p \geq q$, then by (63) and (67) the equality (69) gives a bijective correspondence between all spectral functions $\bar{\sigma}(\cdot)$ of system (20), (63) and all spectral functions $\sigma(\cdot)$ of system (49), (50). This and the first statement of the proposition yield the second statement. $\square$

Below unless otherwise stated we assume that $\mathcal{F}$ is system of intervals (48) with $m_+ = 0$ and $p \geq q$. This means that

$$\mathcal{F} = \left\{ (0, b_1), \dots, (0, b_p), \overbrace{(-c_1, 0), \dots, (-c_r, 0)}^q, \underbrace{\mathbb{R}_-, \dots, \mathbb{R}_-}_{m_- \text{ times}} \right\}, \quad (77)$$

and according to (68) deficiency indices $N_{\pm}$ of the respective system (20) are

$$\begin{aligned} N_+ &= q + p = n, \\ N_- &= p + r_-. \end{aligned} \quad (78)$$

Lemma 15. Let $\varepsilon_j = \{\delta_{kj}\}_{k=1}^n$, $j \in \overline{1, n}$, be the standard basis in $\mathbb{C}^n$, let $\varepsilon'_j = X^* \varepsilon_j$ and let

$$\begin{aligned} u_j(t) &= e^{\Phi_{j+}(t)-b_j} \varepsilon'_j, \quad j \in \overline{1, p}; \\ v_k(t) &= e^{\Phi_{k-}(t)-c_k} \varepsilon'_{p+k}, \quad k \in \overline{1, r_-}, \\ t &\in \mathbb{R}_+, \end{aligned} \quad (79)$$

where $\Phi_{j+}(t)$ and $\Phi_{k-}(t)$ are given by (54). Then $u_j, v_k \in \text{dom } T_{\max}$ and the equalities

$$\begin{aligned} G_0 y &= \left\{ [y, u_j]_{\infty} \right\}_{j=1}^p \in \mathbb{C}^p, \\ G_1 y &= \left\{ [y, v_k]_{\infty} \right\}_{k=1}^{r-} \in \mathbb{C}^{r-}, \end{aligned} \quad (80)$$

$y \in \text{dom } T_{\max}$

define linear mappings $G_0 : \text{dom } T_{\max} \rightarrow \mathbb{C}^p$ and $G_1 : \text{dom } T_{\max} \rightarrow \mathbb{C}^{r-}$ such that the mapping $G := (G_0, G_1)^{\top} : \text{dom } T_{\max} \rightarrow \mathbb{C}^p \oplus \mathbb{C}^{r-}$ is surjective and the identity

$$[y, z]_{\infty} = i(G_0 y, G_0 z)_{\mathbb{C}^p} - i(G_1 y, G_1 z)_{\mathbb{C}^{r-}}, \quad (81)$$

$y, z \in \text{dom } T_{\max}$

is valid. Moreover, if

$$G_0 = (G_{01}, G_{02}, G_{03})^{\top} : \text{dom } T_{\max} \rightarrow \underbrace{\mathbb{C}^{r-} \oplus \mathbb{C}^{m-} \oplus \mathbb{C}^v}_{\mathbb{C}^q} \quad (82)$$

is the block representation of G_0 , then the equalities

$$\Gamma_{0\infty} = \left(\frac{1}{\sqrt{2}} (G_{01} + G_1), G_{02} \right)^{\top} : \text{dom } T_{\max} \rightarrow \underbrace{\mathbb{C}^{r-} \oplus \mathbb{C}^{m-}}_{\mathbb{C}^q} \quad (83)$$

$$\hat{\Gamma}_{\infty} = G_{03} : \text{dom } T_{\max} \rightarrow \mathbb{C}^v \quad (84)$$

$$\Gamma_{1\infty} \left(-\frac{i}{\sqrt{2}} (G_{01} - G_1), -\frac{i}{2} G_{02} \right)^{\top} : \text{dom } T_{\max} \rightarrow \underbrace{\mathbb{C}^{r-} \oplus \mathbb{C}^{m-}}_{\mathbb{C}^q} \quad (85)$$

together with (26) define the operator Γ_{∞} such that (27) and (28) hold.

Proof. Let

$$\begin{aligned} \tilde{u}_j(t) &= e^{\Phi_{j+}(t)-b_j} \varepsilon_j, \quad j \in \overline{1, p}; \\ \tilde{v}_k(t) &= e^{\Phi_{k-}(t)-c_k} \varepsilon_{p+k}, \quad k \in \overline{1, r-}. \end{aligned} \quad (86)$$

Then by (55) $\tilde{u}_j(t) = e^{-b_j} \tilde{y}_{j+}(t, i)$, $\tilde{v}_k(t) = e^{-c_k} \tilde{y}_{k-}(t, -i)$, and in view of (56) and (58) $\tilde{u}_j, \tilde{v}_k \in \text{dom } \tilde{T}_{\max}$. Since $u_j(t) = X^* \tilde{u}_j(t)$ and $v_k(t) = X^* \tilde{v}_k(t)$, it follows that

$$\begin{aligned} u_j &= U \tilde{u}_j, \quad j \in \overline{1, p}; \\ v_k &= U \tilde{v}_k, \quad k \in \overline{1, r-}, \end{aligned} \quad (87)$$

where U is the unitary operator (65). Hence $u_j, v_k \in \text{dom } T_{\max}$.

Clearly,

$$(\tilde{J} \varepsilon_j, \varepsilon_k) = \begin{cases} i \delta_{jk}, & j, k \in \overline{1, p} \\ 0, & j \in \overline{1, p}, k \in \overline{p+1, n} \\ -i \delta_{jk}, & j, k \in \overline{p+1, n}. \end{cases} \quad (88)$$

This and (66), (87) yield

$$\begin{aligned} (Ju_j(t), u_l(t)) &= (\tilde{J} \tilde{u}_j(t), \tilde{u}_l(t)) \\ &= i e^{\Phi_{j+}(t)-b_j} e^{\Phi_{l+}(t)-b_l} \delta_{jl}, \quad j, l \in \overline{1, p} \\ (Ju_j(t), v_k(t)) &= (\tilde{J} \tilde{u}_j(t), \tilde{v}_k(t)) = 0, \\ & \quad j \in \overline{1, p}, k \in \overline{1, r-} \end{aligned} \quad (89)$$

$$\begin{aligned} (Jv_k(t), v_m(t)) &= (\tilde{J} \tilde{v}_k(t), \tilde{v}_m(t)) \\ &= -i e^{\Phi_{k-}(t)-c_k} e^{\Phi_{m-}(t)-c_m} \delta_{km}, \\ & \quad k, m \in \overline{1, r-}. \end{aligned}$$

Since by (51) and (52) $\Phi_{j+}(\infty) = b_j$, $j \in \overline{1, p}$, and $\Phi_{k-}(\infty) = c_k$, $k \in \overline{1, r-}$, it follows that

$$[u_j, u_l]_{\infty} = i \delta_{jl}, \quad j, l \in \overline{1, p}; \quad (90)$$

$$[u_j, v_k]_{\infty} = 0, \quad j \in \overline{1, p}, k \in \overline{1, r-}$$

$$[v_k, v_m]_{\infty} = -i \delta_{km}, \quad k, m \in \overline{1, r-}. \quad (91)$$

Next, denote by $\mathcal{D}_0$ the kernel of the form $[\cdot, \cdot]_{\infty}$. According to [13, Proposition 5.5] the indices of inertia $\nu_{\infty \pm}$ of the form $[\cdot, \cdot]_{\infty}$ are $\nu_{\infty+} = N_+ - q$, $\nu_{\infty-} = N_- - p$ and (78) yields $\nu_{\infty+} = p$, $\nu_{\infty-} = r$. Therefore by (90) and (91) system $\{u_j\}_{j=1}^p \cup \{v_k\}_{k=1}^{r-}$ forms a basis in $\text{dom } T_{\max} / \mathcal{D}_0$ and hence each $y \in \text{dom } T_{\max}$ admits a unique representation

$$y = y_0 + \sum_{j=1}^p \alpha_{jy} u_j + \sum_{k=1}^{r-} \beta_{ky} v_k, \quad (92)$$

where $y_0 \in \mathcal{D}_0$ and $\alpha_{jy}, \beta_{ky} \in \mathbb{C}$. Let $G_0 : \text{dom } T_{\max} \rightarrow \mathbb{C}^p$ and $G_1 : \text{dom } T_{\max} \rightarrow \mathbb{C}^{r-}$ be the operators given by

$$\begin{aligned} G_0 y &= \{i \alpha_{jy}\}_{j=1}^p \in \mathbb{C}^p, \\ G_1 y &= \{-i \beta_{ky}\}_{k=1}^{r-} \in \mathbb{C}^{r-}. \end{aligned} \quad (93)$$

Then by (90) and (91) for any $y, z \in \text{dom } T_{\max}$ one has

$$\begin{aligned} [y, z]_{\infty} &= i \sum_{j=1}^p \alpha_{jy} \overline{\alpha_{jz}} - i \sum_{k=1}^{r-} \beta_{ky} \overline{\beta_{kz}} \\ &= i(G_0 y, G_0 z)_{\mathbb{C}^p} - i(G_1 y, G_1 z)_{\mathbb{C}^{r-}}. \end{aligned} \quad (94)$$

Moreover, by (93) the operator $G = (G_0, G_1)^{\top}$ is surjective. Note also that by (90) and (91) $[y, u_j]_{\infty} = i \alpha_{jy}$, $[y, v_k]_{\infty} = -i \beta_{ky}$, and hence (93) admits the representation (80). This proves the first statement of the lemma.

Now assume that $\Gamma_{0\infty}, \hat{\Gamma}_{\infty}$ and $\Gamma_{1\infty}$ are the operators (83)–(85), Γ_{∞} is given by (26), and P is the orthoprojector in $\mathbb{C}^q$ onto $\mathbb{C}^{m-}$. Then

$$P \Gamma_{1\infty} = -\frac{i}{2} G_{02} = -\frac{i}{2} P \Gamma_{0\infty}, \quad (95)$$

which in view of surjectivity of G yields (27). Finally, for any $y, z \in \text{dom } T_{\max}$ one has

$$\begin{aligned}
 & (\Gamma_{0\infty} y, \Gamma_{1\infty} z) - (\Gamma_{1\infty} y, \Gamma_{0\infty} z) + i(\widehat{\Gamma}_{\infty} y, \widehat{\Gamma}_{\infty} z) \\
 &= \frac{i}{2} (G_{01} y + G_1 y, G_{01} z - G_1 z) + \frac{i}{2} (G_{02} y, G_{02} z) \\
 &\quad + \frac{i}{2} (G_{01} y - G_1 y, G_{01} z + G_1 z) \\
 &\quad + \frac{i}{2} (G_{02} y, G_{02} z) + i(G_{03} y, G_{03} z) \\
 &= i(G_{01} y, G_{01} z) - i(G_1 y, G_1 z) + i(G_{02} y, G_{02} z) \\
 &\quad + i(G_{03} y, G_{03} z) = i(G_0 y, G_0 z) - i(G_1 y, G_1 z) \\
 &= [y, z]_{\infty}.
 \end{aligned} \tag{96}$$

This proves (28). $\square$

3.2. Parametrization of Spectral Functions. A parametrization of all spectral functions $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$ is given by the following theorem.

Theorem 16. Assume that $p, r_-, m_- \in \mathbb{N} \cup \{0\}$, $q := r_- + m_- \leq p$, $v := p - q$, $n := p + q$, and $\widetilde{\mathcal{F}}$ is a system of intervals (77). Let $\mathcal{H} := \mathbb{C}^{r_-} \oplus \mathbb{C}^v \oplus \mathbb{C}^q$, let $\psi_j : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^q)$, $j \in \overline{1, 3}$ and $\psi_4 : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^v)$ be operator (matrix) functions given by

$$\begin{aligned}
 \psi_1(\lambda) &= \text{diag}(ie^{-i\lambda b_1}, \dots, ie^{-i\lambda b_{r_-}}, i\sqrt{2}e^{-i\lambda b_{r_-+1}}, \dots, \\
 &\quad i\sqrt{2}e^{-i\lambda b_q})
 \end{aligned} \tag{97}$$

$$\psi_2(\lambda) = \text{diag}\left(-ie^{i\lambda c_1}, \dots, -ie^{i\lambda c_{r_-}}, \underbrace{0, \dots, 0}_{m_- \text{ times}}\right) \tag{98}$$

$$\begin{aligned}
 \psi_3(\lambda) &= \text{diag}\left(ie^{-i\lambda b_1}, \dots, ie^{-i\lambda b_{r_-}}, \frac{i}{\sqrt{2}}e^{-i\lambda b_{r_-+1}}, \dots, \frac{i}{\sqrt{2}} \right. \\
 &\quad \left. \cdot e^{-i\lambda b_q}\right)
 \end{aligned} \tag{99}$$

$$\psi_4(\lambda) = \text{diag}(ie^{-i\lambda b_{q+1}}, \dots, ie^{-i\lambda b_p}), \tag{100}$$

and let

$$\begin{aligned}
 \omega_1(\lambda) &= \frac{1}{\sqrt{2}} \begin{pmatrix} \psi_1(\lambda) & 0 & \psi_2(\lambda) \\ 0 & -i\sqrt{2}(\psi_4(\lambda) - I_v) & 0 \\ iI_q & 0 & -iI_q \end{pmatrix} \\
 &\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^v \oplus \mathbb{C}^q}{\mathbb{C}^n}\right)
 \end{aligned} \tag{101}$$

$$\omega_2(\lambda)$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} i\psi_1(\lambda) & 0 & -i\psi_2(\lambda) \\ 0 & \sqrt{2}(\psi_4(\lambda) + I_v) & 0 \\ I_q & 0 & I_q \end{pmatrix} \tag{102}$$

$$\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^v \oplus \mathbb{C}^q}{\mathbb{C}^n}\right)$$

$$\omega_3(\lambda)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -i\psi_3(\lambda) & 0 & i\psi_2(\lambda) \\ 0 & -\frac{1}{\sqrt{2}}(\psi_4(\lambda) + I_v) & 0 \\ -I_q & 0 & -I_q \end{pmatrix} \tag{103}$$

$$\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^v \oplus \mathbb{C}^q}{\mathbb{C}^n}\right)$$

$$\omega_4(\lambda)$$

$$= \frac{1}{2\sqrt{2}} \begin{pmatrix} \psi_3(\lambda) & 0 & \psi_2(\lambda) \\ 0 & -\frac{i}{\sqrt{2}}(\psi_4(\lambda) - I_v) & 0 \\ iI_q & 0 & -iI_q \end{pmatrix} \tag{104}$$

$$\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^v \oplus \mathbb{C}^q}{\mathbb{C}^n}\right)$$

(in formulas (97)–(104) $\lambda \in \mathbb{C}_+$). Then the equalities

$$\begin{aligned}
 m_\tau(\lambda) &= (C_0(\lambda) \omega_1(\lambda) + C_1(\lambda) \omega_3(\lambda))^{-1} \\
 &\quad \cdot (C_0(\lambda) \omega_2(\lambda) + C_1(\lambda) \omega_4(\lambda)), \quad \lambda \in \mathbb{C}_+
 \end{aligned} \tag{105}$$

$$\sigma_\tau(s) = \lim_{\delta \rightarrow +0} \lim_{y \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \text{Im } m_\tau(x + iy) dx \tag{106}$$

establish a bijective correspondence $\sigma(s) = \sigma_\tau(s)$ between all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \widetilde{R}_{\mathcal{H}}(\mathbb{C}^n)$ and all spectral functions $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$.

Proof. First by using Theorem 10 we parameterize spectral functions of system (20) with J of the form (25) and $H(t)$ defined by (63) and (50).

Let Γ_∞ be the operator defined by (83)–(85) and (26) and let $Y(\cdot, \lambda)$ be the matrizant of system (20). Since $N_+ = n$ (see (78)) and by Lemma 15 Γ_∞ satisfies (27) and (28), the equality $B(\lambda) = \Gamma_\infty Y(\lambda)$, $\lambda \in \mathbb{C}_+$, defines the monodromy matrix $B(\lambda)$. To calculate $B(\lambda)$ we first calculate the operators $D_0(\lambda) = G_0 Y(\lambda) \in \mathbf{B}(\mathbb{C}^n, \mathbb{C}^p)$ and $D_1(\lambda) = G_1 Y(\lambda) \in \mathbf{B}(\mathbb{C}^n, \mathbb{C}^{r_-})$, $\lambda \in \mathbb{C}_+$ (here G_0 and G_1 are operators (80)). Let

$\tilde{u}_j, \tilde{v}_k \in \text{dom } \tilde{T}_{\max}$ be given by (86) and let $\tilde{G}_0 : \text{dom } \tilde{T}_{\max} \rightarrow \mathbb{C}^p$ and $\tilde{G}_1 : \text{dom } \tilde{T}_{\max} \rightarrow \mathbb{C}^{r_-}$ be operators defined by

$$\begin{aligned}\tilde{G}_0 \tilde{y} &= \{[\tilde{y}, \tilde{u}_j]_{\infty}\}_{j=1}^p \in \mathbb{C}^p, \\ \tilde{G}_1 \tilde{y} &= \{[\tilde{y}, \tilde{v}_k]_{\infty}\}_{k=1}^{r_-} \in \mathbb{C}^{r_-}, \\ \tilde{y} &\in \text{dom } \tilde{T}_{\max}.\end{aligned}\quad (107)$$

Moreover, let $\tilde{Y}(\cdot, \lambda)$ be the matrizant (53) of system (49), (50), let $\tilde{Y}(\lambda)$ be the respective operator (22), and let $\tilde{D}_0(\lambda) = \tilde{G}_0 \tilde{Y}(\lambda) \in \mathbf{B}(\mathbb{C}^n, \mathbb{C}^p)$, $\tilde{D}_1(\lambda) = \tilde{G}_1 \tilde{Y}(\lambda) \in \mathbf{B}(\mathbb{C}^n, \mathbb{C}^{r_-})$, $\lambda \in \mathbb{C}_+$. Since $(\tilde{y}_{j+})^\top$ and $(\tilde{y}_{k-})^\top$ (see (55)) are columns of the matrizant (53), it follows that

$$\begin{aligned}\tilde{D}_m(\lambda) &= ((\tilde{G}_m \tilde{y}_{1+})^\top, \dots, (\tilde{G}_m \tilde{y}_{p+})^\top, (\tilde{G}_m \tilde{y}_{1-})^\top, \dots, \\ &(\tilde{G}_m \tilde{y}_{q-})^\top), \quad m \in \{0, 1\}.\end{aligned}\quad (108)$$

Moreover, by (88)

$$\begin{aligned}[\tilde{y}_{j+}, \tilde{u}_l]_{\infty} &= \lim_{t \rightarrow +\infty} (\tilde{Y}_{j+}(t, \lambda), \tilde{u}_l(t)) \\ &= \lim_{t \rightarrow +\infty} (ie^{-i\lambda \Phi_{j+}(t)} e^{\Phi_{l+}(t) - b_l} \delta_{jl}) \\ &= ie^{-i\lambda b_j} \delta_{jl}, \quad j, l \in \overline{1, p}\end{aligned}$$

$$[\tilde{y}_{j+}, \tilde{v}_k]_{\infty} = \lim_{t \rightarrow +\infty} (\tilde{Y}_{j+}(t, \lambda), \tilde{v}_k(t)) = 0, \quad j \in \overline{1, p}, \quad k \in \overline{1, r_-}$$

$$[\tilde{y}_{k-}, \tilde{u}_j]_{\infty} = \lim_{t \rightarrow +\infty} (\tilde{Y}_{k-}(t, \lambda), \tilde{u}_j(t)) = 0, \quad k \in \overline{1, q}, \quad j \in \overline{1, p}$$

$$\begin{aligned}[\tilde{y}_{k-}, \tilde{v}_m]_{\infty} &= \lim_{t \rightarrow +\infty} (\tilde{Y}_{k-}(t, \lambda), \tilde{v}_m(t)) \\ &= \lim_{t \rightarrow +\infty} (-ie^{i\lambda \Phi_{k-}(t)} e^{\Phi_{m-}(t) - c_m} \delta_{km}) \\ &= -ie^{i\lambda c_k} \delta_{km}, \quad k \in \overline{1, q}, \quad m \in \overline{1, r_-}.\end{aligned}\quad (109)$$

And, consequently,

$$\begin{aligned}\tilde{G}_0 \tilde{y}_{j+} &= \{ie^{-i\lambda b_j} \delta_{jl}\}_{l=1}^p, \\ \tilde{G}_1 \tilde{y}_{j+} &= 0, \\ j &\in \overline{1, p}\end{aligned}\quad (110)$$

$$\begin{aligned}\tilde{G}_0 \tilde{y}_{k-} &= 0, \\ \tilde{G}_1 \tilde{y}_{k-} &= \{-ie^{i\lambda c_k} \delta_{km}\}_{m=1}^{r_-}, \\ k &\in \overline{1, q}.\end{aligned}$$

Hence

$$\begin{aligned}\tilde{D}_0(\lambda) &= \begin{pmatrix} S_1(\lambda) & 0 & 0 & 0 & 0 \\ 0 & S_2(\lambda) & 0 & 0 & 0 \\ 0 & 0 & S_3(\lambda) & 0 & 0 \end{pmatrix} : \underbrace{\mathbb{C}^{r_-} \oplus \mathbb{C}^{m_-}}_{\mathbb{C}^q} \oplus \mathbb{C}^y \oplus \underbrace{\mathbb{C}^{r_-} \oplus \mathbb{C}^{m_-}}_{\mathbb{C}^q} \longrightarrow \underbrace{\mathbb{C}^{r_-} \oplus \mathbb{C}^{m_-} \oplus \mathbb{C}^y}_{\mathbb{C}^p}, \\ \tilde{D}_1(\lambda) &= (0, 0, 0, S_4(\lambda), 0) : \underbrace{\mathbb{C}^{r_-} \oplus \mathbb{C}^{m_-}}_{\mathbb{C}^q} \oplus \mathbb{C}^y \oplus \underbrace{\mathbb{C}^{r_-} \oplus \mathbb{C}^{m_-}}_{\mathbb{C}^q} \longrightarrow \mathbb{C}^{r_-},\end{aligned}\quad (111)$$

where

$$\begin{aligned}S_1(\lambda) &= \text{diag}(ie^{-i\lambda b_1}, \dots, ie^{-i\lambda b_{r_-}}) \in \mathbf{B}(\mathbb{C}^{r_-}) \\ S_2(\lambda) &= \text{diag}(ie^{-i\lambda b_{r_+1}}, \dots, ie^{-i\lambda b_q}) \in \mathbf{B}(\mathbb{C}^{m_-}) \\ S_3(\lambda) &= \text{diag}(ie^{-i\lambda b_{q+1}}, \dots, ie^{-i\lambda b_p}) \in \mathbf{B}(\mathbb{C}^y) \\ S_4(\lambda) &= \text{diag}(-ie^{i\lambda c_1}, \dots, -ie^{i\lambda c_{r_-}}) \in \mathbf{B}(\mathbb{C}^{r_-}).\end{aligned}\quad (112)$$

It follows from (87) and (66) that $G_j = \tilde{G}_j U^{-1} \upharpoonright \text{dom } T_{\max}$, $j \in \{0, 1\}$, where U is the unitary operator (65).

This and (67) imply that $D_j(\lambda) = \tilde{D}_j(\lambda)X$, $j \in \{0, 1\}$, and, consequently,

$$\begin{aligned}D_0(\lambda) &= \begin{pmatrix} G_{01}Y(\lambda) \\ G_{02}Y(\lambda) \\ G_{03}Y(\lambda) \end{pmatrix} \\ &= \begin{pmatrix} \frac{1}{\sqrt{2}}S_1(\lambda) & 0 & 0 & \frac{i}{\sqrt{2}}S_1(\lambda) & 0 \\ 0 & \frac{1}{\sqrt{2}}S_2(\lambda) & 0 & 0 & \frac{i}{\sqrt{2}}S_2(\lambda) \\ 0 & 0 & S_3(\lambda) & 0 & 0 \end{pmatrix} \quad (113) \\ D_1(\lambda) &= G_1Y(\lambda) = \begin{pmatrix} \frac{1}{\sqrt{2}}S_4(\lambda) & 0 & 0 & -\frac{i}{\sqrt{2}}S_4(\lambda) & 0 \end{pmatrix}\end{aligned}$$

(here G_{0j} are entries of the block representation (82) of G_0). Therefore by (83)–(85) the monodromy matrix $B(\lambda) = \Gamma_\infty Y(\lambda)$ is

$$B(\lambda) = \begin{pmatrix} B_{11}(\lambda) & 0 & B_{13}(\lambda) \\ 0 & B_{22}(\lambda) & 0 \\ B_{31}(\lambda) & 0 & B_{33}(\lambda) \end{pmatrix} : \mathbb{C}^q \oplus \mathbb{C}^\nu \oplus \mathbb{C}^q \longrightarrow \mathbb{C}^q \oplus \mathbb{C}^\nu \oplus \mathbb{C}^q, \quad (114)$$

where

$$B_{11}(\lambda) = \text{diag}\left(\frac{1}{2}(S_1(\lambda) + S_4(\lambda)), \frac{1}{\sqrt{2}}S_2(\lambda)\right) \quad (115)$$

$$B_{13}(\lambda) = \text{diag}\left(\frac{i}{2}(S_1(\lambda) - S_4(\lambda)), \frac{i}{\sqrt{2}}S_2(\lambda)\right) \quad (116)$$

$$B_{22}(\lambda) = S_3(\lambda), \quad (117)$$

$$B_{31}(\lambda) = \text{diag}\left(-\frac{i}{2}(S_1(\lambda) - S_4(\lambda)), -\frac{i}{2\sqrt{2}}S_2(\lambda)\right) \quad (118)$$

$$B_{33}(\lambda) = \text{diag}\left(\frac{1}{2}(S_1(\lambda) + S_4(\lambda)), \frac{1}{2\sqrt{2}}S_2(\lambda)\right),$$

and the equalities (30)–(33) take the form

$$w_1(\lambda) = \begin{pmatrix} B_{11}(\lambda) & 0 & B_{13}(\lambda) \\ 0 & -i(B_{22}(\lambda) - I_\nu) & 0 \\ 0 & 0 & -I_q \end{pmatrix} \quad (119)$$

$$w_2(\lambda) = \frac{1}{2} \begin{pmatrix} B_{13}(\lambda) & 0 & -B_{11}(\lambda) \\ 0 & B_{22}(\lambda) + I_\nu & 0 \\ I_q & 0 & 0 \end{pmatrix} \quad (120)$$

$$w_3(\lambda) = \begin{pmatrix} B_{31}(\lambda) & 0 & B_{33}(\lambda) \\ 0 & -\frac{1}{2}(B_{22}(\lambda) + I_\nu) & 0 \\ -I_q & 0 & 0 \end{pmatrix} \quad (121)$$

$$w_4(\lambda) = \frac{1}{2} \begin{pmatrix} B_{33}(\lambda) & 0 & -B_{31}(\lambda) \\ 0 & -\frac{i}{2}(B_{22}(\lambda) - I_\nu) & 0 \\ 0 & 0 & -I_q \end{pmatrix}. \quad (122)$$

According to Theorem 10 equalities (34) and (35) with $w_j(\lambda)$ of the form (119)–(122) give a bijective correspondence $\tilde{\sigma}(s) = \tilde{\sigma}_\tau(s)$ between all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \tilde{\mathcal{R}}_{\mathcal{H}}(\mathbb{C}^n)$ and all spectral functions $\tilde{\sigma}(s)$ of system (20). Combining of this statement with Proposition 14 implies that equality (105) with $\omega_j(\lambda) = w_j(\lambda)X^*$, $j \in \overline{1, 4}$, and the equality (106) establish a bijective correspondence $\sigma(s) = \sigma_\tau(s)$ between all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \tilde{\mathcal{R}}_{\mathcal{H}}(\mathbb{C}^n)$ and all spectral functions $\sigma(\cdot) \in SF_n(\mathcal{F})$. Moreover, the immediate calculation shows that $\omega_j(\lambda)$ are of the form (101)–(104). $\square$

In the case $m_- = 0$ spectral functions $\sigma(\cdot) \in SF_n(\mathcal{F})$ can be parameterized in a somewhat other form. Namely, the following theorem is valid.

Theorem 17. Assume that $p, q \in \mathbb{N} \cup \{0\}$, $q \leq p$, $\nu := p - q$, $n = p + q$, and

$$\mathcal{F} = \{(0, b_1), \dots, (0, b_p), (-c_1, 0), \dots, (-c_q, 0)\}, \quad (123)$$

where $b_j, c_j \in (0, \infty)$. Let

$$\begin{aligned} \mathcal{P}_1(\lambda) &= \text{diag}(ie^{-i\lambda b_1}, \dots, ie^{-i\lambda b_p}), \\ \mathcal{P}_2(\lambda) &= \text{diag}(-ie^{i\lambda c_1}, \dots, -ie^{i\lambda c_q}) \end{aligned} \quad (124)$$

$$\mathcal{P}_3(\lambda) = \text{diag}(ie^{-i\lambda b_{q+1}}, \dots, ie^{-i\lambda b_p}), \quad \lambda \in \mathbb{C}_+$$

and let

$$\begin{aligned} Q_1(\lambda) &= \begin{pmatrix} 0 & 0 & i\sqrt{2}\mathcal{P}_2(\lambda) \\ 0 & \frac{1}{2}(\mathcal{P}_3(\lambda) - 3I_\nu) & 0 \\ -\sqrt{2}I_q & 0 & 0 \end{pmatrix} \\ &\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^\nu \oplus \mathbb{C}^q}{\mathbb{C}^n}\right) \end{aligned} \quad (125)$$

$$\begin{aligned} Q_2(\lambda) &= \frac{1}{2} \begin{pmatrix} 0 & 0 & \sqrt{2}\mathcal{P}_2(\lambda) \\ 0 & \frac{i}{2}(\mathcal{P}_3(\lambda) + 3I_\nu) & 0 \\ i\sqrt{2}I_q & 0 & 0 \end{pmatrix} \\ &\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^\nu \oplus \mathbb{C}^q}{\mathbb{C}^n}\right) \end{aligned} \quad (126)$$

$$\begin{aligned} Q_3(\lambda) &= \begin{pmatrix} i\sqrt{2}\mathcal{P}_1(\lambda) & 0 & 0 \\ 0 & \frac{1}{2}(3\mathcal{P}_3(\lambda) - I_\nu) & 0 \\ 0 & 0 & \sqrt{2}I_q \end{pmatrix} \\ &\in \mathbf{B}\left(\frac{\mathbb{C}^q \oplus \mathbb{C}^\nu \oplus \mathbb{C}^q}{\mathbb{C}^n}\right) \end{aligned} \quad (127)$$

$$\begin{aligned}
Q_4(\lambda) &= \frac{1}{2} \begin{pmatrix} -\sqrt{2}\mathcal{P}_1(\lambda) & 0 & 0 \\ 0 & \frac{i}{2}(3\mathcal{P}_3(\lambda) + I_\nu) & 0 \\ 0 & 0 & i\sqrt{2}I_q \end{pmatrix} \\
&\in \mathbf{B} \left(\frac{\mathbb{C}^q \oplus \mathbb{C}^\nu \oplus \mathbb{C}^q}{\mathbb{C}^n} \right)
\end{aligned} \quad (128)$$

(in formulas (125)–(128) $\lambda \in \mathbb{C}_+$). Then the equalities

$$\begin{aligned}
m_K(\lambda) &= (K(\lambda)Q_1(\lambda) + Q_3(\lambda))^{-1} \\
&\cdot (K(\lambda)Q_2(\lambda) + Q_4(\lambda)), \quad \lambda \in \mathbb{C}_+
\end{aligned} \quad (129)$$

$$\sigma_K(s) = \lim_{\delta \rightarrow +0} \lim_{y \rightarrow +0} \frac{1}{\pi} \int_{-\delta}^{s-\delta} \operatorname{Im} m_K(x+iy) dx \quad (130)$$

establish a bijective correspondence $\sigma(s) = \sigma_K(s)$ between all contractive operator-functions $K(\cdot) \in \mathcal{S}(\mathbb{C}^n)$ and all spectral functions $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$.

Proof. Clearly, system of intervals (123) is system (77) with $m_- = 0$ and $r_- = q$. Therefore the operator-functions $\psi_j(\lambda)$ (see (97)–(100)) are $\psi_1(\lambda) = \psi_3(\lambda) = \mathcal{P}_1(\lambda)$, $\psi_2(\lambda) = \mathcal{P}_2(\lambda)$, $\psi_4(\lambda) = \mathcal{P}_3(\lambda)$, and hence

$$\begin{aligned}
Q_1(\lambda) &= i\omega_1(\lambda) + \omega_3(\lambda), \\
Q_2(\lambda) &= i\omega_2(\lambda) + \omega_4(\lambda) \\
Q_3(\lambda) &= i\omega_1(\lambda) - \omega_3(\lambda), \\
Q_4(\lambda) &= i\omega_2(\lambda) - \omega_4(\lambda),
\end{aligned} \quad (131)$$

where $\omega_j(\lambda)$ are given by (101)–(104). Moreover, the subspace $\mathcal{H}$ in Theorem 16 coincides with $\mathbb{C}^n$. Therefore $\widetilde{R}_{\mathcal{H}}(\mathbb{C}^n) = \widetilde{R}(\mathbb{C}^n)$ and according to Proposition 8 the equalities

$$\begin{aligned}
C_0(\lambda) &= i(K(\lambda) + I_n), \\
C_1(\lambda) &= K(\lambda) - I_n,
\end{aligned} \quad (132)$$

$\lambda \in \mathbb{C}_+$

give a bijective correspondence between all operator-functions $K(\cdot) \in \mathcal{S}(\mathbb{C}^n)$ and all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \widetilde{R}(\mathbb{C}^n)$. Substituting (132) into (105) one gets

$$\begin{aligned}
m_K(\lambda) & (= m_\tau(\lambda)) \\
&= (i(K(\lambda) + I_n)\omega_1(\lambda) + (K(\lambda) - I_n)\omega_3(\lambda))^{-1} \\
&\cdot (i(K(\lambda) + I_n)\omega_2(\lambda) + (K(\lambda) - I_n)\omega_4(\lambda)) \\
&= (K(\lambda)Q_1(\lambda) + Q_3(\lambda))^{-1} \\
&\cdot (K(\lambda)Q_2(\lambda) + Q_4(\lambda)).
\end{aligned} \quad (133)$$

Thus (105) admits the representation (129), which in view of Theorem 16 yields the required statement. $\square$

3.3. Particular Cases. In this subsection by using Theorems 16 and 17 we obtain parametrization of spectral functions $\sigma(\cdot) \in SF_n(\widetilde{\mathcal{F}})$ in certain particular cases.

Corollary 18. Assume that $p \in \mathbb{N}$,

$$\widetilde{\mathcal{F}} = \{(0, b_1), \dots, (0, b_p)\} \quad (134)$$

($b_j \in (0, \infty)$) and

$$\mathcal{Q}(\lambda) = \operatorname{diag}(e^{-i\lambda b_1}, \dots, e^{-i\lambda b_p}), \quad \lambda \in \mathbb{C}_+. \quad (135)$$

Then the equality

$$m_K(\lambda) = \frac{i}{2} (\mathcal{Q}(\lambda) - K(\lambda))^{-1} (\mathcal{Q}(\lambda) + K(\lambda)), \quad \lambda \in \mathbb{C}_+ \quad (136)$$

together with (130) gives a bijective correspondence $\sigma(s) = \sigma_K(s)$ between all operator-functions $K(\cdot) \in \mathcal{S}(\mathbb{C}^p)$ and all spectral functions $\sigma(\cdot) \in SF_p(\widetilde{\mathcal{F}})$.

Proof. Clearly, system of intervals (134) is system (123) with $q = 0$, $n = p$, and hence the corresponding operator-functions $Q_j(\lambda)$ (see (125)–(128)) are

$$Q_1(\lambda) = \frac{1}{2} (i\mathcal{Q}(\lambda) - 3I_p), \quad (137)$$

$$Q_2(\lambda) = \frac{i}{4} (i\mathcal{Q}(\lambda) + 3I_p)$$

$$Q_3(\lambda) = \frac{1}{2} (3i\mathcal{Q}(\lambda) - I_p), \quad (138)$$

$$Q_4(\lambda) = \frac{i}{4} (3i\mathcal{Q}(\lambda) + I_p).$$

Let $N(\cdot) \in \mathcal{S}(\mathbb{C}^p)$ and let $m(\lambda) = m_N(\lambda)$ be given by (129) with $N(\lambda)$ instead of $K(\lambda)$. It follows from (137) and (138) that

$$\begin{aligned}
&N(\lambda)Q_1(\lambda) + Q_3(\lambda) \\
&= \frac{i}{2} (N(\lambda) + 3I_p) (\mathcal{Q}(\lambda) - K(\lambda)) \\
&N(\lambda)Q_2(\lambda) + Q_4(\lambda) \\
&= -\frac{1}{4} (N(\lambda) + 3I_p) (\mathcal{Q}(\lambda) + K(\lambda)),
\end{aligned} \quad (139)$$

where

$$\begin{aligned}
K(\lambda) &= -i(N(\lambda) + 3I_p)^{-1} (3N(\lambda) + I_p) \\
&= -i(3N(\lambda) + I_p)(N(\lambda) + 3I_p)^{-1}.
\end{aligned} \quad (140)$$

Therefore $m(\lambda) = m_K(\lambda)$, where $m_K(\lambda)$ is given by (136) with $K(\lambda)$ of the form (140).

Let X and $\widehat{J}$ be operators in $\mathbb{C}^p \oplus \mathbb{C}^p$ defined by

$$\begin{aligned}
X &= \frac{1}{\sqrt{8}} \begin{pmatrix} 3I_p & I_p \\ I_p & 3I_p \end{pmatrix}, \\
\widehat{J} &= \begin{pmatrix} I_p & 0 \\ 0 & -I_p \end{pmatrix}.
\end{aligned} \quad (141)$$

Then $X^* \hat{J} X = \hat{J}$ and according to [14] equality (140) gives a bijection $N(\lambda) \rightarrow K(\lambda)$ of the set $\mathcal{S}(\mathbb{C}^p)$ onto itself. This and Theorem 17 yield the desired statement. $\square$

Corollary 19. Assume that $p \in \mathbb{N}$, $n = 2p$, and

$$\mathcal{F} = \{(0, b_1), \dots, (0, b_p), (-c_1, 0), \dots, (-c_p, 0)\} \quad (142)$$

($b_j, c_j \in (0, \infty)$). Let $\mathcal{P}_1(\lambda)$ and $\mathcal{P}_2(\lambda)$ be given by (124) and let $\mathcal{Q}_j : \mathbb{C}_+ \rightarrow \mathbf{B}(\mathbb{C}^p \oplus \mathbb{C}^p)$ be the operator-functions defined by the block representations

$$\begin{aligned} \mathcal{Q}_1(\lambda) &= \begin{pmatrix} 0 & i\sqrt{2}\mathcal{P}_2(\lambda) \\ -\sqrt{2}I_p & 0 \end{pmatrix}, \\ \mathcal{Q}_2(\lambda) &= \begin{pmatrix} i\sqrt{2}\mathcal{P}_1(\lambda) & 0 \\ 0 & \sqrt{2}I_p \end{pmatrix}, \end{aligned} \quad (143)$$

$\lambda \in \mathbb{C}_+.$

Then statement of Theorem 17 holds with the equality

$$\begin{aligned} m_K(\lambda) &= -\frac{i}{2} (K(\lambda) \mathcal{Q}_1(\lambda) + \mathcal{Q}_2(\lambda))^{-1} \\ &\quad \cdot (K(\lambda) \mathcal{Q}_1(\lambda) - \mathcal{Q}_2(\lambda)), \quad \lambda \in \mathbb{C}_+ \end{aligned} \quad (144)$$

instead of (129).

Proof. Since system of intervals (142) is system (123) with $q = p$, the operator-functions $Q_j(\lambda)$ (see (125)–(128)) are

$$\begin{aligned} Q_1(\lambda) &= \mathcal{Q}_1(\lambda), \\ Q_2(\lambda) &= -\frac{i}{2} \mathcal{Q}_1(\lambda), \\ Q_3(\lambda) &= \mathcal{Q}_2(\lambda), \\ Q_4(\lambda) &= \frac{i}{2} \mathcal{Q}_2(\lambda). \end{aligned} \quad (145)$$

Therefore equality (129) admits the representation (144), which proves the required statement. $\square$

Proposition 20. Assume that $p \in \mathbb{N}$, $n = 2p$,

$$\mathcal{F} = \left\{ (0, b_1), \dots, (0, b_p), \underbrace{\mathbb{R}_-, \dots, \mathbb{R}_-}_{p \text{ times}} \right\} \quad (146)$$

($b_j \in (0, \infty)$), and $\mathcal{Q}(\lambda)$ is given by (135). Then the equalities

$$m_{K_1}(\lambda) = \frac{i}{2} (\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} (\mathcal{Q}(\lambda) + K_1(\lambda)) \quad (147)$$

$$N_K(\lambda) = K_2(\lambda) (\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} \mathcal{Q}(\lambda) \quad (148)$$

$$\begin{aligned} m_K(\lambda) &= \begin{pmatrix} m_{K_1}(\lambda) & 0 \\ N_K(\lambda) & \frac{i}{2} I_p \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^p \rightarrow \\ &\quad \mathbb{C}^p \oplus \mathbb{C}^p, \quad \lambda \in \mathbb{C}_+ \end{aligned} \quad (149)$$

together with (130) establish a bijective correspondence $\sigma(s) = \sigma_K(s)$ between all operator-functions $K(\cdot) \in \mathcal{S}(\mathbb{C}^p, \mathbb{C}^n)$ with the block representation $K(\lambda) = (K_1(\lambda), K_2(\lambda))^T : \mathbb{C}^p \rightarrow \mathbb{C}^p \oplus \mathbb{C}^p$, $\lambda \in \mathbb{C}_+$, and all spectral functions $\sigma(\cdot) \in SF_n(\mathcal{F})$.

Proof. Clearly, system of intervals (146) is system (77) with $r_- = 0$ and $q = m_- = p$. Therefore for this system Theorem 16 holds with $\psi_1(\lambda) = i\sqrt{2}\mathcal{Q}(\lambda)$, $\psi_2(\lambda) = 0$, $\psi_3(\lambda) = (i/\sqrt{2})\mathcal{Q}(\lambda)$, and

$$\omega_1(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} i\sqrt{2}\mathcal{Q}(\lambda) & 0 \\ iI_p & -iI_p \end{pmatrix}, \quad (150)$$

$$\omega_2(\lambda) = \frac{1}{2\sqrt{2}} \begin{pmatrix} -\sqrt{2}\mathcal{Q}(\lambda) & 0 \\ I_p & I_p \end{pmatrix}$$

$$\omega_3(\lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}}\mathcal{Q}(\lambda) & 0 \\ -I_p & -I_p \end{pmatrix}, \quad (151)$$

$$\omega_4(\lambda) = \frac{1}{2\sqrt{2}} \begin{pmatrix} \frac{i}{\sqrt{2}}\mathcal{Q}(\lambda) & 0 \\ iI_p & -iI_p \end{pmatrix},$$

where $\omega_j(\lambda) \in \mathbf{B}(\mathbb{C}^p \oplus \mathbb{C}^p)$. Moreover, the subspace $\mathcal{H}$ in Theorem 16 is $\mathcal{H} = \{0\} \oplus \mathbb{C}^p$ and according to Proposition 8 the equalities

$$C_0(\lambda) = \begin{pmatrix} \frac{1}{\sqrt{2}}I_p & -K_1(\lambda) \\ 0 & K_2(\lambda) + iI_p \end{pmatrix}, \quad (152)$$

$$C_1(\lambda) = \begin{pmatrix} i\sqrt{2}I_p & iK_1(\lambda) \\ 0 & -iK_2(\lambda) - I_p \end{pmatrix}$$

give a bijective correspondence between all operator-functions $K(\cdot) \in \mathcal{S}(\mathbb{C}^p, \mathbb{C}^n)$ with the block representation $K(\lambda) = (K_1(\lambda), K_2(\lambda))^T : \mathbb{C}^p \rightarrow \mathbb{C}^p \oplus \mathbb{C}^p$ and all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \tilde{R}_{\{0\} \oplus \mathbb{C}^p}(\mathbb{C}^n)$.

It follows from (152) and (150), (151) that

$$\begin{aligned} &(C_0(\lambda) \omega_1(\lambda) + C_1(\lambda) \omega_3(\lambda))^{-1} \\ &= \left[\sqrt{2} \begin{pmatrix} i(\mathcal{Q}(\lambda) - K_1(\lambda)) & 0 \\ iK_2(\lambda) & I_p \end{pmatrix} \right]^{-1} \\ &= \frac{1}{\sqrt{2}} \begin{pmatrix} -i(\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} & 0 \\ -K_2(\lambda) (\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} & I_p \end{pmatrix} \end{aligned} \quad (153)$$

$$C_0(\lambda) \omega_2(\lambda) + C_1(\lambda) \omega_4(\lambda)$$

$$= \frac{1}{\sqrt{2}} \begin{pmatrix} -(\mathcal{Q}(\lambda) + K_1(\lambda)) & 0 \\ K_2(\lambda) & iI_p \end{pmatrix}$$

and substitution of these equalities into (105) gives

$$m_K(\lambda) := m_\tau(\lambda) = \begin{pmatrix} m_{K_1}(\lambda) & 0 \\ N_K(\lambda) & \frac{i}{2} I_p \end{pmatrix}, \quad (154)$$

where $m_{K_1}(\lambda)$ is of the form (147) and

$$\begin{aligned} N_K(\lambda) &= \frac{1}{2} K_2(\lambda) \\ &\cdot \left[(\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} (\mathcal{Q}(\lambda) + K_1(\lambda)) + I_p \right] = \frac{1}{2} \\ &\cdot K_2(\lambda) (\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} \\ &\times (\mathcal{Q}(\lambda) + K_1(\lambda) + \mathcal{Q}(\lambda) - K_1(\lambda)) = K_2(\lambda) \\ &\cdot (\mathcal{Q}(\lambda) - K_1(\lambda))^{-1} \mathcal{Q}(\lambda). \end{aligned} \quad (155)$$

Thus $m_K(\lambda)$ is of the form (149), which in view of Theorem 16 proves the required statement. $\square$

Theorem 21. Assume that $p \in \mathbb{N}$ and $\widetilde{\mathcal{F}}$ is a system of intervals given by

$$\widetilde{\mathcal{F}} = \left\{ \underbrace{\mathbb{R}_+, \dots, \mathbb{R}_+}_{p \text{ times}}, \underbrace{\mathbb{R}_-, \dots, \mathbb{R}_-}_{p \text{ times}} \right\}. \quad (156)$$

Then the equalities

$$\begin{aligned} \Sigma_F(x) &= \frac{1}{2\pi} \begin{pmatrix} I_p & F^*(x) \\ F(x) & I_p \end{pmatrix}, \quad x \in \mathbb{R}, \text{ and} \\ \sigma_F(s) &= \int_0^s \Sigma_F(x) dx \end{aligned} \quad (157)$$

give a bijective correspondence between all operator-functions $F \in \mathcal{S}_{\mathbb{R}}(\mathbb{C}^p)$ and all spectral functions $\sigma(\cdot) \in SF_n(\mathcal{F})$.

Proof. Clearly, system of intervals (156) is system (48) with $r_+ = r_- = 0$ and $m_+ = m_- = q = p$. Therefore $n = 2p$, $\nu(= p - q) = 0$, and (62) takes the form

$$\begin{aligned} X &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_p & iI_p \\ I_p & -iI_p \end{pmatrix}, \\ X^* &= \frac{1}{\sqrt{2}} \begin{pmatrix} I_p & I_p \\ -iI_p & iI_p \end{pmatrix}. \end{aligned} \quad (158)$$

Consider canonical system (20) associated with $\widetilde{\mathcal{F}}$ (recall that J and $H(t)$ in this system are given by (25) and (63), respectively). Clearly, $J = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}$; that is, system (20) is a Hamiltonian system. Moreover, by (68) deficiency indices $N_{\pm}$ of this system are $N_+ = N_- = p$. Therefore according to [15] the equalities

$$\begin{aligned} \Gamma_0 y &= -y_1(0), \\ \Gamma_1 y &= y_0(0), \end{aligned} \quad (159)$$

$$y \in \text{dom } T_{\max}$$

define a boundary triplet $\Pi = \{\mathbb{C}^p, \Gamma_0, \Gamma_1\}$ for $T_{\max}$ (in (159) $y_0(t)$ and $y_1(t)$ are components of the representation $y(t) = \{y_0(t), y_1(t)\} \in \mathbb{C}^p \oplus \mathbb{C}^p$, $y \in \text{dom } T_{\max}$).

Let $M(\cdot)$ be the Weyl function of the triplet Π . Since $T_{\min}$ is a densely defined operator, it follows from [16, Corollary 4.8] that the equality

$$\widehat{m}_{\tau}(\lambda) = \begin{pmatrix} M(\lambda) (C_0(\lambda) - C_1(\lambda) M(\lambda))^{-1} C_0(\lambda) & -\frac{1}{2} I_p - M(\lambda) (C_0(\lambda) - C_1(\lambda) M(\lambda))^{-1} C_1(\lambda) \\ \frac{1}{2} I_p - (C_0(\lambda) - C_1(\lambda) M(\lambda))^{-1} C_0(\lambda) & (C_0(\lambda) - C_1(\lambda) M(\lambda))^{-1} C_1(\lambda) \end{pmatrix} \quad (160)$$

($\lambda \in \mathbb{C}_+$) together with (35) gives a bijective correspondence $\widehat{\sigma}(s) = \widehat{\sigma}_{\tau}(s)$ between all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \widetilde{R}(\mathbb{C}^p)$ and all spectral functions $\widehat{\sigma}(\cdot)$ of system (20). Therefore by Proposition 14 the equality

$$m_{\tau}(\lambda) = X \widehat{m}_{\tau}(\lambda) X^* \quad (161)$$

together with (106) gives a bijective correspondence $\sigma(s) = \sigma_{\tau}(s)$ between all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \widetilde{R}(\mathbb{C}^p)$ and all spectral functions $\sigma(\cdot) \in SF_n(\mathcal{F})$.

Let us calculate $M(\lambda)$. Assume that $\widetilde{y}_{k-} = \widetilde{y}_{k-}(\cdot, \lambda)$, $k \in \overline{1, p}$, $\lambda \in \mathbb{C}_+$, be vector-functions defined by the second equality in (55) and let $y_{k-} = y_{k-}(t, \lambda) := X^* \widetilde{y}_{k-}(t, \lambda)$ (this means that $y_{k-} = U \widetilde{y}_{k-}$, where U is the unitary operator (65)). It follows from (57) that $\{y_{k-}\}_{k=1}^p$ is a basis in $\mathfrak{N}_{\lambda}(T_{\min})$. Let $Z(\cdot, \lambda) \in \mathbf{B}(\mathbb{C}^p, \mathbb{C}^n)$ be the operator solution of system (20) given by

$$\begin{aligned} Z(t, \lambda) &= (y_{1-}(t, \lambda), y_{2-}(t, \lambda), \dots, y_{p-}(t, \lambda)), \\ &\lambda \in \mathbb{C}_+. \end{aligned} \quad (162)$$

Clearly,

$$Z(0, \lambda) = \frac{1}{\sqrt{2}} \begin{pmatrix} I_p \\ iI_p \end{pmatrix} : \mathbb{C}^p \longrightarrow \mathbb{C}^p \oplus \mathbb{C}^p \quad (163)$$

and the equality $(Z(\lambda)x)(t) = Z(t, \lambda)x$, $x \in \mathbb{C}^p$, defines a bijective linear mapping $Z(\lambda)$ from $\mathbb{C}^p$ onto $\mathfrak{N}_{\lambda}(T_{\min})$. It follows from (159) and (163) that

$$\begin{aligned} \Gamma_0 Z(\lambda) &= -\frac{i}{\sqrt{2}} I_p, \\ \Gamma_1 Z(\lambda) &= \frac{1}{\sqrt{2}} I_p, \end{aligned} \quad (164)$$

$\lambda \in \mathbb{C}_+.$

Hence $\Gamma_1 Z(\lambda) = i\Gamma_0 Z(\lambda)$, $\lambda \in \mathbb{C}_+$, and by definition (18) of the Weyl function one has

$$M(\lambda) = iI_p, \quad \lambda \in \mathbb{C}_+. \quad (165)$$

Moreover, according to Proposition 8 the equalities

$$\begin{aligned} C_0(\lambda) &= i(K(\lambda) + I_p), \\ C_1(\lambda) &= K(\lambda) - I_p, \end{aligned} \quad (166)$$

$$\lambda \in \mathbb{C}_+$$

give a bijective correspondence between all operator-functions $K \in \mathcal{S}(\mathbb{C}^p)$ and all pairs $\tau = \{C_0(\cdot), C_1(\cdot)\} \in \tilde{\mathcal{R}}(\mathbb{C}^p)$. Substituting (165) and (166) into (160) one obtains

$$\begin{aligned} \widehat{m}_K(\lambda) &:= \widehat{m}_\tau(\lambda) \\ &= \frac{1}{2} \begin{pmatrix} i(K(\lambda) + I_p) & -K(\lambda) \\ -K(\lambda) & -i(K(\lambda) - I_p) \end{pmatrix} \end{aligned} \quad (167)$$

and the immediate calculation with taking (158) and (161) into account gives

$$m_K(\lambda) := m_\tau(\lambda) = \frac{i}{2} \begin{pmatrix} I_p & 0 \\ 2K(\lambda) & I_p \end{pmatrix}. \quad (168)$$

Thus equalities (168) and (130) give a bijective correspondence $\sigma(s) = \sigma_K(s)$ between all operator-functions $K \in \mathcal{S}(\mathbb{C}^p)$ and all spectral functions $\sigma(\cdot) \in SF_n(\tilde{\mathcal{F}})$. Next, in view of (168) one has

$$\begin{aligned} \operatorname{Im} m_K(x + iy) &= \frac{1}{2} \begin{pmatrix} I_p & K^*(x + iy) \\ K(x + iy) & I_p \end{pmatrix}, \\ x &\in \mathbb{R}, \quad y > 0. \end{aligned} \quad (169)$$

Since for each finite interval $\Delta \subset \mathbb{R}$ the function $K(x + iy)$ is bounded on $\Delta \times \mathbb{R}$, it follows from the Lebesgue theorem that $\sigma_K(s)$ admits the representation $\sigma_K(s) = \int_{[0,s]} \Sigma_F(x) dx$, where $\Sigma_F(x)$ is given by the first equality in (157) with $F(x) = \lim_{y \rightarrow +0} K(x + iy)$. Letting now $\sigma_F(s) = \sigma_K(s)$, we obtain the required statement. $\square$

Corollary 22. Let either $\tilde{\mathcal{F}} = \underbrace{\{\mathbb{R}_+, \dots, \mathbb{R}_+\}}_{p \text{ times}}$ or $\tilde{\mathcal{F}} = \underbrace{\{\mathbb{R}, \dots, \mathbb{R}\}}_{p \text{ times}}$. Then the set $SF_p(\tilde{\mathcal{F}})$ consists of the unique spectral function $\sigma(s) = (1/2\pi)sI_p$, $s \in \mathbb{R}$.

Proof. First assume that $\tilde{\mathcal{F}} = \underbrace{\{\mathbb{R}_+, \dots, \mathbb{R}_+\}}_{p \text{ times}}$. Let $\tilde{\mathcal{F}}_- = \underbrace{\{\mathbb{R}_-, \dots, \mathbb{R}_-\}}_{p \text{ times}}$, let $\tilde{\mathcal{F}}' = \tilde{\mathcal{F}} \cup \tilde{\mathcal{F}}_-$ (that is, $\tilde{\mathcal{F}}'$ is given by the right hand side of (156)), and let in (157) $F(x) = 0$. Then by Theorem 21 the equality $\tilde{\sigma}(s) = \operatorname{diag}(\sigma(s), \sigma(s))$ defines a spectral function $\tilde{\sigma}(\cdot) \in SF_{2p}(\tilde{\mathcal{F}}')$. This implies that $\sigma(\cdot) \in SF_p(\tilde{\mathcal{F}}) \cap SF_p(\tilde{\mathcal{F}}_-)$.

Conversely, let $\sigma_1(\cdot) \in SF_p(\tilde{\mathcal{F}})$. Then the equality $\tilde{\sigma}_1(s) = \operatorname{diag}(\sigma_1(s), \sigma(s))$ defines a spectral function $\tilde{\sigma}_1(\cdot) \in SF_{2p}(\tilde{\mathcal{F}}')$. Clearly, $\tilde{\sigma}_1(\cdot) = \sigma_F(\cdot)$ with $F(x) = 0$ (see (157)) and hence $\sigma_1(s) = \sigma(s)$. This proves the required statement for $\tilde{\mathcal{F}} = \{\mathbb{R}_+, \dots, \mathbb{R}_+\}$. In the case $\tilde{\mathcal{F}} = \{\mathbb{R}, \dots, \mathbb{R}\}$ the statement of the corollary is implied by the Plancherel theorem and the obvious inclusion $SF_p(\mathbb{R}, \dots, \mathbb{R}) \subset SF_p(\mathbb{R}_+, \dots, \mathbb{R}_+)$. $\square$

Remark 23. The fact that $\sigma(s) = (1/2\pi)sI_p$ is a spectral function of the classes $SF_p(\mathbb{R}_+, \dots, \mathbb{R}_+)$ and $SF_p(\mathbb{R}, \dots, \mathbb{R})$ directly follows from the Plancherel theorem. Corollary 22 claims that there are no other spectral functions in these classes.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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