Research Article

Existence of Solutions for Singular Second-Order Ordinary Differential Equations with Periodic and Deviated Nonlocal Multipoint Boundary Conditions

Ahmed M. A. El-Sayed ¹ and Fatma M. Gaafar ²

¹Faculty of Science, Alexandria University, Alexandria, Egypt

²Faculty of Science, Damanhour University, El Beheira, Egypt

Correspondence should be addressed to Fatma M. Gaafar; fatmagaafar2@yahoo.com

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This paper studies the existence of continuous solutions for a class of nonlinear singular second-order ordinary differential equations subject to one of the following boundary conditions: periodic-deviated multipoint boundary conditions, periodic-integral boundary conditions, and periodic-nonlocal integral conditions in the Riemann-Stieltjes sense. An existence result based on the Schauder fixed point theorem and Leray-Schauder continuation principle is used to obtain at least one continuous solution for the singular second-order ordinary differential problems. Two examples are given to show the application of our results.

1. Introduction

In this paper, we study the existence of at least one continuous solution for the second-order differential equation

$$x'' = f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1), \quad (1)$$

where $f : [0, 1] \times R^2 \rightarrow R$ is a given function satisfying Carathéodory's conditions (i.e., for a.e. $t \in [0, 1]$ the function $f(t, \cdot, \cdot)$ is continuous on R^2 and for each $(u, v) \in R^2$, the function $f(\cdot, u, v)$ is measurable on $[0, 1]$) and is allowed to be singular at $t = 0$ and $t = 1$.

We consider the following nonlocal and multipoint boundary conditions

periodic-deviated multipoint boundary conditions of the form:

$$\begin{aligned} x(0) &= x(1), \\ \sum_{i=1}^m a_i x(\phi(\xi_i)) &= 0, \end{aligned} \quad (2)$$

periodic and integral boundary conditions:

$$\begin{aligned} x(0) &= x(1), \\ \int_0^1 x(\phi(\theta)) d\theta &= 0, \end{aligned} \quad (3)$$

and periodic and nonlocal integral conditions:

$$\begin{aligned} x(0) &= x(1), \\ \int_c^d x(\theta) d\eta(\theta) &= 0, \end{aligned} \quad (4)$$

$0 < c < d < 1$

where $\phi : [0, 1] \rightarrow [0, 1]$ is a deviated continuous given function, i.e., $\phi(\xi_i) \leq \xi_i$, $\xi_i \in [0, 1]$, $a_i \in R$ with $\sum_{i=1}^m a_i \neq 0$, $i = 1, 2, 3, \dots, m$, and $\eta(\theta)$ is an increasing function and the integral is meant in the Riemann-Stieltjes sense for $0 < c < d < 1$.

For the function $\phi(t)$ we can take

- (i) $\phi(t) = k \sin t$, $t \in [0, \pi/2]$, where $0 < k \leq 1$.
- (ii) $\phi(t) = kt^\gamma$, $t \in [0, 1]$, $k \in (0, 1]$, and $\gamma \geq 1$.

The considered problem is to determine sufficient conditions on f guaranteeing that the boundary value problem (1), (2) or (1), (3) or (1), (4) has a solution.

Note. One must notice that the following periodic and multipoint boundary conditions are special cases of our periodic and multipoint boundary conditions:

$$\begin{aligned} x(0) &= x(1), \quad \sum_{i=1}^m a_i x(\xi_i) = 0, \quad \xi_i \in (0, 1). \\ x(0) &= x(1), \quad x(\xi) = 0, \quad \xi \in (0, 1). \\ x(0) &= x(1), \quad x(\xi) = -x(\eta), \quad \xi, \eta \in (0, 1). \end{aligned} \quad (5)$$

We list the following assumptions which we use in this paper:

(H_0) $a_i \in \mathbb{R}$, $\sum_{i=1}^m a_i \neq 0$ and $\phi(\xi_i) \leq \xi_i$, where $0 < \xi_1 < \xi_2 < \dots < \xi_m < 1$.

(H_1) There exists $p(t), r(t) \in L^1_{loc}(0, 1)$ with $t(1 - t)p(t), t(1 - t)r(t), q(t) \in L^1[0, 1]$ such that f has a growth condition of the form

$$\begin{aligned} |f(t, u, v)| &\leq p(t)|u|^{\sigma_1} + q(t)|v|^{\sigma_2} + r(t), \\ \text{a.e. } t \in [0, 1], \quad (u, v) \in \mathbb{R}^2 \text{ and } 0 < \sigma_1, \sigma_2 < 1, \end{aligned} \quad (6)$$

where $L^1_{loc}(0, 1) = \{x \mid x \in L^1[a, b] \text{ for every closed interval } [a, b] \subset (0, 1)\}$.

We point out that our results can not be extended to the cases in [1, 2]. On the one hand the cases σ_1 and $\sigma_2 = 1$ which are used in them are not included in our hypothesis and on the other hand periodic and deviated multipoint boundary conditions are different from boundary conditions of them. Hence we consider a new class of assumptions and periodic and deviated multipoint boundary conditions.

The investigation of multipoint boundary value problems was started by Il'in and Moiseev [3]. From that time, many researchers investigate wide-ranging nonlinear multipoint nonlocal boundary value problems (see Chen et al. [4], Guo et al. [5], Ma [6], Webb [7], Webb and Zima [8], and Zhang and Liu [9] and the references cited therein).

The singular ordinary differential equations emerge in the fields of the theory of boundary layer, Newtonian fluid mechanics, gas dynamics, and so on. The theory of singular boundary value problems has turned into an essential region of examination during the last few decades; one may refer to Agarwal and O'Regan [10], Agarwal et al. [11], Cheng [12], Cui and Zou [13], Du and Zhao [14], Erbe and Mathsen [15], Feng and Ge [16], Han and Wu [17], Jiang et al. [18], Kiguradze [19], Li et al. [20], Li and Yan [21], B. Liu et al. [22], L. S. Liu et al. [23], Liua [24], Ma and Thompson [2], Moshinsky [25], Pandey and Barnwal [26], Sun and Zhang [27], Suna [28], Timoshenko [29], Wei [30], Yan et al. [31], Yaoa and Lina [32], Zhao [33], Zhang and Sun [34], and Zhang and Feng [35], and the references therein.

The existence of solutions of ordinary differential equation with integral and periodic boundary conditions has

been widely considered in recent years; they constitute very important and interesting problems because they have various applications in thermoelasticity, chemical engineering, population dynamics, and underground water flow and include nonlocal and multipoint boundary conditions (see, for example, Feng et al. [36], Feng and Cong [37], Hua et al. [38], Jiang et al. [18], Nanware and Dhaigude [39], Song et al. [40], Staněk [41], Webb and Infante [42], Yan et al. [31], Yang [43], Yao [44], Zhang and Ge [45], and Zhang and Xu [46], and the references therein).

2. Preliminary Lemmas

We shall use the classical Banach spaces $L^1[0, 1]$, $C[0, 1]$, and $C^k[0, 1]$. We denote by $AC[0, 1]$ the space of absolute continuous functions on $[0, 1]$ and denoted $AC^k[a, b] = \{x \in C^k[a, b] \mid x^{(k)} \in AC[a, b]\}$. Let $AC_{loc}(0, 1)$ be the space consisting of functions that are absolutely continuous on every closed interval $[a, b] \subset (0, 1)$.

Let $\mathcal{E}$ be the Banach space

$$\mathcal{E} = \{y \in L^1_{loc}(0, 1) \mid t(1 - t)y(t) \in L^1[0, 1]\}, \quad (7)$$

with the norm

$$\|y\|_{\mathcal{E}} = \int_0^1 \tau(1 - \tau)|y(\tau)|d\tau < \infty. \quad (8)$$

Let X be the Banach space

$$\begin{aligned} X = \left\{ x \in C^1(0, 1) \mid x \in C[0, 1], \lim_{t \rightarrow 1} (1 - t) \right. \\ \left. \cdot x'(t) \text{ and } \lim_{t \rightarrow 0} tx'(t) \text{ exist} \right\} \end{aligned} \quad (9)$$

equipped with the norm

$$\|x\|_X = \|x\| + \|x\|_1, \quad (10)$$

where $\|x\| = \max_{t \in [0, 1]} |x(t)|$ and $\|x\|_1 = \max_{t \in [0, 1]} |t(1 - t)x'(t)|$.

For convenience we set the following notation: $\mathcal{A} = 1 + \sum_{i=1}^m |a_i| / |\sum_{i=1}^m a_i|$.

Lemma 1 (see [47]). *Let $y \in \mathcal{E}$. Then*

- (i) $\int_0^t \tau y(\tau) d\tau \int_t^1 (1 - \tau)y(\tau) d\tau \in L^1[0, 1]$, and

$$\begin{aligned} \int_0^1 \int_0^t \tau y(\tau) d\tau dt &= \int_0^1 \int_t^1 (1 - \tau)y(\tau) d\tau dt \\ &= \int_0^1 \tau(1 - \tau)y(\tau) d\tau \end{aligned} \quad (11)$$
- (ii) $\lim_{t \rightarrow 0} t \int_t^1 (1 - \tau)y(\tau) d\tau = 0$, $\lim_{t \rightarrow 1} (1 - t) \int_0^t \tau y(\tau) d\tau = 0$.

Theorem 2 (Schauder's fixed point theorem). *Let K be a closed convex subset of a Banach space X . If $N : K \rightarrow K$ is continuous and $N(K)$ is relatively compact, then N has a fixed point in K .*

Theorem 3 (Leray-Schauder continuation principle; see, e.g., [48]). *Let X be a Banach space and $N : X \rightarrow X$ be a compact map. Assume that there exists $D > 0$ so that if $u = \lambda Nu$ for $\lambda \in (0, 1)$, then $\|u\|_X \leq D$. Then $u = Nu$ is solvable.*

The following lemmas play a pivotal role to define the solutions for the given problems.

Lemma 4. *The singular second-order ordinary differential equation with periodic and deviated multipoint boundary conditions (1)-(2) is equivalent to the functional integral equation:*

$$\begin{aligned} x(t) = & -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\ & - t \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\ & + \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\ & \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\ & \cdot \left. \int_{\phi(\xi_i)}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right]. \end{aligned} \quad (12)$$

Proof. The solution of (1) is given by

$$\begin{aligned} x(t) = & x(0) + tx'(0) \\ & + \int_0^t (t-\tau) f(\tau, x(\tau), x'(\tau)) d\tau. \end{aligned} \quad (13)$$

Condition $x(0) = x(1)$ gives

$$x'(0) = - \int_0^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau, \quad (14)$$

and we have

$$\begin{aligned} x(t) = & x(0) - t \int_0^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\ & + \int_0^t (t-\tau) f(\tau, x(\tau), x'(\tau)) d\tau, \\ = & x(0) - (1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\ & - t \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau. \end{aligned} \quad (15)$$

Also condition $\sum_{i=1}^m a_i x(\phi(\xi_i)) = 0$ gives

$$\begin{aligned} x(0) = & \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\ & \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\ & \cdot \left. \int_{\phi(\xi_i)}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right], \end{aligned} \quad (16)$$

and substituting it in (15) we get (12).

Conversely, (cf. (12)) $x(t) \in AC_{loc}(0, 1)$. Moreover

$$\begin{aligned} x'(t) = & \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\ & - \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau. \end{aligned} \quad (17)$$

Integrating $x'(t)$ from 0 to t ($t \in (0, 1)$) and interchanging the order of integration to get

$$\begin{aligned} x(t) - x(0) = & \int_0^t x'(\rho) d\rho \\ = & \int_0^t \int_0^\rho \tau f(\tau, x(\tau), x'(\tau)) d\tau d\rho \\ & - \int_0^t \int_\rho^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau d\rho, \\ = & -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau - t \int_t^1 (1-\tau) \\ & \cdot f(\tau, x(\tau), x'(\tau)) d\tau, \\ = & x(t) - \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\ & \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau - \sum_{i=1}^m a_i \phi(\xi_i) \\ & \cdot \left. \int_{\phi(\xi_i)}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right], \end{aligned} \quad (18)$$

hence we get

$$\begin{aligned} x(0) = & \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\ & \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\ & \cdot \left. \int_{\phi(\xi_i)}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right]. \end{aligned} \quad (19)$$

Similarly integrate $x'(t)$ from t ($t \in (0, 1)$) to 1 and interchange the order of integration to get

$$\begin{aligned} x(1) - x(t) = & \int_t^1 x'(\rho) d\rho \\ = & \int_t^1 \int_0^\rho \tau f(\tau, x(\tau), x'(\tau)) d\tau d\rho \\ & - \int_t^1 \int_\rho^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau d\rho, \\ = & (1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau - t \int_t^1 (1-\tau) \end{aligned}$$

$$\begin{aligned}
& \cdot f(\tau, x(\tau), x'(\tau)) d\tau, \\
& = -x(t) + \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\
& \quad \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\
& \quad \cdot \left. \int_{\phi(\xi_i)}^1 (1 - \tau) f(\tau, x(\tau), x'(\tau)) d\tau \right]. \quad (20)
\end{aligned}$$

Hence we have

$$\begin{aligned}
x(1) &= \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\
& \quad \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\
& \quad \cdot \left. \int_{\phi(\xi_i)}^1 (1 - \tau) f(\tau, x(\tau), x'(\tau)) d\tau \right]. \quad (21)
\end{aligned}$$

Then from (19) and (21) we have $x(0) = x(1)$; also from (12) we have $\sum_{i=1}^m a_i x(\phi(\xi_i)) = 0$.

This proves that the solution of singular second-order ordinary differential equation with periodic and deviated multipoint boundary conditions (1)-(2) is equivalent to the functional integral equation (12). $\square$

Define a nonlinear operator F by

$$(Fx)(t) = f(t, x(t), x'(t)). \quad (22)$$

From (H_1) , we have

$$\begin{aligned}
\|Fx\|_{\mathcal{E}} &= \int_0^1 t(1-t) |f(t, x(t), x'(t))| dt, \\
&\leq \int_0^1 t(1-t) \\
&\quad \cdot (|p(t)| |x(t)|^{\sigma_1} + |q(t)| |x'(t)|^{\sigma_2} + |r(t)|) dt, \\
&\leq \|x\|^{\sigma_1} \int_0^1 t(1-t) |p(t)| dt + \int_0^1 t^{1-\sigma_2} (1-t)^{1-\sigma_2} \\
&\quad \cdot |q(t)| t^{\sigma_2} (1-t)^{\sigma_2} |x'(t)|^{\sigma_2} dt + \int_0^1 t(1-t) \\
&\quad \cdot |r(t)| dt \leq \|p\|_{\mathcal{E}} \|x\|^{\sigma_1} + \|q\|_{L^1} \|x\|_1^{\sigma_2} + \|r\|_{\mathcal{E}} \\
&< \infty, \quad (23)
\end{aligned}$$

and we conclude that $F : X \rightarrow \mathcal{E}$ is well defined.

For $x \in X$, we define an operator N by

$$\begin{aligned}
(Nx)(t) &= -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\
&\quad - t \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\
&\quad + \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \\
&\quad \cdot \int_0^{\phi(\xi_i)} \tau f(\tau, x(\tau), x'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\
&\quad \cdot \left. \int_{\phi(\xi_i)}^1 (1 - \tau) f(\tau, x(\tau), x'(\tau)) d\tau \right]. \quad (24)
\end{aligned}$$

For $x \in X$ and from (H_0) and (H_1) , we have

$$\begin{aligned}
|(Nx)(t)| &\leq (1-t) \int_0^t \tau |f(\tau, x(\tau), x'(\tau))| d\tau \\
&\quad + t \int_t^1 (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau \\
&\quad + \frac{1}{|\sum_{i=1}^m a_i|} \left[\sum_{i=1}^m |a_i| (1 - \phi(\xi_i)) \right. \\
&\quad \cdot \int_0^{\phi(\xi_i)} \tau |f(\tau, x(\tau), x'(\tau))| d\tau + \sum_{i=1}^m |a_i| \phi(\xi_i) \\
&\quad \cdot \left. \int_{\phi(\xi_i)}^1 (1 - \tau) |f(\tau, x(\tau), x'(\tau))| d\tau \right], \\
&\leq \int_0^t \tau (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau + \int_t^1 \tau (1-\tau) \\
&\quad \cdot |f(\tau, x(\tau), x'(\tau))| d\tau + \frac{1}{|\sum_{i=1}^m a_i|} \left(\sum_{i=1}^m |a_i| \right. \\
&\quad \cdot \left[\int_0^{\phi(\xi_i)} \tau (1 - \phi(\xi_i)) |f(\tau, x(\tau), x'(\tau))| d\tau \right. \\
&\quad \left. \left. + \int_{\phi(\xi_i)}^1 \phi(\xi_i) (1 - \tau) |f(\tau, x(\tau), x'(\tau))| d\tau \right] \right), \\
&\leq \left(1 + \frac{\sum_{i=1}^m |a_i|}{|\sum_{i=1}^m a_i|} \right) \int_0^1 \tau (1-\tau) \\
&\quad \cdot |f(\tau, x(\tau), x'(\tau))| d\tau, \\
&= \mathcal{A} \|Fx\|_{\mathcal{E}}, \\
&\leq \mathcal{A} (\|p\|_{\mathcal{E}} \|x\|^{\sigma_1} + \|q\|_{L^1} \|x\|_1^{\sigma_2} + \|r\|_{\mathcal{E}}) < \infty. \quad (25)
\end{aligned}$$

Then $(Nx) : [0, 1] \rightarrow \mathcal{E}$ is well defined and bounded.

Lemma 5. Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ be a function satisfying Carathéodory's conditions and conditions (H_0) and (H_1) are fulfilled; then $Nx \in X$ and

$$(Nx)''(t) = f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1). \quad (26)$$

Proof. Firstly, to show $N : X \rightarrow C[0, 1]$, we have $Nx \in AC_{loc}(0, 1)$ and by Lemma 1 and from (19), (21), and (24) we have $\lim_{t \rightarrow 0} (Nx)(t) = (Nx)(0)$ and $\lim_{t \rightarrow 1} (Nx)(t) = (Nx)(1)$; then $(Nx)(t) \in C[0, 1]$; moreover we have

$$\begin{aligned} (Nx)'(t) &= \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\ &\quad - \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau. \end{aligned} \quad (27)$$

Now since

$$\begin{aligned} &\int_0^1 |(Nx)'(t)| dt \\ &\leq \int_0^1 \int_0^t \tau |f(\tau, x(\tau), x'(\tau))| d\tau dt \\ &\quad + \int_0^1 \int_t^1 (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau dt, \\ &= \int_0^1 \tau |f(\tau, x(\tau), x'(\tau))| \int_\tau^1 dt d\tau \\ &\quad + \int_0^1 (1-\tau) |f(\tau, x(\tau), x'(\tau))| \int_0^\tau dt d\tau, \\ &= 2 \int_0^1 \tau (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau = 2 \|Fx\|_{\mathcal{G}} \\ &\leq 2 (\|p\|_{\mathcal{G}} \|x\|^{\sigma_1} + \|q\|_{L^1} \|x\|_1^{\sigma_2} + \|r\|_{\mathcal{G}}) < \infty, \end{aligned} \quad (28)$$

then we have $Nx \in AC[0, 1]$. Now we have (27) and condition (H_1) together with the fact that $\tau f(\tau, x(\tau), x'(\tau)) \in L^1[0, r]$ and $(1-\tau)f(\tau, x(\tau), x'(\tau)) \in L^1[r, 1]$ for each $r \in (0, 1)$; we have $(Nx)'(t) \in AC_{loc}(0, 1)$ and

$$(Nx)''(t) = f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1). \quad (29)$$

Now to prove that $\lim_{t \rightarrow 1} (1-t)(Nx)'(t)$ and $\lim_{t \rightarrow 0} t(Nx)'(t)$ exist, we prove $t(1-t)(Nx)'(t) \in AC[0, 1]$, since

$$\begin{aligned} &\int_0^1 \left| \left[t(1-t)(Nx)'(t) \right]' \right| dt = \int_0^1 \left| (1-t)(Nx)'(t) \right. \\ &\quad \left. - t(Nx)'(t) + t(1-t)(Nx)''(t) \right| dt \\ &\leq \int_0^1 \left| (1-t)(Nx)'(t) \right| + t \left| (Nx)'(t) \right| \end{aligned}$$

$$\begin{aligned} &+ t(1-t) \left| (Nx)''(t) \right| dt = \int_0^1 \left| (Nx)'(t) \right| \\ &+ t(1-t) \left| (Nx)''(t) \right| dt \\ &\leq \int_0^1 \int_0^t \tau |f(\tau, x(\tau), x'(\tau))| d\tau dt \\ &+ \int_0^1 \int_t^1 (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau dt \\ &+ \int_0^1 t(1-t) |f(t, x(t), x'(t))| dt = 3 \int_0^1 \tau (1 \\ &- \tau) |f(\tau, x(\tau), x'(\tau))| d\tau \leq 3 (\|p\|_{\mathcal{G}} \|x\|^{\sigma_1} \\ &+ \|q\|_{L^1} \|x\|_1^{\sigma_2} + \|r\|_{\mathcal{G}}) < \infty. \end{aligned} \quad (30)$$

Then $Nx \in X$ which completes the proof of the lemma. $\square$

Lemma 6. $(Nx)(t) : X \rightarrow X$ is a completely continuous operator.

Proof. By the definition of N and (H_1) , let $\{x_k\} \subset X$ be a convergent sequence to x in X ; then $\lim_{k \rightarrow \infty} \|x_k - x\| = 0$. Since f satisfies Carathéodory's conditions, we have

$$\lim_{k \rightarrow \infty} f(t, x_k(t), x'_k(t)) = f(t, x(t), x'(t)) \quad (31)$$

for a.e. $t \in [0, 1]$.

From conditions (H_0) and (H_1) together with the fact that $\tau f(\tau, x(\tau), x'(\tau)) \in L^1[0, r]$ and $(1-\tau)f(\tau, x(\tau), x'(\tau)) \in L^1[r, 1]$ for each $r \in (0, 1)$, we obtain by applying Lebesgue dominated convergence theorem (cf. (24))

$$\|(Nx_k)(t) - (Nx)(t)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty. \quad (32)$$

Also we have (cf. (27))

$$\|(Nx_k)'(t) - (Nx)'(t)\| \rightarrow 0 \quad \text{as } k \rightarrow \infty \quad (33)$$

Therefore we obtain $\lim_{k \rightarrow \infty} \|(Nx_k)(t) - (Nx)(t)\|_X = 0$; that is, N is a continuous operator.

Secondly, let $B \subset X$ be a bounded set. We will show that $(Nx)(B)$ is a relatively compact subset of X . Let $\{x_k\} \subset B$; let $0 \leq t_1 < t_2 \leq 1$. From (27) and condition (H_1) , we have

$$\begin{aligned} &|(Nx_k)(t_2) - (Nx_k)(t_1)| = \left| \int_{t_1}^{t_2} (Nx_k)'(t) dt \right| \\ &\leq \int_{t_1}^{t_2} |(Nx_k)'(t)| dt, \\ &\leq \int_{t_1}^{t_2} \int_0^t \tau |f(\tau, x_k(\tau), x'_k(\tau))| d\tau + \int_{t_1}^{t_2} \int_t^1 (1-\tau) \\ &\quad \cdot |f(\tau, x_k(\tau), x'_k(\tau))| d\tau dt, \end{aligned}$$

$$\begin{aligned}
& \leq \int_{t_1}^{t_2} \int_0^t \tau \left(|p(\tau)| |x_k(\tau)|^{\sigma_1} + |q(\tau)| |x'_k(\tau)|^{\sigma_2} \right. \\
& \quad \left. + |r(\tau)| \right) d\tau dt + \int_{t_1}^{t_2} \int_t^1 (1-\tau) \\
& \quad \cdot \left(|p(\tau)| |x_k(\tau)|^{\sigma_1} + |q(\tau)| |x'_k(\tau)|^{\sigma_2} \right. \\
& \quad \left. + |r(\tau)| \right) d\tau dt, \\
& \leq \int_{t_1}^{t_2} \left\{ \|x_k\|^{\sigma_1} \int_0^t \tau |p(\tau)| d\tau + \|x_k\|_1^{\sigma_2} \right. \\
& \quad \cdot \int_0^t \frac{\tau |q(\tau)|}{\tau^{\sigma_2} (1-\tau)^{\sigma_2}} d\tau + \int_0^t \tau |r(\tau)| d\tau \Big\} dt \\
& \quad + \int_{t_1}^{t_2} \left\{ \|x_k\|^{\sigma_1} \int_t^1 (1-\tau) |p(\tau)| d\tau + \|x_k\|_1^{\sigma_2} \right. \\
& \quad \cdot \int_t^1 \frac{(1-\tau) |q(\tau)|}{\tau^{\sigma_2} (1-\tau)^{\sigma_2}} d\tau + \int_t^1 (1-\tau) |r(\tau)| d\tau \Big\} dt \\
& \leq \int_{t_1}^{t_2} \left\{ \|x_k\|^{\sigma_1} \int_0^t \tau |p(\tau)| d\tau + \|x_k\|_1^{\sigma_2} \right. \\
& \quad \cdot \int_0^t \frac{\tau |q(\tau)|}{\tau (1-\tau)} d\tau + \int_0^t \tau |r(\tau)| d\tau \Big\} dt \\
& \quad + \int_{t_1}^{t_2} \left\{ \|x_k\|^{\sigma_1} \int_t^1 (1-\tau) |p(\tau)| d\tau + \|x_k\|_1^{\sigma_2} \right. \\
& \quad \cdot \int_t^1 \frac{(1-\tau) |q(\tau)|}{\tau (1-\tau)} d\tau \\
& \quad \left. + \int_t^1 (1-\tau) |r(\tau)| d\tau \right\} dt.
\end{aligned} \tag{34}$$

Similarly we have

$$\begin{aligned}
& \left| t_2 (1-t_2) (Nx_k)'(t_2) - t_1 (1-t_1) (Nx_k)'(t_1) \right| \\
& = \left| \int_{t_1}^{t_2} [t(1-t) (Nx_k)''(t)]' dt \right|, \\
& = \left| \int_{t_1}^{t_2} [t(1-t) (Nx_k)''(t) - t (Nx_k)'(t) \right. \\
& \quad \left. + (1-t) (Nx_k)'(t)] dt \right|, \\
& \leq \int_{t_1}^{t_2} [t(1-t) |(Nx_k)''(t)| + t |(Nx_k)'(t)| + (1-t) \\
& \quad \cdot |(Nx_k)'(t)|] dt, \\
& = \int_{t_1}^{t_2} [t(1-t) |(Nx_k)''(t)| + |(Nx_k)'(t)|] dt.
\end{aligned} \tag{35}$$

From (26) we obtain

$$\begin{aligned}
& t(1-t) |(Nx_k)''(t)| = t(1-t) |f(t, x_k(t), x'_k(t))|, \\
& \leq t(1-t) \\
& \quad \cdot (|p(t)| |x_k(t)|^{\sigma_1} + |q(t)| |x'_k(t)|^{\sigma_2} + |r(t)|), \\
& \leq t(1-t) |p(t)| \|x_k\|^{\sigma_1} + t^{1-\sigma_2} (1-t)^{1-\sigma_2} |q(t)| \\
& \quad \cdot \|x_k\|_1^{\sigma_2} + t(1-t) |r(t)|, \\
& \leq t(1-t) |p(t)| \|x_k\|^{\sigma_1} + |q(t)| \|x_k\|_1^{\sigma_2} + t(1-t) \\
& \quad \cdot |r(t)|.
\end{aligned} \tag{36}$$

Then from (35) and by similar way as (34) we obtain

$$\begin{aligned}
& \left| t_2 (1-t_2) (Nx_k)'(t_2) - t_1 (1-t_1) (Nx_k)'(t_1) \right| \\
& \leq \|x_k\|^{\sigma_1} \int_{t_1}^{t_2} t(1-t) |p(t)| dt + \|x_k\|_1^{\sigma_2} \int_{t_1}^{t_2} |q(t)| dt \\
& \quad + \int_{t_1}^{t_2} t(1-t) |r(t)| dt \\
& \quad + \int_{t_1}^{t_2} \left\{ \|x_k\|^{\sigma_1} \int_0^t \tau |p(\tau)| d\tau \right. \\
& \quad \left. + \|x_k\|_1^{\sigma_2} \int_0^t \frac{\tau |q(\tau)|}{\tau (1-\tau)} d\tau + \int_0^t \tau |r(\tau)| d\tau \right\} dt \\
& \quad + \int_{t_1}^{t_2} \left\{ \|x_k\|^{\sigma_1} \int_t^1 (1-\tau) |p(\tau)| d\tau \right. \\
& \quad \left. + \|x_k\|_1^{\sigma_2} \int_t^1 \frac{|q(\tau)|}{\tau} d\tau \right. \\
& \quad \left. + \int_t^1 (1-\tau) |r(\tau)| d\tau \right\} dt.
\end{aligned} \tag{37}$$

Since $t(1-t)p(t), q(t), t(1-t)r(t) \in L^1[0, 1]$ and by using (i) of Lemma 1, thus from (34) and (37) we have the fact that $(Nx_k)(t)$ is equicontinuous on $[0, 1]$. By Arzelà-Ascoli theorem, $\{Nx(t)\}$ and $\{t(1-t)(Nx)'(t)\}$ are relatively compact and we have that fact that $N : X \rightarrow X$ is a completely continuous operator. $\square$

3. Existence Results

Theorem 7. Let $f : [0, 1] \times \mathbb{R}^2 \rightarrow \mathbb{R}$ satisfy the Carathéodory conditions. Assume that assumptions (H_0) and (H_1) hold. Then problem (1)-(2) has at least one solution.

Proof. In order to apply Theorem 2, we define a ball $B_k = \{x \in X : \|x\|_X \leq k, t \in [0, 1]\}$, where

$$\begin{aligned}
& k \geq \max \left\{ (3(1+\mathcal{A}) \|p\|_{\mathcal{G}})^{1/(1-\sigma_1)}, \right. \\
& \quad \left. (3(1+\mathcal{A}) \|q\|_{L^1})^{1/(1-\sigma_2)}, 3(1+\mathcal{A}) \|r\|_{\mathcal{G}} \right\}.
\end{aligned} \tag{38}$$

Firstly, we prove that $N : B_k \longrightarrow B_k$. For $x \in B_k$, and from (25), we have

$$\begin{aligned} |(Nx)(t)| &\leq \mathcal{A} (\|p\|_{\mathcal{G}} \|x\|^{\sigma_1} + \|q\|_{L^1} \|x\|^{\sigma_2} + \|r\|_{\mathcal{G}}), \\ &\leq \mathcal{A} (\|p\|_{\mathcal{G}} k^{\sigma_1} + \|q\|_{L^1} k^{\sigma_2} + \|r\|_{\mathcal{G}}). \end{aligned} \quad (39)$$

Also from (27)

$$\begin{aligned} &|t(1-t)(Nx)'(t)| \\ &\leq t(1-t) \int_0^t \tau |f(\tau, x(\tau), x'(\tau))| d\tau \\ &\quad + t(1-t) \int_t^1 (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau, \\ &\leq t \int_0^t \tau (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\ &\quad + (1-t) \int_t^1 \tau (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau, \\ &\leq \int_0^1 \tau (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau \\ &\leq \|p\|_{\mathcal{G}} \|x\|^{\sigma_1} + \|q\|_{L^1} \|x\|^{\sigma_2} + \|r\|_{\mathcal{G}} \\ &\leq \|p\|_{\mathcal{G}} k^{\sigma_1} + \|q\|_{L^1} k^{\sigma_2} + \|r\|_{\mathcal{G}}. \end{aligned} \quad (40)$$

Then

$$\begin{aligned} \|Nx\|_X &= \max_{t \in [0,1]} |Nx(t)| + \max_{t \in [0,1]} |t(1-t)(Nx)'(t)|, \\ &\leq \left(2 + \frac{\sum_{i=1}^m |a_i|}{\left|\sum_{i=1}^m a_i\right|}\right) (\|p\|_{\mathcal{G}} k^{\sigma_1} + \|q\|_{L^1} k^{\sigma_2} + \|r\|_{\mathcal{G}}), \\ &= (1 + \mathcal{A}) (\|p\|_{\mathcal{G}} k^{\sigma_1} + \|q\|_{L^1} k^{\sigma_2} + \|r\|_{\mathcal{G}}) \\ &\leq \frac{k}{3} + \frac{k}{3} + \frac{k}{3} = k. \end{aligned} \quad (41)$$

Thus we have $N : B_k \longrightarrow B_k$ and Lemma 6 provides that $(Nx)(t) : X \longrightarrow X$ is a completely continuous operator. Hence Theorem 2 implies the existence of a solution of problem (1)-(2) in B_k . $\square$

Theorem 8. Let $f : [0, 1] \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function satisfying the Carathéodory conditions. Assume that assumptions (H_0) and (H_1) hold. Then problem (1)-(2) has at least one solution in X .

Proof. From Lemma 4, we have the fact that $x(t)$ is a solution of problem (1)-(2) if and only if $x(t)$ satisfies the integral equation (12). We apply Theorem 3 to obtain the existence of at least one solution of (1)-(2) in X .

To do this, by Lemma 6, it suffices to verify that the set of all possible solutions of the family of equations

$$x'' = \lambda f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1), \quad (42)$$

$$u(0) = u(1), \quad \sum_{i=1}^m a_i x(\phi(\xi_i)) = 0, \quad (43)$$

is a priori bounded in X by a constant independent of $\lambda \in (0, 1)$.

Let $y(t) \in X$ be a solution of (42)-(43) for some $\lambda \in (0, 1)$. Then, for $t \in [0, 1]$, we have

$$\begin{aligned} |y(t)| &= \left| - (1-t) \int_0^t \tau \lambda f(\tau, y(\tau), y'(\tau)) d\tau \right. \\ &\quad \left. - t \int_t^1 (1-\tau) \lambda f(\tau, y(\tau), y'(\tau)) d\tau \right. \\ &\quad \left. + \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \right. \right. \\ &\quad \cdot \int_0^{\phi(\xi_i)} \tau \lambda f(\tau, y(\tau), y'(\tau)) d\tau + \sum_{i=1}^m a_i \phi(\xi_i) \\ &\quad \cdot \left. \int_{\phi(\xi_i)}^1 (1-\tau) \lambda f(\tau, y(\tau), y'(\tau)) d\tau \right] \Bigg|, \\ &= \left| - (1-t) \int_0^t \tau y''(\tau) d\tau - t \int_t^1 (1-\tau) y''(\tau) d\tau \right. \\ &\quad \left. + \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1 - \phi(\xi_i)) \int_0^{\phi(\xi_i)} \tau y''(\tau) d\tau \right. \right. \\ &\quad \left. \left. + \sum_{i=1}^m a_i \phi(\xi_i) \int_{\phi(\xi_i)}^1 (1-\tau) y''(\tau) d\tau \right] \right|, \\ &\leq \int_0^t \tau (1-\tau) |y''(\tau)| d\tau + \int_t^1 \tau (1-\tau) |y''(\tau)| d\tau \\ &\quad + \frac{1}{\left|\sum_{i=1}^m a_i\right|} \left(\sum_{i=1}^m |a_i| \right. \\ &\quad \cdot \left[\int_0^{\phi(\xi_i)} \tau (1-\phi(\xi_i)) |y''(\tau)| d\tau \right. \\ &\quad \left. \left. + \int_{\phi(\xi_i)}^1 \phi(\xi_i) (1-\tau) |y''(\tau)| d\tau \right] \right), \\ &= \left(1 + \frac{\sum_{i=1}^m |a_i|}{\left|\sum_{i=1}^m a_i\right|} \right) \int_0^1 \tau (1-\tau) |y''(\tau)| d\tau, \end{aligned} \quad (44)$$

which implies that

$$\|y\| \leq \mathcal{A} \|y''\|_{\mathcal{G}}. \quad (45)$$

Similarly (cf. (17)),

$$\begin{aligned} &|t(1-t)y'(t)| \\ &= \left| t(1-t) \int_0^t \tau \lambda f(\tau, y(\tau), y'(\tau)) d\tau \right. \\ &\quad \left. - t(1-t) \int_t^1 (1-\tau) \lambda f(\tau, y(\tau), y'(\tau)) d\tau \right|, \end{aligned}$$

$$\begin{aligned}
&= \left| t(1-t) \int_0^t \tau y''(\tau) d\tau \right. \\
&\quad \left. - t(1-t) \int_t^1 (1-\tau) y''(\tau) d\tau \right|, \\
&\leq t \int_0^t \tau(1-\tau) |y''(\tau)| d\tau + (1-t) \\
&\quad \cdot \int_t^1 \tau(1-\tau) |y''(\tau)| d\tau, \\
&\leq \int_0^1 \tau(1-\tau) |y''(\tau)| d\tau \leq \|y''\|_{\mathcal{E}},
\end{aligned} \tag{46}$$

which implies that

$$\|y\|_1 \leq \|y''\|_{\mathcal{E}}. \tag{47}$$

Now from (42), (45), and (47) and condition (H_1) , it follows that

$$\begin{aligned}
|t(1-t) y''(t)| &= |\lambda t(1-t) f(t, y(t), y'(t))| \\
&\leq t(1-t) \\
&\quad \cdot (|p(t)| |y(t)|^{\sigma_1} + |q(t)| |y'(t)|^{\sigma_2} + |r(t)|).
\end{aligned} \tag{48}$$

And we have

$$\begin{aligned}
&\int_0^1 |t(1-t) y''(t)| dt \leq \int_0^1 t(1-t) \\
&\quad \cdot (|p(t)| |y(t)|^{\sigma_1} + |q(t)| |y'(t)|^{\sigma_2} + |r(t)|) dt, \\
&\leq \|p\|_{\mathcal{E}} \|y\|^{\sigma_1} + \|q\|_{L^1} \|y\|_1^{\sigma_2} + \|r\|_{\mathcal{E}}, \\
&\|y''\|_{\mathcal{E}} \leq \|p\|_{\mathcal{E}} \mathcal{A}^{\sigma_1} \|y''\|_{\mathcal{E}}^{\sigma_2} + \|q\|_{L^1} \|y''\|_{\mathcal{E}}^{\sigma_2} + \|r\|_{\mathcal{E}}.
\end{aligned} \tag{49}$$

Since $\sigma_1, \sigma_2 \in (0, 1)$, from the above inequality, there exists a constant $\mathcal{B}$, such that $\|y''\|_{\mathcal{E}} \leq \mathcal{B}$ which implies from (45) that

$$\|y\| \leq \mathcal{A}\mathcal{B}. \tag{50}$$

Also from (47) we have

$$\|y\|_1 \leq \mathcal{B}. \tag{51}$$

Therefore

$$\|y\|_X \leq \left(2 + \frac{\sum_{i=1}^m |a_i|}{|\sum_{i=1}^m a_i|} \right) \mathcal{B} = (1 + \mathcal{A}) \mathcal{B}. \tag{52}$$

Then all conditions of Theorem 3 are satisfied and then problem (1)-(2) has at least one solution in X . $\square$

Corollary 9. *Under the assumptions of Theorem 8, the singular second-order ordinary differential equation with periodic and multipoint nonlocal boundary conditions*

$$\begin{aligned}
x''(t) &= f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1), \\
x(0) &= x(1), \quad \sum_{i=1}^m a_i x(\xi_i) = 0, \quad \xi_i \in (0, 1),
\end{aligned} \tag{53}$$

has at least one solution in X represented by

$$\begin{aligned}
x(t) &= -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\
&\quad - t \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\
&\quad + \frac{1}{\sum_{i=1}^m a_i} \left[\sum_{i=1}^m a_i (1-\xi_i) \int_0^{\xi_i} \tau f(\tau, x(\tau), x'(\tau)) d\tau \right. \\
&\quad \left. + \sum_{i=1}^m a_i \xi_i \int_{\xi_i}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \right].
\end{aligned} \tag{54}$$

Proof. Letting $\phi(\xi_i) = \xi_i \in (0, 1)$ in problem (1)-(2), then from Theorem 8 we derive that singular second-order ordinary differential problem (53) has at least one solution given by (54). $\square$

Example 10. Consider the singular second-order ordinary differential problem

$$\begin{aligned}
x'' &= \frac{\lambda_1 e^t \sin \pi t}{t(1-t) \sqrt{\pi + |x|}} x^{1/2} \\
&\quad + \frac{1 + \cos(x^2 + x'^3)}{3 + e^t} x^{(1/6)} + \frac{\lambda_2 e^{-2t}}{t(1-t)(1+t^2)},
\end{aligned} \tag{55}$$

a.e. $t \in (0, 1)$,

$$x(0) = x(1),$$

$$2x\left(\sin \frac{\pi}{8}\right) - x\left(\sin \frac{\pi}{6}\right) + x\left(\sin \frac{\pi}{4}\right) = 0,$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$. Here, $f(t, u, v) = (\lambda_1 e^t \sin \pi t / (t(1-t) \sqrt{\pi + |u|}) u^{1/2} + ((1 + \cos(u^2 + v^3)) / (3 + e^t)) v^{1/6} + \lambda_2 e^{-2t} / (t(1-t)(1+t^2)))$. Clearly

$$|f(t, u, v)| \leq p(t) |u|^{1/2} + q(t) |v|^{1/6} + r(t), \tag{56}$$

where $p(t) = |\lambda_1| e^t / (t(1-t))$, $q(t) = 1/2$, $r(t) = |\lambda_2| / (t(1-t)(1+t^2))$, and $\phi(\xi_i) = \sin \xi_i$ for $\xi_i \in [\pi/8, \pi/4] \subset (0, 1)$. We can see that $p(t), r(t) \in \mathcal{E}$ and $q(t) \in L^1$. Indeed we have $\|p\|_{\mathcal{E}} = |\lambda_1|(e-1)$, $\|q\|_{L^1} = 1/2$, $\|r\|_{\mathcal{E}} = |\lambda_2| \pi/4$, $\phi(\xi_i) \leq \xi_i$, and $\sum_{i=1}^3 a_i = 2 \neq 0$. Assumptions (H_0) and (H_1) hold; therefore, by Theorem 8, singular boundary value problem (55) has a continuous solution in X .

4. Problems with Periodic and Integral Boundary Conditions

Following exactly the same arguments used in Sections 2 and 3, we can prove analogous results for the problems (1), (3) and (1), (4), so we omit the proof of main results of this section.

Firstly, we deal with the problem (1), (3). By analogous method as in Lemma 4, we have the following result.

Lemma 11. *The singular second-order ordinary differential equation with periodic and integral boundary conditions (1), (3) is equivalent to the functional integral equation:*

$$\begin{aligned}
 x(t) &= -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\
 &\quad - t \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\
 &\quad + \int_0^1 (1-\phi(\theta)) \int_0^{\phi(\theta)} \tau f(\tau, x(\tau), x'(\tau)) d\tau d\theta \\
 &\quad + \int_0^1 \phi(\theta) \int_{\phi(\theta)}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau d\theta.
 \end{aligned} \tag{57}$$

We define an operator N_1 associated with the boundary value problem (1), (3) as follows: for $x \in X$

$$\begin{aligned}
 (N_1 x)(t) &= -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau \\
 &\quad - t \int_t^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau \\
 &\quad + \int_0^1 (1-\phi(\theta)) \int_0^{\phi(\theta)} \tau f(\tau, x(\tau), x'(\tau)) d\tau d\theta \\
 &\quad + \int_0^1 \phi(\theta) \int_{\phi(\theta)}^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau d\theta.
 \end{aligned} \tag{58}$$

From (H_1) , we have

$$\begin{aligned}
 |(N_1 x)(t)| &\leq (1-t) \int_0^t \tau |f(\tau, x(\tau), x'(\tau))| d\tau \\
 &\quad + t \int_t^1 (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau + \int_0^1 (1 \\
 &\quad - \phi(\theta)) \int_0^{\phi(\theta)} \tau |f(\tau, x(\tau), x'(\tau))| d\tau d\theta \\
 &\quad + \int_0^1 \phi(\theta) \int_{\phi(\theta)}^1 (1-\tau) \\
 &\quad \cdot |f(\tau, x(\tau), x'(\tau))| d\tau d\theta, \\
 &\leq \int_0^t \tau (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau + \int_t^1 \tau (1-\tau) \\
 &\quad \cdot |f(\tau, x(\tau), x'(\tau))| d\tau
 \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^1 \left[\int_0^{\phi(\theta)} \tau (1-\phi(\theta)) |f(\tau, x(\tau), x'(\tau))| d\tau \right. \\
 &\quad \left. + \int_{\phi(\theta)}^1 \phi(\theta) (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau \right] d\theta, \\
 &= \int_0^1 \tau (1-\tau) |f(\tau, x(\tau), x'(\tau))| d\tau + \int_0^1 \int_0^1 \tau (1 \\
 &\quad - \tau) |f(\tau, x(\tau), x'(\tau))| d\tau d\theta, \\
 &= 2 \|Fx\|_{\mathcal{E}} \leq 2 (\|p\|_{\mathcal{E}} \|x\|^{\sigma_1} + \|q\|_{L^1} \|x\|_1^{\sigma_2} + \|r\|_{\mathcal{E}}) \\
 &< \infty.
 \end{aligned} \tag{59}$$

Then $(N_1 x) : [0, 1] \rightarrow R$ is well defined and bounded.

Lemma 12. *Let $f : [0, 1] \times R^2 \rightarrow R$ be a function satisfying Carathéodory's conditions and (H_1) is fulfilled; then $(N_1 x)(t) : X \rightarrow X$ is a completely continuous operator and*

$$(Nx)''(t) = f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1). \tag{60}$$

Proof. The proof is similar to that of Lemmas 5 and 6. $\square$

Using the operator (58) and the method of proof for the results obtained in Section 3, we can establish the following result for the problem (1), (3).

Theorem 13. *Let $f : [0, 1] \times R^2 \rightarrow R$ be a function satisfying the Carathéodory conditions. Assume that the assumption (H_1) holds. Then there exists at least one solution $x \in X$ of the nonlinear singular ordinary differential equation with periodic and integral conditions (1), (3) represented by (57).*

The existence of solutions for the problem (1), (4) can be proved by a similar way.

Theorem 14. *Let $f : [0, 1] \times R^2 \rightarrow R$ be a function satisfying the Carathéodory conditions. Assume that the assumption (H_1) holds. Then there exists at least one solution $x \in X$ of the nonlinear singular ordinary differential equation with periodic and nonlocal integral conditions*

$$\begin{aligned}
 x''(t) &= f(t, x(t), x'(t)), \quad \text{a.e. } t \in (0, 1), \\
 x(0) &= x(1), \quad \int_c^d x(\theta) d\eta(\theta) = 0, \quad 0 < c < d < 1,
 \end{aligned} \tag{61}$$

represented by

$$\begin{aligned}
 x(t) &= -(1-t) \int_0^t \tau f(\tau, x(\tau), x'(\tau)) d\tau - t \int_t^1 (1 \\
 &\quad - \tau) f(\tau, x(\tau), x'(\tau)) d\tau \\
 &\quad + \frac{1}{\eta(d) - \eta(c)} \left[\int_c^d (1-\theta)
 \end{aligned}$$

$$\cdot \int_0^\theta \tau f(\tau, x(\tau), x'(\tau)) d\tau d\eta(\theta) + \int_c^d \theta \int_\theta^1 (1-\tau) f(\tau, x(\tau), x'(\tau)) d\tau d\eta(\theta) \Big], \quad (62)$$

where $\eta(\theta)$ is an increasing function and the integral is meant in the Riemann-Stieltjes sense for $0 < c < d < 1$.

Example 15. Consider the singular second-order ordinary differential problem

$$x'' = \frac{\cos t^2}{4t(1-t)} \sin\left(\frac{|x|^{1/4}}{2}\right) + \frac{(\sqrt{t}+1)x'^{3/2}}{3(1+|x'|)} + \frac{(1+\cos^2 x)e^t}{t(1-t)}, \quad \text{a.e. } t \in (0, 1), \quad (63)$$

under periodic and integral conditions

$$x(0) = x(1), \quad \int_0^1 x\left(\frac{2}{3}\theta^5\right) d\theta = 0, \quad (64)$$

or under periodic and nonlocal integral conditions

$$x(0) = x(1), \quad \int_{1/3}^{1/2} x(\theta) \sin \pi \theta d\theta = 0. \quad (65)$$

Here, $f(t, u, v) = (\cos t^2/4t(1-t)) \sin(|u|^{1/4}/2) + (\sqrt{t}+1)v^{3/2}/3(1+|v|) + (1+\cos^2 x)e^t/t(1-t)$. Clearly

$$|f(t, u, v)| \leq p(t)|u|^{1/4} + q(t)|v|^{1/2} + r(t), \quad (66)$$

where $p(t) = 1/8t(1-t)$, $q(t) = (\sqrt{t}+1)/3$, and $r(t) = 2e^t/t(1-t)$, and we have $\phi(\theta) = (2/3)\theta^5$ and $d\eta(\theta) = \sin \pi \theta d\theta$. Clearly, $\|p\|_{\mathcal{E}} = 1/8$, $\|q\|_{L^1} = 5/9$, and $\|r\|_{\mathcal{E}} = 2(e-1)$. Assumption (H_1) holds; therefore, by Theorems 13 and 14, singular boundary value problem (63) under the condition (64) or (65) has a continuous solution in X .

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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