Research Article

Orlicz-Aleksandrov-Fenchel Inequality for Orlicz Multiple Mixed Volumes

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Our main aim is to generalize the classical mixed volume $V(K_1, \dots, K_n)$ and Aleksandrov-Fenchel inequality to the Orlicz space. In the framework of Orlicz-Brunn-Minkowski theory, we introduce a new affine geometric quantity by calculating the Orlicz first-order variation of the mixed volume and call it *Orlicz multiple mixed volume* of convex bodies $K_1, \dots, K_n$, and L_n , denoted by $V_\varphi(K_1, \dots, K_n, L_n)$, which involves $(n + 1)$ convex bodies in $\mathbb{R}^n$. The fundamental notions and conclusions of the mixed volume and Aleksandrov-Fenchel inequality are extended to an Orlicz setting. The related concepts and inequalities of L_p -multiple mixed volume $V_p(K_1, \dots, K_n, L_n)$ are also derived. The Orlicz-Aleksandrov-Fenchel inequality in special cases yields L_p -Aleksandrov-Fenchel inequality, Orlicz-Minkowski inequality, and Orlicz isoperimetric type inequalities. As application, a new Orlicz-Brunn-Minkowski inequality for Orlicz harmonic addition is established, which implies Orlicz-Brunn-Minkowski inequalities for the volumes and quermassintegrals.

1. Introduction

One of the most important operations in geometry is vector addition. As an operation between sets K and L , defined by

$$K + L = \{x + y : x \in K, y \in L\}, \quad (1)$$

it is usually called Minkowski addition, and combined with volume, plays an important role in the Brunn-Minkowski theory. During the last few decades, the theory has been extended to L_p -Brunn-Minkowski theory. The set, called L_p addition, was introduced by Firey in [1, 2]. The operation, denoted by $+_p$, is defined by

$$h(K +_p L, x)^p = h(K, x)^p + h(L, x)^p, \quad (2)$$

for all $x \in \mathbb{R}^n$, $1 \leq p \leq \infty$, and compact convex sets K and L in $\mathbb{R}^n$ containing the origin. When $p = \infty$, (2) is interpreted as $h(K +_\infty L, x) = \max\{h(K, x), h(L, x)\}$, as is customary. Here the functions are the support functions. If K is a nonempty closed (not necessarily bounded) convex set in $\mathbb{R}^n$, then

$$h(K, x) = \max\{x \cdot y : y \in K\}, \quad (3)$$

for $x \in \mathbb{R}^n$, defines the support function $h(K, x)$ of K . L_p addition, volume, and combine inequalities lead to the L_p -Brunn-Minkowski theory, which are the fundamental and core content in the L_p Brunn-Minkowski theory. For some important results and more information from this theory, we refer to [3–22] and the references therein.

L_p spaces have a natural generalization, known as the Orlicz spaces. In recent years, progress towards an Orlicz-Brunn-Minkowski theory is initiated by Lutwak, Yang, and Zhang [23, 24] to initiate an extension of the L_p Brunn-Minkowski theory to an Orlicz-Brunn-Minkowski theory. They successively established the fundamental affine inequalities for these bodies. The definition of a corresponding addition is shown in later work of Gardner, Hug, and Weil [25] and they developed a very general and comprehensive Orlicz-Brunn-Minkowski theory. And they constructed a general framework for the Orlicz-Brunn-Minkowski theory and made clear for the first time the relation to Orlicz spaces and norms. The Orlicz addition of convex bodies was also introduced from different angles and established the Orlicz-Brunn-Minkowski inequality for the Orlicz addition (see [26]). The Orlicz centroid inequality for star bodies was introduced which is an extension from convex to star bodies

(see [27]). The existence of even Orlicz-Minkowski problem is demonstrated by Haberl, Lutwak, Yang, and Zhang [28]. Ludwig [29], Ludwig, and Reitzner [11] introduced what soon came to be seen as the Orlicz affine area; for related work, see [30]. In these papers the notions of L_p -addition, L_p -mixed volume, L_p -affine surface area, L_p -centroid body, and L_p -projection body and inequalities were extended to an Orlicz setting. Advances in the theory and its dual can be found in [31–48].

In 2014, Gardner, Hug, and Weil [25] introduced the Orlicz addition $K +_{\varphi} L$ of compact convex sets K and L in $\mathbb{R}^n$ containing the origin, implicitly, by

$$h(K +_{\varphi} L, u) = \inf \left\{ \lambda > 0 : \varphi \left(\frac{h(K, u)}{\lambda} \right) + \varphi \left(\frac{h(L, u)}{\lambda} \right) \leq 1 \right\}, \quad (4)$$

where $\varphi : [0, \infty) \rightarrow (0, \infty)$ is a convex and increasing function such that $\varphi(0) = 0$ and $\varphi(1) = 1$. Let Φ denote the set of convex functions $\varphi : [0, \infty) \rightarrow [0, \infty)$ that is increasing and satisfies $\varphi(0) = 0$ and $\varphi(1) = 1$. For $\varphi(t) = t^p$ and $p \geq 1$, the new addition $K +_{\varphi} L$ becomes $K +_p L$. The Orlicz mixed volume with respect to the Orlicz addition, denoted by $V_{\varphi}(K, L)$, is defined by

$$V_{\varphi}(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi \left(\frac{h(L, u)}{h(K, u)} \right) h(K, u) dS(K; u), \quad (5)$$

where $S(K; u)$ is the surface area measure of K , K is a convex body containing the origin in its interior, and L is a compact convex set containing the origin. For $\varphi(t) = t^p$ and $p \geq 1$, $V_{\varphi}(K, L)$ becomes $V_p(K, L)$, defined by (see [49])

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} dS(K; u). \quad (6)$$

Associated with the convex bodies $K_1, \dots, K_{n-1}$ in $\mathbb{R}^n$ is a unique positive Borel measure on S^{n-1} , $S(K_1, \dots, K_{n-1}; \cdot)$; call it the mixed area measure of $K_1, \dots, K_{n-1}$. For any convex body K_n , one has the integral representation (see, e.g., [5], p. 354).

$$\begin{aligned} V(K_1, \dots, K_n) &= \frac{1}{n} \int_{S^{n-1}} h(K_n, u) dS(K_1, \dots, K_{n-1}; u). \end{aligned} \quad (7)$$

The integration is with respect to the mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ on S^{n-1} . The mixed area measure $S(K_1, \dots, K_{n-1}; \cdot)$ is symmetric in its (first $n-1$) arguments. When $K_1 = \dots = K_{n-i-1} = K$ and $K_{n-i} = \dots = K_{n-1} = B$, the mixed area measure $S(K, \dots, K, B, \dots, B; \cdot)$ with i copies of B and $(n-i-1)$ copies of K will be written as $S_i(K, \cdot)$. For $K_1 = \dots = K_{n-1} = K$, $S(K_1, \dots, K_{n-1}; \cdot)$ reduces to the surface area measure $S(K; \cdot)$. The classical Aleksandrov-Fenchel inequality is as follows: if $K_1, \dots, K_n$ are convex bodies containing the origin and $1 \leq r \leq n$ (see, e.g., [50, p. 401]), then

$$V(K_1, \dots, K_n) \geq \prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}. \quad (8)$$

In the paper, we further improve the Orlicz-Brunn-Minkowski theory. Our main aim is to introduce a new mixed volume of convex bodies $K_1, \dots, K_n, L_n$, denoted as $V_{\varphi}(K_1, \dots, K_n, L_n)$, which involves $(n+1)$ convex bodies in $\mathbb{R}^n$, and call it Orlicz multiple mixed volume for convex bodies $K_1, \dots, K_n$, and L_n . The fundamental notions and conclusions of the mixed volume and Aleksandrov-Fenchel inequality are extended to an Orlicz setting. The new Orlicz-Aleksandrov-Fenchel inequality in special cases yields L_p -Aleksandrov-Fenchel inequality, Orlicz-Minkowski inequality, and Orlicz isoperimetric type inequality. As application, we prove Orlicz-Brunn-Minkowski inequality for the mixed volumes, which implies the Orlicz-Brunn-Minkowski inequalities for the volumes and quermassintegrals.

Following the spirit of introduction of Aleksandrov [51], Fenchel and Jessen's mixed quermassintegrals (see [52]), and introduction of Lutwak's L_p -mixed quermassintegrals (see [49]), we are based on the study of Orlicz first-order variational of mixed volumes. In Section 4, we prove that the Orlicz first-order variation of mixed volumes can be expressed as follows: for $\varphi \in \Phi$ and $\varepsilon > 0$

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n) \\ = \frac{1}{\varphi'_l(1)} \cdot V_{\varphi}(K_1, \dots, K_n, L_n), \end{aligned} \quad (9)$$

where $K_1, \dots, K_n$ are convex bodies containing the origin, L_n is a convex body containing the origin in its interior, $V(K_1, \dots, K_n)$ is the usual mixed volume, and $\varphi'_l(1)$ denotes the value of left derivative of convex function φ at point 1. Here, $K +_{\varphi, \varepsilon} L$ denotes the Orlicz linear combination of convex bodies K and L . If K, L are convex bodies containing the origin, $\alpha, \beta \geq 0$, and $\varphi \in \Phi$, then Orlicz linear combination $+_{\varphi}(K, L, \alpha, \beta)$ is defined by ([25])

$$\begin{aligned} \alpha \cdot \varphi \left(\frac{h(K, x)}{h(+_{\varphi}(K, L, \alpha, \beta), x)} \right) + \beta \\ \cdot \varphi \left(\frac{h(L, x)}{h(+_{\varphi}(K, L, \alpha, \beta), x)} \right) = 1. \end{aligned} \quad (10)$$

For $\alpha = 1$ and $\beta = \varepsilon \geq 0$, the Orlicz linear combination $+_{\varphi}(K, L, 1, \varepsilon)$ is denoted by $K +_{\varphi, \varepsilon} L$. This first-order variational equation then provides the new geometric quantity $V_{\varphi}(K_1, \dots, K_n, L_n)$; it is called it Orlicz multiple mixed volume of convex bodies $K_1, \dots, K_n, L_n$ and defined by

$$\begin{aligned} V_{\varphi}(K_1, \dots, K_n, L_n) \\ := \varphi'_l(1) \cdot \left(\frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n) \right). \end{aligned} \quad (11)$$

We prove also the new affine geometric quantity $V_{\varphi}(K_1, \dots, K_n, L_n)$ which has an integral representation.

$$V_\varphi(K_1, \dots, K_n, L_n) = \frac{1}{n} \cdot \int_{S^{n-1}} \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) h(L_n, u) \cdot dS(K_1, \dots, K_{n-1}; u). \quad (12)$$

When $\varphi(t) = t^p$ and $p \geq 1$, $V_\varphi(K_1, \dots, K_n, L_n)$ becomes a new mixed volume $V_p(K_1, \dots, K_n, L_n)$; call it L_p -multiple mixed volume, which is first published and named here.

$$V_p(K_1, \dots, K_n, L_n) = \frac{1}{n} \cdot \int_{S^{n-1}} h(K_n, u)^p h(L_n, u)^{1-p} dS(K_1, \dots, K_{n-1}; u). \quad (13)$$

For $p = 1$, $V_p(K_1, \dots, K_n, L_n)$ becomes $V(K_1, \dots, K_n)$, and for $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = L$ and $L_n = K$, $V_p(K_1, \dots, K_n, L_n)$ becomes the p -mixed quermassintegral $W_{p,i}(K, L)$ of K and L , defined by, for all $0 \leq i < n$ (see [49]),

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} dS_i(K; u). \quad (14)$$

Putting $K_n = L$, $L_n = K$, and $K_1 = \dots = K_{n-1} = K$ in (12), $V_\varphi(K_1, \dots, K_n, L_n)$ becomes the well-know Orlicz mixed volume $V_\varphi(K, L)$. This shows that the well-known mixed volumes $V_p(K, L)$, $W_{p,i}(K, L)$, and $V_\varphi(K, L)$ are all the special cases of $V_\varphi(K_1, \dots, K_n, L_n)$, where $p \geq 1$ and $\varphi \in \Phi$. In fact, the Orlicz mixed quermassintegral $W_{\varphi,i}(K, L)$ is also a special case of $V_\varphi(K_1, \dots, K_n, L_n)$. The Orlicz mixed quermassintegral of K and L , $W_{\varphi,i}(K, L)$, is defined by (see Section 4)

$$W_{\varphi,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K, u). \quad (15)$$

In Section 5, we establish the following Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed volume. If $K_1, \dots, K_n$ are convex bodies containing the origin, L_n is a convex body containing the origin in its interior, $1 \leq r \leq n$, and $\varphi \in \Phi$, then

$$V_\varphi(K_1, \dots, K_n, L_n) \geq V(K_1, \dots, K_{n-1}, L_n) \cdot \varphi\left(\frac{\prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}}{V(K_1, \dots, K_{n-1}, L_n)}\right). \quad (16)$$

When $\varphi(t) = t^p$ and $p \geq 1$, (16) becomes a new L_p -Aleksandrov-Fenchel type inequality. If $K_1, \dots, K_n$ are convex bodies containing the origin, L_n is a convex body containing the origin in its interior, $1 \leq r \leq n$, and $p \geq 1$, then

$$V_p(K_1, \dots, K_n, L_n) \geq \frac{\prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{p/r}}{V(K_1, \dots, K_{n-1}, L_n)^{p-1}}. \quad (17)$$

This L_p type inequality has never been discovered before, which is first published and named here. Obviously, for $p = 1$, (17) becomes (8). When $r = n$, $K_1 = \dots = K_{n-1} = K$, $K_n = L$, and $L_n = K$, (16) becomes the following inequality which was established by Gardner, Hug, and Weil [25]. If K is a convex body containing the origin in its interior, L is a convex body containing the origin, and $\varphi \in \Phi$, then

$$V_\varphi(K, L) \geq V(K) \cdot \varphi\left(\left(\frac{V(L)}{V(K)}\right)^{1/n}\right). \quad (18)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic. In Section 5, we show also that (16) in special case yields the following result. If K is a convex body containing the origin in its interior, L is a convex body containing the origin, $0 \leq i < n$, and $\varphi \in \Phi$, then (see [43])

$$W_{\varphi,i}(K, L) \geq W_i(K) \cdot \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \quad (19)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic, where $W_i(K)$ is the usual quermassintegral of convex body K .

The classical isoperimetric inequality states that if K is convex body (see, e.g., [50, p.382]), then

$$\left(\frac{S(K)}{\omega_n}\right)^n \geq \left(\frac{V(K)}{\kappa_n}\right)^{n-1}, \quad (20)$$

with equality if and only if K is ball, where $S(K)$ is the surface area of K and κ_n denotes volume of the unit ball B and its surface area by ω_n . The Orlicz isoperimetric inequality is established in Section 5. If K is a convex body containing the origin, $\varphi \in \Phi$, and $0 \leq i < n$, then

$$\frac{V_\varphi(K[n-i-1], B[i+1], K)}{W_i(K)} \geq \varphi\left(\left(\frac{\kappa_n}{W_i(K)}\right)^{1/(n-i)}\right). \quad (21)$$

If φ is strictly convex, equality holds if and only if K is a ball, where $V_\varphi(K[n-i-1], B[i+1], K)$ denotes the Orlicz multiple mixed volume $V_\varphi(\underbrace{K, \dots, K}_{n-i-1}, \underbrace{B, \dots, B}_{i+1}, K)$. When $\varphi(t) = t^p$ and $p \geq 1$, (21) becomes the following L_p -isoperimetric inequality. If K is a convex body containing the origin, $p \geq 1$, and $0 \leq i < n$, then

$$\left(\frac{nW_{p,i}(K, B)}{\omega_n}\right)^{n-i} \geq \left(\frac{W_i(K)}{\kappa_n}\right)^{n-i-p}, \quad (22)$$

with equality if and only if K is a ball, where $W_{p,i}(K, B)$ denotes the mixed p -quermassintegral of K and B . For $i = 0$ and $p = 1$, (22) becomes isoperimetric inequality (20).

In Section 6, we establish the following Orlicz-Brunn-Minkowski type inequality. If $K_1, \dots, K_n, L_n$ are convex bodies containing the origin and $\varphi \in \Phi$, then for $\varepsilon > 0$

$$1 \geq \varphi \left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, K_n +_{\varphi, \varepsilon} L_n)} \right) + \varepsilon \cdot \varphi \left(\frac{V(K_1, \dots, K_{n-1}, L_n)}{V(K_1, \dots, K_{n-1}, K_n +_{\varphi, \varepsilon} L_n)} \right), \quad (23)$$

If φ is strictly convex, equality holds if and only if K_n and L_n are homothetic. Inequality (23) in special case yields the following inequality, which was established by Gardner, Hug, and Weil [25]. If K, L are convex bodies containing the origin and $\varphi \in \Phi$

$$1 \geq \varphi \left(\left(\frac{V(K)}{V(K +_{\varphi} L)} \right)^{1/n} \right) + \varphi \left(\left(\frac{V(L)}{V(K +_{\varphi} L)} \right)^{1/n} \right). \quad (24)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic (also see [26]). In fact, the following inequality is also the special case of (23). If K, L are convex bodies containing the origin, $\varphi \in \Phi$, and $0 \leq i < n$, then (see [43])

$$1 \geq \varphi \left(\left(\frac{W_i(K)}{W_i(K +_{\varphi} L)} \right)^{1/(n-i)} \right) + \varphi \left(\left(\frac{W_i(L)}{W_i(K +_{\varphi} L)} \right)^{1/(n-i)} \right). \quad (25)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

This paper is organized as follows. In Section 2, we collect some basic concepts and facts that will be used in the proofs of our results. The Orlicz addition and Orlicz linear combination are introduced in Section 3. In Section 4, we introduce the Orlicz multiple mixed volumes, and some of its basic properties and lemmas are shown and hence L_p -multiple mixed volume $V_p(K_1, \dots, K_n, L_n)$ of convex bodies $K_1, \dots, K_n, L_n$ is also derived. In Section 5, Orlicz-Aleksandrov-Fenchel inequality is established, which in special case yields an L_p -Aleksandrov-Fenchel inequality and Orlicz-Minkowski inequality, respectively. The Orlicz isoperimetric and Urysohn's inequalities are given, which imply the L_p -isoperimetric inequality and L_p -Urysohn's inequality, respectively. In Section 6, a new Orlicz-Brunn-Minkowski inequality for the Orlicz addition is established, which is a generalization of the Orlicz-Brunn-Minkowski inequality for volumes and quermassintegrals.

2. Notations and Preliminaries

The setting for this paper is n -dimensional Euclidean space $\mathbb{R}^n$. Let $\mathcal{K}^n$ denote the set of convex bodies (compact convex

subsets with nonempty interiors) in $\mathbb{R}^n$, let $\mathcal{K}_o^n$ be the class of members of $\mathcal{K}^n$ containing the origin, and let $\mathcal{K}_{oo}^n$ be those sets in $\mathcal{K}^n$ containing the origin in their interiors. We reserve the letter $u \in S^{n-1}$ for unit vectors. S^{n-1} is the unit sphere. For a compact set K , we write $V(K)$ for the (n -dimensional) Lebesgue measure of K and call this the volume of K . A nonempty closed convex set is uniquely determined by its support function. Support function is homogeneous of degree 1; that is,

$$h(K, rx) = rh(K, x), \quad (26)$$

for all $x \in \mathbb{R}^n$ and $r \geq 0$. Obviously, for a pair of compact convex sets K and L , we have

$$h(K, x) \leq h(L, x), \quad \text{if and only if } K \subset L. \quad (27)$$

A function is a support function of a compact convex set if and only if it is positively homogeneous degree one and subadditive. Let d denote the Hausdorff metric on $\mathcal{K}^n$, i.e., for $K, L \in \mathcal{K}^n$,

$$d(K, L) = \|h(K, u) - h(L, u)\|_{\infty}, \quad (28)$$

where $\|\cdot\|_{\infty}$ denotes the sup-norm on the space of continuous functions $C(S^{n-1})$.

2.1. Mixed Volumes. If K_i ($i = 1, 2, \dots, r$) are compact convex subset and λ_i ($i = 1, 2, \dots, r$) are nonnegative real numbers, then fundamental importance is the fact that the volume of $\sum_{i=1}^r \lambda_i K_i$ is a homogeneous polynomial in λ_i given by (see, e.g., [53])

$$V(\lambda_1 K_1 + \dots + \lambda_n K_n) = \sum_{i_1, \dots, i_n} \lambda_{i_1} \dots \lambda_{i_n} V_{i_1 \dots i_n}, \quad (29)$$

where the sum is taken over all n -tuples $(i_1, \dots, i_n)$ of positive integers not exceeding r . The coefficient $V_{i_1 \dots i_n}$ depends only on the bodies $K_{i_1}, \dots, K_{i_n}$ and is uniquely determined by (29); it is called the mixed volume of $K_{i_1}, \dots, K_{i_n}$ and is written as $V(K_{i_1}, \dots, K_{i_n})$. If $K_1 = \dots = K_{n-i} = K$ and $K_{n-i+1} = \dots = K_n = L$, then the mixed volume $V(K_1, \dots, K_n)$ is written as $V_i(K, L)$. If $K_1 = \dots = K_{n-i} = K$, $K_{n-i+1} = \dots = K_n = B$, the mixed volume $V_i(K, B)$ is written as $W_i(K)$ and called quermassintegrals (or i th mixed quermassintegrals) of K . If $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, and $K_n = L$, the mixed volume $V(K, \dots, K, B, \dots, B, L)$, with i copies of B and $(n-i-1)$ copies of K , is written as $W_i(K, L)$ and called as the mixed quermassintegrals.

Aleksandrov [51] (also see Fenchel and Jessen [52], Busemann [54], and Schneider [55]) have shown that, for $K \in \mathcal{K}^n$ and $i = 0, 1, \dots, n-1$, there exists a regular Borel measure (i th mixed surface area measure) $S_i(K, \cdot)$ on S^{n-1} , such that the mixed quermassintegrals $W_i(K, L)$ have the following representation:

$$\begin{aligned} W_i(K, L) &= \frac{1}{n-i} \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K + \varepsilon L) - W_i(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{S^{n-1}} h(L, u) dS_i(K, u). \end{aligned} \quad (30)$$

A Minkowski inequality for mixed quermassintegrals states that, for $K, L \in \mathcal{K}^n$ and $0 \leq i < n-1$,

$$W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1} W_i(L), \quad (31)$$

with equality if and only if K and L are homothetic (see [54, 56]). A Brunn-Minkowski inequality for mixed quermassintegrals states that, for $K, L \in \mathcal{K}^n$ and $0 \leq i < n-1$,

$$W_i(K+L)^{1/(n-i)} \geq W_i(K)^{1/(n-i)} + W_i(L)^{1/(n-i)}, \quad (32)$$

with equality if and only if K and L are homothetic (see [2]).

2.2. L_p -Mixed Volumes. The best-known inequality concerning volumes of compact convex sets is the Brunn-Minkowski inequality, stating that if K and L are compact convex sets in $\mathbb{R}^n$, then

$$V(K+L)^{1/n} \geq V(K)^{1/n} + V(L)^{1/n}, \quad (33)$$

with equality if and only if K and L are homothetic.

The mixed volume $V_1(K, L)$ of compact convex sets K, L is defined by

$$\begin{aligned} V_1(K, L) &:= \frac{1}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K + \varepsilon L) - V(K)}{\varepsilon} \\ &= \frac{1}{n} \int_{S^{n-1}} h(L, u) dS(K, u). \end{aligned} \quad (34)$$

The Minkowski's first inequality for K, L states that

$$V_1(K, L) \geq V(K)^{(n-1)/n} V(L)^{1/n}, \quad (35)$$

with equality if and only if K and L are homothetic.

From the Brunn-Minkowski inequality, using only a little calculus, one can also derive Minkowski's second inequality (see [57, p.370])

$$V_1(K, L)^2 \geq V(K) V(K, \dots, K, L, L), \quad (36)$$

with equality if and only if K and L are homothetic and where K appears $(n-2)$ times.

The middle expression in (34) is the first variation of the volume of K with respect to L and the right-hand side of (34) is its integral representation. The L_p -Brunn-Minkowski theory received its greatest single impetus when Lutwak [49] found the appropriate L_p versions of (34) and (35) and their ingredients. By replacing Minkowski addition and scalar multiplication in (34) by L_p addition and its scalar addition and its scalar multiplication,

$${}_p\varepsilon \cdot L = \varepsilon^{1/p} L, \quad (37)$$

where $p \geq 1$. He established the L_p Minkowski mixed volume inequality.

$$V_p(K, L) \geq V(K)^{(n-p)/n} V(L)^{p/n}, \quad (38)$$

with equality if and only if K and L are homothetic, where

$$V_p(K, L) := \frac{p}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K + {}_p\varepsilon \cdot L) - V(K)}{\varepsilon}, \quad (39)$$

and showed that the L_p -mixed volume has the following integral representation:

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} dS(K, u). \quad (40)$$

In particular, in the L_p -Brunn-Minkowski theory, $S(K, \cdot)$ is replaced by the p -surface area measure $S_p(K, \cdot)$ given by

$$dS_p(K, \cdot) = h(K, u)^{1-p} dS(K, u). \quad (41)$$

Hence, the integral representation (40) becomes

$$V_p(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_p(K, u). \quad (42)$$

2.3. The Mixed p -Quermassintegrals. Mixed quermassintegrals are, of course, the first variation of the ordinary quermassintegrals, with respect to Minkowski addition. The mixed quermassintegrals $W_{p,0}(K, L), W_{p,1}(K, L), \dots, W_{p,n-1}(K, L)$ are the first variation of the ordinary quermassintegrals, with respect to Fiery addition: for $K, L \in \mathcal{K}_{oo}^n$ and real $p \geq 1$, defined by (see, e.g., [49])

$$W_{p,i}(K, L) = \frac{p}{n-i} \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K + {}_p\varepsilon \cdot L) - W_i(K)}{\varepsilon}. \quad (43)$$

For $p \geq 1$ and $0 \leq i < n$ and each $K \in \mathcal{K}_{oo}^n$, there exists a regular Borel measure $S_{p,i}(K, \cdot)$ on S^{n-1} , such that the mixed quermassintegral $W_{p,i}(K, L)$ has the following integral representation:

$$W_{p,i}(K, L) = \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u), \quad (44)$$

for all $L \in \mathcal{K}_{oo}^n$. The measure $S_{p,i}(K, \cdot)$ is absolutely continuous with respect to $S_i(K, \cdot)$ and has Radon-Nikodym derivative

$$\frac{dS_{p,i}(K, \cdot)}{dS_i(K, \cdot)} = h(K, \cdot)^{1-p}, \quad (45)$$

where $S_i(K, \cdot)$ is a regular Boel measure on S^{n-1} . The measure $S_{n-1}(K, \cdot)$ is independent of the body K and is just ordinary Lebesgue measure, S , on S^{n-1} . $S_i(B, \cdot)$ denotes the i -th surface area measure of the unit ball in $\mathbb{R}^n$. In fact, $S_i(B, \cdot) = S$ for all i . The surface area measure $S_0(K, \cdot)$ will frequently be written simply as $S(K, \cdot)$. When $i = 0$, $S_{p,i}(K, \cdot)$ is just the p -surface area measure $S_p(K, \cdot)$ (see [15, 16]). Obviously, putting $i = 0$ in (44), the mixed p -quermassintegrals $W_{p,i}(K, L)$ become the L_p -mixed volume $V_p(K, L)$.

A fundamental inequality for mixed p -quermassintegrals states that (see [49]), for $K, L \in \mathcal{K}_{oo}^n$, $p \geq 1$ and $0 \leq i < n$,

$$W_{p,i}(K, L)^{n-i} \geq W_i(K)^{n-i-p} W_i(L)^p, \quad (46)$$

with equality if and only if K and L are homothetic. The L_p -Brunn-Minkowski inequality for quermassintegrals was also established. If $K, L \in \mathcal{K}_{oo}^n$ and $p \geq 1$ and $0 \leq i < n$, then

$$W_i(K + {}_pL)^{p/(n-i)} \geq W_i(K)^{p/(n-i)} + W_i(L)^{p/(n-i)}, \quad (47)$$

with equality if and only if K and L are homothetic.

3. Orlicz Linear Combination

Throughout the paper, let Φ_m , $m \in \mathbb{N}$, denote the set of convex functions $\varphi : [0, \infty)^m \rightarrow [0, \infty)$ that are strictly increasing in each variable and satisfy $\varphi(0, \dots, 0) = 0$ and $\varphi(0, \dots, 1, \dots, 0) = 1$. When $m = 1$, we shall write Φ instead of Φ_1 .

Let $m \geq 2$, $\varphi \in \Phi_m$, $K_j \in \mathcal{K}_o^n$, and $j = 1, \dots, m$; the Orlicz addition of $K_1, \dots, K_m$, denoted by $+_\varphi(K_1, \dots, K_m)$, is defined by ([25])

$$h(+_\varphi(K_1, \dots, K_m), x) = \inf \left\{ \lambda > 0 : \varphi \left(\frac{h(K_1, x)}{\lambda}, \dots, \frac{h(K_m, x)}{\lambda} \right) \leq 1 \right\}, \quad (48)$$

for $x \in \mathbb{R}^n$.

Equivalently, the Orlicz radial addition $+_\varphi(K_1, \dots, K_m)$ can be defined implicitly (and uniquely) by

$$\varphi \left(\frac{h(K_1, x)}{h(+_\varphi(K_1, \dots, K_m), x)}, \dots, \frac{h(K_m, x)}{h(+_\varphi(K_1, \dots, K_m), x)} \right) = 1, \quad (49)$$

if $h(K_1, x) + \dots + h(K_m, x) > 0$ and by $h(+_\varphi(K_1, \dots, K_m), x) = 0$, if $h(K_1, x) = \dots = h(K_m, x) = 0$, for all $x \in \mathbb{R}^n$.

An important special case is obtained when

$$\varphi(x_1, \dots, x_m) = \sum_{j=1}^m \varphi(x_j), \quad (50)$$

for some fixed $\varphi \in \Phi$ such that $\varphi(1) = 1$, and in this case write $+_\varphi(K_1, \dots, K_m) = K_1 +_\varphi \dots +_\varphi K_m$. This means that $K_1 +_\varphi \dots +_\varphi K_m$ is defined either by

$$h(K_1 +_\varphi \dots +_\varphi K_m, x) = \inf \left\{ \lambda > 0 : \sum_{j=1}^m \varphi \left(\frac{h(K_j, x)}{\lambda} \right) \leq 1 \right\}, \quad (51)$$

for all $x \in \mathbb{R}^n$, or by the corresponding special case of (49). From (51), it follows easy that

$$\sum_{j=1}^m \varphi \left(\frac{h(K_j, x)}{\lambda} \right) = 1, \quad (52)$$

if and only if

$$\lambda = h(K_1 +_\varphi \dots +_\varphi K_m, x). \quad (53)$$

The Orlicz addition $+_\varphi$ is continuous, monotonic, $GL(n)$ covariant, and projection covariant. The Orlicz linear combination was defined by (see [25])

$$h(+_\varphi(K_1, \dots, K_m, \alpha_1, \dots, \alpha_m), x) = \inf \left\{ \lambda > 0 : \sum_{j=1}^m \alpha_j \varphi \left(\frac{h(K_j, x)}{\lambda} \right) \leq 1 \right\}, \quad (54)$$

for all $x \in \mathbb{R}^n$, where $K_1, \dots, K_m \in \mathcal{K}_o^n$, $\alpha_j \geq 0$, and $\varphi \in \Phi$. Definition (54) corresponds to taking the function φ in (48) and (49) to be

$$\varphi(x_1, \dots, x_m) = \sum_{j=1}^m \alpha_j \varphi(x_j), \quad (55)$$

where $\varphi \in \Phi_m$.

4. Orlicz Multiple Mixed Volumes

Let us introduce Orlicz multiple mixed volume.

Definition 1. For $\varphi \in \Phi$, we define Orlicz multiple mixed volume, denoted by $V_\varphi(K_1, \dots, K_n, L_n)$, defined by

$$V_\varphi(K_1, \dots, K_n, L_n) = \frac{1}{n} \cdot \int_{S^{n-1}} \varphi \left(\frac{h(K_n, u)}{h(L_n, u)} \right) h(L_n, u) \cdot dS(K_1, \dots, K_{n-1}; u), \quad (56)$$

for all $K_1, \dots, K_n \in \mathcal{K}_o^n$ and $L_n \in \mathcal{K}_{oo}^n$.

To derive this Definition 1, we need the following lemmas.

Lemma 2 ([25]). *If $K, L \in \mathcal{K}_o^n$, then*

$$K +_{\varphi, \varepsilon} L \rightarrow K \quad (57)$$

in the Hausdorff metric as $\varepsilon \rightarrow 0^+$.

Lemma 3. *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $\varphi \in \Phi$, then for $\varepsilon > 0$*

$$\begin{aligned} \varphi'_l(1) \cdot \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n) &= \frac{1}{n} \\ &\cdot \int_{S^{n-1}} \varphi \left(\frac{h(K_n, u)}{h(L_n, u)} \right) h(L_n, u) \\ &\cdot dS(K_1, \dots, K_{n-1}; u). \end{aligned} \quad (58)$$

Proof. Suppose $\varepsilon > 0$, $K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $u \in S^{n-1}$; let

$$h_\varepsilon = h(L_n +_{\varphi, \varepsilon} K_n, u). \quad (59)$$

Since

$$\frac{h(L_n, u)}{h_\varepsilon} = \varphi^{-1} \left(1 - \varepsilon \varphi \left(\frac{h(K_n, u)}{h_\varepsilon} \right) \right) \quad (60)$$

and from Lemma 2 and noting that φ is continuous function, we obtain

$$\begin{aligned} \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n) &= \frac{1}{n} \lim_{\varepsilon \rightarrow 0^+} \int_{S^{n-1}} \frac{h_\varepsilon}{\varepsilon} \\ &\cdot \frac{h_\varepsilon - h(L_n, u)}{h_\varepsilon} dS(K_1, \dots, K_{n-1}; u) = \frac{1}{n} \\ &\cdot \lim_{\varepsilon \rightarrow 0^+} \int_{S^{n-1}} h_\varepsilon \cdot \varphi \left(\frac{h(K_n, u)}{h_\varepsilon} \right) \end{aligned}$$

$$\begin{aligned}
& \cdot \frac{1 - \varphi^{-1}(1 - \varepsilon \varphi(h(K_n, u)/h_\varepsilon))}{1 - (1 - \varepsilon \varphi(h(K_n, u)/h_\varepsilon))} \\
& \cdot dS(K_1, \dots, K_{n-1}; u) = \frac{1}{n} \int_{S^{n-1}} h(L_n, u) \\
& \cdot \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) \lim_{y \rightarrow 1^-} \frac{1 - y}{\varphi(1) - \varphi(y)} dS(K_1, \dots, \\
& K_{n-1}; u) = \frac{1}{n \varphi'_l(1)} \int_{S^{n-1}} \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) h(L_n, u) \\
& \cdot dS(K_1, \dots, K_{n-1}; u), \tag{61}
\end{aligned}$$

where

$$y = \varphi^{-1}\left(1 - \varepsilon \varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right)\right). \tag{62}$$

For any $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $\varphi \in \Phi$, we see that either side of (61) is equal to $V_\varphi(K_1, \dots, K_n, L_n)$, and so this new Orlicz multiple dual mixed volume $V_\varphi(K_1, \dots, K_n, L_n)$ has been born. Hence, the equation in Definition 1 follows immediately. $\square$

This theorem plays a central role in our deriving the Orlicz multiple mixed volume. Hence, we give the second kind of different proof.

Second Proof. One has

$$\frac{dh_\varepsilon}{d\varepsilon} = \frac{d}{d\varepsilon} \left(\frac{h(L_n, u)}{\varphi^{-1}\left(1 - \varepsilon \varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right)\right)} \right) = \frac{h(L_n, u) \frac{d\varphi^{-1}(y)}{dy} \left[\varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right) - \varepsilon \cdot \frac{d\varphi(z)}{dz} \frac{h(K_n, u)}{h_\varepsilon^2} \frac{dh_\varepsilon}{d\varepsilon} \right]}{\left(\varphi^{-1}\left(1 - \varepsilon \varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right)\right) \right)^2}. \tag{63}$$

where

and

$$z = \frac{h(K_n, u)}{h_\varepsilon}. \tag{65}$$

$$y = 1 - \varepsilon \varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right), \tag{64}$$

Hence

$$\frac{dh_\varepsilon}{d\varepsilon} = \frac{h(L_n, u) \frac{d\varphi^{-1}(y)}{dy} \varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right)}{\left(\varphi^{-1}\left(1 - \varepsilon \varphi\left(\frac{h(K_n, u)}{h_\varepsilon}\right)\right) \right)^2} + \varepsilon \cdot \frac{h(K_n, u) h(L_n, u)}{h_\varepsilon^2} \frac{d\varphi^{-1}(y)}{dy} \frac{d\varphi(z)}{dz}. \tag{66}$$

Therefore

$$\begin{aligned}
& \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n) \\
& = \frac{1}{n} \lim_{n\varepsilon \rightarrow 0^+} \int_{S^{n-1}} \frac{h_\varepsilon - h(L_n, u)}{\varepsilon} dS(K_1, \dots, K_{n-1}; u) \tag{67} \\
& = \frac{1}{n} \lim_{n\varepsilon \rightarrow 0^+} \int_{S^{n-1}} \frac{dh_\varepsilon}{d\varepsilon} dS(K_1, \dots, K_{n-1}; u).
\end{aligned}$$

On the other hand, noting that $y \rightarrow 1^-$ as $\varepsilon \rightarrow 0^+$, we have

$$\frac{d\varphi^{-1}(y)}{d\varepsilon} = \lim_{y \rightarrow 1^-} \frac{\varphi^{-1}(y) - \varphi^{-1}(1)}{y - 1} = \frac{1}{\varphi'_l(1)}. \tag{68}$$

From (64), (66), (67), (68), and Lemma 2 and noticing that φ and φ^{-1} are continuous functions, (58) yields easy.

Lemma 4. If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $\varphi \in \Phi$, then

$$\begin{aligned}
& V_\varphi(K_1, \dots, K_n, L_n) \\
& = \varphi'_l(1) \cdot \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n). \tag{69}
\end{aligned}$$

Proof. This follows immediately from Definition 1 and Lemma 3. $\square$

Lemma 5 ([25]). If $K_i, L_i \in \mathcal{K}_o^n$ and $K_i \rightarrow K$, $L_i \rightarrow L$ as $i \rightarrow \infty$, then

$$K_i +_{\varphi, \varepsilon} L_i \rightarrow K +_{\varphi, \varepsilon} L, \quad \text{as } i \rightarrow \infty, \tag{70}$$

for $\varepsilon > 0$.

Let us list some important properties of the Orlicz multiple mixed volume.

Lemma 6. If $K_1, \dots, K_n, L_n, K, L \in \mathcal{K}_o^n$, $\lambda_1, \dots, \lambda_n \geq 0$, and $\varphi \in \Phi$, then one has the following:

- (1) $V_\varphi(K_1, \dots, K_n, L_n) \geq 0$.
- (2) $V_\varphi(K, K, \dots, K) = V(K)$.
- (3) for $K \subset L$

$$V_\varphi(K, K_2, \dots, K_n, L_n) \leq V_\varphi(L, K_2, \dots, K_n, L_n). \quad (71)$$

(4)

$$\begin{aligned} V_\varphi(\lambda_1 K_1, \dots, \lambda_n K_n, \lambda_n L_n) \\ = \lambda_1 \cdots \lambda_n V_\varphi(K_1, \dots, K_n, L_n). \end{aligned} \quad (72)$$

(5) The Orlicz multiple mixed volume $V_\varphi(K_1, \dots, K_n, L_n)$ is continuous function of $K_1, \dots, K_n, L_n$.

(6)

$$\begin{aligned} V_\varphi(\lambda_1 K + \lambda_2 L, K_2, \dots, K_n, L_n) \\ = \lambda_1 V_\varphi(K, K_2, \dots, K_n, L_n) \\ + \lambda_2 V_\varphi(L, K_2, \dots, K_n, L_n). \end{aligned} \quad (73)$$

This shows that the Orlicz multiple mixed volume $V_\varphi(K_1, \dots, K_n, L_n)$ is linear in its $(n-1)$ variables in front.

Proof. From Definition 1, it immediately gives (1), (2), and (3). Combining Definition 1 with the fact (see [58])

$$\begin{aligned} S(\lambda_1 K_1, \dots, \lambda_{n-1} K_{n-1}, \cdot) \\ = \lambda_1 \cdots \lambda_{n-1} S(K_1, \dots, K_{n-1}, \cdot), \end{aligned} \quad (74)$$

it yields (4) directly.

Suppose $L_{in} \rightarrow L_n$, $K_{ij} \rightarrow K_j$ as $i \rightarrow \infty$ where $j = 1, \dots, n$; combining Definition 1 and Lemma 5 with the facts

$$\begin{aligned} V(K_{i1}, \dots, K_{in}, K_{in+\varphi, \varepsilon} L_{in}) &\rightarrow \\ V(K_1, \dots, K_n, K_{n+\varphi, \varepsilon} L_n) & \end{aligned} \quad (75)$$

and

$$V(K_{i1}, \dots, K_{in}, L_{in}) \rightarrow V(K_1, \dots, K_n, L_n) \quad (76)$$

as $i \rightarrow \infty$, it yields (5) directly.

Combining Definition 1 and noting that the mixed area measure is Minkowski linear in each of its arguments (see [59])

$$\begin{aligned} S(K + L, K_2, \dots, K_{n-1}, \cdot) \\ = S(K, K_2, \dots, K_{n-1}, \cdot) + S(L, K_2, \dots, K_{n-1}, \cdot), \end{aligned} \quad (77)$$

it yields (6) directly. $\square$

Lemma 7 ([25]). Suppose $K, L \in \mathcal{K}_o^n$ and $\varepsilon > 0$. If $\varphi \in \Phi$, then for $A \in GL(n)$

$$A(K +_{\varphi, \varepsilon} L) = AK +_{\varphi, \varepsilon} AL. \quad (78)$$

We easily find that Orlicz multiple dual mixed volume $V_\varphi(K_1, K_2, \dots, K_n, L_n)$ is invariant under simultaneous unimodular centro-affine transformation.

Lemma 8. If $K_1, \dots, K_n, L_n \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, then, for $A \in SL(n)$,

$$V_\varphi(AK_1, \dots, AK_n, AL_n) = V_\varphi(K_1, \dots, K_n, L_n). \quad (79)$$

Proof. From (69) and Lemma 7, we have, for $A \in SL(n)$,

$$\begin{aligned} V_\varphi(AK_1, \dots, AK_n, AL_n) &= \varphi'_l(1) \lim_{\varepsilon \rightarrow 0^+} \frac{V(AK_1, \dots, AK_{n-1}, AL_n +_{\varphi, \varepsilon} AK_n) - V(AK_1, \dots, AK_{n-1}, AL_n)}{\varepsilon} \\ &= \varphi'_l(1) \lim_{\varepsilon \rightarrow 0^+} \frac{V(AK_1, \dots, AK_{n-1}, A(L_n +_{\varphi, \varepsilon} K_n)) - V(K_1, \dots, K_{n-1}, L_n)}{\varepsilon} \\ &= \varphi'_l(1) \lim_{\varepsilon \rightarrow 0^+} \frac{V(K_1, \dots, K_{n-1}, L_n +_{\varphi, \varepsilon} K_n) - V(K_1, \dots, K_{n-1}, L_n)}{\varepsilon} = V_\varphi(K_1, \dots, K_n, L_n). \end{aligned} \quad (80)$$

$\square$

Lemma 9 ([25]). If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$, and $\varphi \in \Phi$, then

$$V_\varphi(K, L) = \frac{\varphi'_l(1)}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K +_{\varphi, \varepsilon} L) - V(K)}{\varepsilon}. \quad (81)$$

Lemma 10. If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$, and $\varphi \in \Phi$, then

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0^+} \frac{V_1(K, K +_{\varphi, \varepsilon} L) - V(K)}{\varepsilon} \\ = \frac{1}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K +_{\varphi, \varepsilon} L) - V(K)}{\varepsilon}. \end{aligned} \quad (82)$$

Proof. Suppose $\varepsilon > 0$, $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$, and $u \in S^{n-1}$; let

$$\bar{h}_\varepsilon = h(K +_{\varphi, \varepsilon} L, u). \quad (83)$$

From (5), (35), (66), (68), and Lemma 9, we obtain

$$\begin{aligned}
 & \lim_{\varepsilon \rightarrow 0^+} \frac{V_1(K, K + \varphi_\varepsilon L) - V(K)}{\varepsilon} \\
 &= \frac{1}{n} \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{\bar{h}_\varepsilon - h(K, u)}{\varepsilon} dS(K; u) \\
 &= \frac{1}{n} \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{d\bar{h}_\varepsilon}{d\varepsilon} dS(K; u) \\
 &= \frac{1}{n\varphi'_l(1)} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS(K; u) \\
 &= \frac{1}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K + \varphi_\varepsilon L) - V(K)}{\varepsilon}.
 \end{aligned} \tag{84}$$

□

Lemma 11. Let $K_1, \dots, K_n \in \mathcal{K}_o^n$ and $L_n \in \mathcal{K}_{oo}^n$. If $K_1 = \dots = K_{n-1} = K$, $K_n = L$, and $L_n = K$, then

$$V_\varphi(K_1, \dots, K_n, L_n) = V_\varphi(K, L). \tag{85}$$

Proof. On the one hand, putting $K_1 = \dots = K_{n-1} = K$, $K_n = L$, and $L_n = K$ in (69) and noting Lemmas 10 and 9, it follows that

$$\begin{aligned}
 & V_\varphi(K_1, \dots, K_n, L_n) \\
 &= \varphi'_l(1) \lim_{\varepsilon \rightarrow 0^+} \frac{V_1(K, K + \varphi_\varepsilon L) - V(K)}{\varepsilon} \\
 &= \frac{\varphi'_l(1)}{n} \lim_{\varepsilon \rightarrow 0^+} \frac{V(K + \varphi_\varepsilon L) - V(K)}{\varepsilon} = V_\varphi(K, L).
 \end{aligned} \tag{86}$$

On the other hand, putting $K_1 = \dots = K_{n-1} = K$, $K_n = L$, and $L_n = K$ in Definition 1 and in view of (5), then

$$\begin{aligned}
 & V_\varphi(K_1, \dots, K_n, L_n) \\
 &= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS(K_1, \dots, K_{n-1}; u) \\
 &= V_\varphi(K, L).
 \end{aligned} \tag{87}$$

Combining (86), (87), and Lemma 3, this shows that

$$V_\varphi(K_1, \dots, K_n, L_n) = V_\varphi(K, L), \tag{88}$$

if $K_1 = \dots = K_{n-1} = K$, $K_n = L$, and $L_n = L$. □

Lemma 12. If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, and $0 \leq i < n$, then

$$\begin{aligned}
 & V_\varphi(K[n-i-1], B[i], L, K) \\
 &= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K; u).
 \end{aligned} \tag{89}$$

Proof. On the one hand, putting $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $L_n = K$, and $K_n = L$ in (69), from (31), (66), and (68), we obtain for $\varphi \in \Phi$

$$\begin{aligned}
 & V_\varphi(K[n-i-1], B[i], L, K) \\
 &= \varphi'_l(1) \lim_{\varepsilon \rightarrow 0^+} \frac{W_i(K, K + \varphi_\varepsilon L) - W_i(K)}{\varepsilon} \\
 &= \frac{1}{n} \varphi'_l(1) \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{h(K + \varphi_\varepsilon L) - h(K, u)}{\varepsilon} dS_i(K; u) \\
 &= \frac{1}{n} \varphi'_l(1) \int_{S^{n-1}} \lim_{\varepsilon \rightarrow 0^+} \frac{d\bar{h}_\varepsilon}{d\varepsilon} dS_i(K; u) \\
 &= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K; u).
 \end{aligned} \tag{90}$$

On the other hand, putting $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $L_n = K$, and $K_n = L$, from Definition 1, we have for $\varphi \in \Phi$

$$\begin{aligned}
 & V_\varphi(K[n-i-1], B[i], L, K) \\
 &= \frac{1}{n} \int_{S^{n-1}} \varphi\left(\frac{h(L, u)}{h(K, u)}\right) h(K, u) dS_i(K; u),
 \end{aligned} \tag{91}$$

where $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$ and $\varphi \in \Phi$, and $0 \leq i < n$. □

Here, we denote the Orlicz multiple mixed volume $V_\varphi(K[n-i-1], B[i], L, K)$ by $W_{\varphi,i}(K, L)$ and call $W_{\varphi,i}(K, L)$ as Orlicz mixed quermassintegral of convex bodies K and L . When $i = 0$, Orlicz mixed quermassintegral $W_{\varphi,i}(K, L)$ becomes Orlicz dual mixed volume $V_\varphi(K, L)$.

Remark 13. When $\varphi(t) = t^p$, $p = 1$, from the integral representation for Orlicz multiple mixed volume $V_\varphi(K_1, \dots, K_n, L_n)$ (see Definition 1), it follows easy that $V_\varphi(K_1, \dots, K_n, L_n) = V(K_1, \dots, K_n)$. On the other hand, when $\varphi(t) = t^p$, $p = 1$, from (69) and noting that $\varphi'_l(1) = 1$ and $K_n + \varphi_\varepsilon L_n = K_n + \varepsilon \cdot K_n$, hence

$$\begin{aligned}
 & V(K_1, \dots, K_n) \\
 &= \lim_{\varepsilon \rightarrow 0^+} \frac{V(K_1, \dots, K_{n-1}, K_n + \varepsilon \cdot K_n) - V(K_1, \dots, K_n)}{\varepsilon}.
 \end{aligned} \tag{92}$$

This is very interesting that the mixed volume is such a limiting form.

On the other hand, taking $L_n = K_n$ in Definition 1 and noting $\varphi(1) = 1$, $V_\varphi(K_1, \dots, K_n, L_n)$ also becomes $V(K_1, \dots, K_n)$.

Remark 14. When $\varphi(t) = t^p$ and $p \geq 1$, we write $V_\varphi(K_1, \dots, K_n, L_n)$ as $V_p(K_1, \dots, K_n, L_n)$ and call $V_p(K_1, \dots, K_n, L_n)$ as L_p -multiple mixed volume of convex bodies $K_1, \dots, K_n, L_n$.

Putting $\varphi(t) = t^p$, $p \geq 1$ in Definition 1, then

$$V_p(K_1, \dots, K_n, L_n) = \frac{1}{n} \cdot \int_{S^{n-1}} h(L_n, u)^{1-p} h(K_n, u)^p dS(K_1, \dots, K_{n-1}; u). \quad (93)$$

On the other hand, when $\varphi(t) = t^p$ and $p \geq 1$, from (69), we get the following a limit of representation of L_p -multiple mixed volume:

$$V_p(K_1, \dots, K_n, L_n) = p \lim_{\varepsilon \rightarrow 0^+} \frac{V(K_1, \dots, K_{n-1}, L_n + \varepsilon \cdot K_n) - V(K_1, \dots, K_{n-1}, L_n)}{\varepsilon}. \quad (94)$$

where $p \geq 1$ and $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$.

Putting $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = L$, and $L_n = K$ in (93) and in view of (45) and (46), we obtain

$$\begin{aligned} V_p(K_1, \dots, K_n, L_n) &= \frac{1}{n} \int_{S^{n-1}} h(L, u)^p h(K, u)^{1-p} dS_i(K, u) \\ &= \frac{1}{n} \int_{S^{n-1}} h(L, u)^p dS_{p,i}(K, u) = W_{p,i}(K, L). \end{aligned} \quad (95)$$

Obviously, when $L_n = K_n$ or $p = 1$, from (93), $V_p(K_1, \dots, K_n, L_n)$ becomes the mixed volume $V(K_1, \dots, K_n)$. When $K_1 = \dots = K_{n-1} = K$, $K_n = L$, and $L_n = K$, $V_p(K_1, \dots, K_n, L_n)$ becomes $V_p(K, L)$.

Lemma 15 (Jensen's inequality). *Let μ be a probability measure on a space X and $g : X \rightarrow I \subset \mathbb{R}$ is a μ -integrable function, where I is a possibly infinite interval. If $\psi : I \rightarrow \mathbb{R}$ is a convex function, then*

$$\int_X \psi(g(x)) d\mu(x) \geq \psi\left(\int_X g(x) d\mu(x)\right). \quad (96)$$

If ψ is strictly convex, equality holds if and only if $g(x)$ is constant for μ -almost all $x \in X$ (see [60, p.165]).

5. Orlicz-Aleksandrov-Fenchel Inequality for Orlicz Multiple Mixed Volumes

Lemma 16. *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $\varphi \in \Phi$, then*

$$\frac{V_\varphi(K_1, \dots, K_n, L_n)}{V(K_1, \dots, K_{n-1}, L_n)} \geq \varphi\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right). \quad (97)$$

If φ is strictly convex, equality holds if and only if K_n and L_n are homothetic.

Proof. For $K_1, \dots, K_n \in \mathcal{K}_o^n$ and any $u \in S^{n-1}$, since

$$\begin{aligned} \frac{1}{nV(K_1, \dots, K_n)} \int_{S^{n-1}} h(K_n, u) dS(K_1, \dots, K_{n-1}; u) \\ = 1, \end{aligned} \quad (98)$$

so $(h(K_n, u)/nV(K_1, \dots, K_n))S(K_1, \dots, K_{n-1}; u)$ is a probability measure on S^{n-1} .

From (7) and (96) and in view of Definition 1, we obtain

$$\begin{aligned} \frac{V_\varphi(K_1, \dots, K_n, L_n)}{V(K_1, \dots, K_{n-1}, L_n)} &= \frac{1}{nV(K_1, \dots, K_{n-1}, L_n)} \\ &\cdot \int_{S^{n-1}} \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) h(L_n, u) dS(K_1, \dots, K_{n-1}; u) \\ &\geq \varphi\left(\frac{1}{nV(K_1, \dots, K_{n-1}, L_n)} \int_{S^{n-1}} h(K_n, u) \right. \\ &\cdot dS(K_1, \dots, K_{n-1}; u) \Big) = \varphi\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right). \end{aligned} \quad (99)$$

Next, we discuss the equality condition of (97). Suppose the equality hold in (97), form the equality condition of Jensen's inequality, it follows that if φ is strictly convex the equality in (97) holds if and only if $h(K_n, u)/h(L_n, u)$ is constant for $S(K, \dots, K; \cdot)$ -almost all $u \in S^{n-1}$. This follows that if φ is strictly convex, the equality in (97) holds if and only if $h(K_n, u)$ and $h(L_n, u)$ proportional for $S(K, \dots, K; \cdot)$ -almost all $u \in S^{n-1}$. This yields that if φ is strictly convex, the equality in (97) holds if and only if K_n and L_n are homothetic.

Conversely, if φ is strictly convex, then there exist $r > 0$ such that

$$\frac{h(K_n, u)}{h(L_n, u)} = r, \quad (100)$$

for $S(K, \dots, K; \cdot)$ -almost all $u \in S^{n-1}$; this shows that

$$r = \frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}. \quad (101)$$

From Definition 1, (7), and (101), we obtain

$$\begin{aligned} \frac{V_\varphi(K_1, \dots, K_n, L_n)}{V(K_1, \dots, K_{n-1}, L_n)} &= \frac{1}{nV(K_1, \dots, K_{n-1}, L_n)} \\ &\cdot \int_{S^{n-1}} \varphi\left(\frac{h(K_n, u)}{h(L_n, u)}\right) h(L_n, u) \\ &\cdot dS(K_1, \dots, K_{n-1}; u) = \frac{1}{nV(K_1, \dots, K_{n-1}, L_n)} \\ &\times \int_{S^{n-1}} \varphi\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right) h(L_n, u) \\ &\cdot dS(K_1, \dots, K_{n-1}; u) = \varphi\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right). \end{aligned} \quad (102)$$

This implies that the equality in (97) holds. $\square$

Theorem 17 (Orlicz-Aleksandrov-Fenchel inequality). *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, $1 \leq r \leq n$, and $\varphi \in \Phi$, then*

$$\begin{aligned} V_\varphi(K_1, \dots, K_n, L_n) \\ \geq V(K_1, \dots, K_{n-1}, L_n) \\ \cdot \varphi\left(\frac{\prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}}{V(K_1, \dots, K_{n-1}, L_n)}\right). \end{aligned} \quad (103)$$

Proof. This follows immediately from Lemma 16 with the Aleksandrov-Fenchel inequality. $\square$

Unfortunately, precise equality for Orlicz-Aleksandrov-Fenchel inequality is also unknown in general, because the precise equality for the classical Aleksandrov-Fenchel inequality is unknown in general; see [38, p.408] for a full discussion.

Corollary 18. *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $\varphi \in \Phi$, then*

$$\begin{aligned} V_\varphi(K_1, \dots, K_n, L_n) \\ \geq V(K_1, \dots, K_{n-1}, L_n) \\ \cdot \varphi\left(\left(\frac{V(K_1) \cdots V(K_n)}{V(K_1, \dots, K_{n-1}, L_n)^n}\right)^{1/n}\right). \end{aligned} \quad (104)$$

If φ is strictly convex, equality holds if and only if $K_1, \dots, K_n, L_n$ are all homothetic of each other.

Proof. This inequality (104) yields immediately from (103) with $r = n$.

Next, we discuss the equality condition of (104). When $r = n$, the Aleksandrov-Fenchel inequality becomes

$$V(K_1, \dots, K_n)^n \geq V(K_1) \cdots V(K_n). \quad (105)$$

Although, the precise equality for the Aleksandrov-Fenchel inequality are unknown in general, but it is well known that the equality in (105) holds if and only if $K_1, \dots, K_n$ are all homothetic of each other. Hence combining the equality conditions of Lemma 16, it follows that the equality condition of (104) holds if and only if φ is strictly convex and $K_1, \dots, K_n, L_n$ are all homothetic of each other. $\square$

Theorem 19. *If $K \in \mathcal{K}_{oo}^n$, $L \in \mathcal{K}_o^n$, $0 \leq i < n$, and $\varphi \in \Phi$, then*

$$W_{\varphi,i}(K, L) \geq W_i(K) \cdot \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \quad (106)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof. This inequality (106) yields immediately from Theorem 17 with $r = n - i - 1$, $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $L_n = K$, and $K_n = L$.

Next, we discuss the equality condition of (106). When $r = n - i - 1$, $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $L_n = K$, and $K_n = L$, (103) becomes

$$\begin{aligned} W_{\varphi,i}(K, L) &\geq W_i(K) \\ &\cdot \varphi\left(\frac{[W_i(K)^{n-i-1} W_i(L)]^{1/(n-i)}}{W_i(K)}\right). \end{aligned} \quad (107)$$

$$= W_i(K) \varphi\left(\left(\frac{W_i(L)}{W_i(K)}\right)^{1/(n-i)}\right). \quad (108)$$

When $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $L_n = K$, and $K_n = L$, (97) becomes

$$W_{\varphi,i}(K, L) \geq W_i(K) \cdot \varphi\left(\frac{W_i(K, L)}{W_i(K)}\right). \quad (109)$$

It is not difficult to see that we derive (107) from (109) by using the well-known inequality and with equality

$$W_i(K, L)^{n-i} \geq W_i(K)^{n-i-1} W_i(L), \quad (110)$$

with equality if K and L are all homothetic. Although, the precise equality for the Aleksandrov-Fenchel inequality are unknown in general, but combining the equality conditions of (97) and (110), it follows that the equality in (106) holds if φ is strictly convex and if and only if K and L are homothetic. $\square$

Through the above discussion, it is clear that the equality in (18) (stated in the introduction) holds if and only if K and L are homothetic if φ is strictly convex. Lutwak's L_p Minkowski inequality (46) follows immediately from (106) with $\varphi(t) = t^p$ and $p \geq 1$.

There is a quadratic inequality including not only Minkowski first inequality (35), but also Minkowski second inequality (36). This is the celebrated quadratic Aleksandrov-Fenchel inequality, stating that, for compact convex sets $K_1, \dots, K_n$ in $\mathbb{R}^n$,

$$\begin{aligned} V(K_1, K_2, K_3, \dots, K_n)^2 \\ \geq V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n). \end{aligned} \quad (111)$$

The Aleksandrov-Fenchel inequality appears in the work of Aleksandrov [51] and Fenchel and Jessen [52] and is essentially the most powerful inequality of its type known (see [57, p.371]). Here, we give the following Orlicz quadratic Aleksandrov-Fenchel inequality.

Theorem 20 (Orlicz quadratic Aleksandrov-Fenchel inequality). *If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, and $\varphi \in \Phi$, then*

$$\begin{aligned} V_\varphi(K_1, \dots, K_n, L_n) &\geq V(K_1, \dots, K_{n-1}, L_n) \\ &\cdot \varphi\left(\left(\frac{V(K_1, K_1, K_3, \dots, K_n) V(K_2, K_2, K_3, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)^2}\right)^{1/2}\right). \end{aligned} \quad (112)$$

Proof. This follows immediately from (103). $\square$

Obviously, when $\varphi(t) = t^p$ and $p = 1$, (112) becomes (111). On the other hand, putting $L_n = K_n$ and combining ϕ which is increasing function in (112), (112) becomes also (111).

Theorem 21 (Orlicz isoperimetric inequality). *If $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $0 \leq i < n$, then*

$$\frac{V_\varphi(K[n-i-1], B[i+1], K)}{W_i(K)} \geq \varphi\left(\left(\frac{\kappa_n}{W_i(K)}\right)^{1/(n-i)}\right). \quad (113)$$

If φ is strictly convex, it equality holds if and only if K is a ball.

Proof. This follows immediately from (103) with $r = n - i - 1$, $K_1 = \dots = K_{n-i-1} = K$, $K_{n-i} = \dots = K_{n-1} = B$, $L_n = K$, and $K_n = B$. $\square$

When $\varphi(t) = t^p$, $p = 1$, and $i = 0$, Orlicz isoperimetric inequality (113) becomes the classical isoperimetric inequality. If K is convex body, then (see, e.g., [50, p.382])

$$\left(\frac{S(K)}{\omega_n}\right)^n \geq \left(\frac{V(K)}{\kappa_n}\right)^{n-1}, \quad (114)$$

with equality if and only if K is ball.

The well-known Urysohn's inequality states that if K is convex body, then

$$\left(\frac{w(K)}{2}\right)^n \geq \frac{V(K)}{\kappa_n}, \quad (115)$$

with equality if and only if K is ball (see [50, p.382]). Here, $w(K)$ is the mean width of convex body K , denoted by

$$w(K) := \frac{2}{\kappa_n} W_{n-1}(K) = \frac{2}{\kappa_n} V(K, B, \dots, B). \quad (116)$$

In the following, we establish an Orlicz Urysohn's inequality.

Theorem 22 (Orlicz Urysohn's inequality). *If $K \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $0 \leq i < n$, then*

$$\frac{V_\varphi(B[n-1], K, B)}{\kappa_n} \geq \varphi\left(\left(\frac{W_i(K)}{\kappa_n}\right)^{1/(n-i)}\right). \quad (117)$$

If φ is strictly convex, equality holds if and only if K is a ball.

Proof. This follows immediately from (103) with $r = n - i - 1$, $K_1 = \dots = K_{n-1} = B$, $L_n = B$, and $K_n = K$. $\square$

When $\varphi(t) = t^p$ and $p \geq 1$, the Orlicz Urysohn's inequality becomes the following L_p -Urysohn's inequality. If $K \in \mathcal{K}_o^n$ and $p \geq 1$ and $0 \leq i < n$, then

$$\left(\frac{W_{p,i}(B, K)}{\kappa_n}\right)^{(n-i)/p} \geq \frac{W_i(K)}{\kappa_n}. \quad (118)$$

If φ is strictly convex, equality holds if and only if K is a ball. Putting $p = 1$ and $i = 0$ in (118), (118) becomes the Urysohn's inequality (115).

The following uniqueness is a direct consequence of the Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed volumes.

Theorem 23. *If $K_1, \dots, K_n, L_n \in \mathcal{M} \subset \mathcal{K}_o^n$, and $\varphi \in \Phi$ are strictly convex and if either*

$$V_\varphi(K_1, \dots, K_{n-1}, L_n, Q) = V_\varphi(K_1, \dots, K_{n-1}, K_n, Q), \quad (119)$$

for all $Q \in \mathcal{M}$,

or

$$\begin{aligned} & \frac{V_\varphi(K_1, \dots, K_{n-1}, Q, L_n)}{V(K_1, \dots, K_{n-1}, L_n)} \\ &= \frac{V_\varphi(K_1, \dots, K_{n-1}, Q, K_n)}{V(K_1, \dots, K_n)}, \quad \text{for all } Q \in \mathcal{M}, \end{aligned} \quad (120)$$

then $K_n = L_n$.

Proof. Suppose (119) hold. Taking L_n for Q , then from Definition 1 and Lemma 16, we obtain

$$\begin{aligned} V(K_1, \dots, K_{n-1}, L_n) &= V_\varphi(K_1, \dots, K_n, L_n) \\ &\geq V(K_1, \dots, K_{n-1}, L_n) \end{aligned} \quad (121)$$

$$\cdot \varphi\left(\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right)\right),$$

with equality if and only if K_n and L_n are homothetic. Hence

$$1 \geq \varphi\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right), \quad (122)$$

with equality if and only if K_n and L_n are homothetic. Since φ is increasing function on $(0, \infty)$, this follows that

$$V(K_1, \dots, K_n) \leq V(K_1, \dots, K_{n-1}, L_n), \quad (123)$$

with equality if and only if K_n and L_n are homothetic. On the other hand, if taking K_n for Q , we similarly get $V(K_1, \dots, K_n) \geq V(K_1, \dots, K_{n-1}, L_n)$, with equality if and only if K_n and L_n are homothetic. Hence $V(K_1, \dots, K_n) = V(K_1, \dots, K_{n-1}, L_n)$, and K_n and L_n are homothetic; it follows that K_n and L_n must be equal.

Suppose (120) hold. Taking K_n for Q , then from Definition 1 and Lemma 16, we obtain

$$\begin{aligned} 1 &= \frac{V_\varphi(K_1, \dots, K_n, L_n)}{V(K_1, \dots, K_{n-1}, L_n)} \\ &\geq \varphi\left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, L_n)}\right), \end{aligned} \quad (124)$$

with equality if and only if K_n and L_n are homothetic. Since φ is increasing function on $(0, \infty)$, this follows that

$$V(K_1, \dots, K_{n-1}, L_n) \geq V(K_1, \dots, K_n), \quad (125)$$

with equality if and only if K_n and L_n are homothetic. On the other hand, if taking L_n for Q , we similarly get $V(K_1, \dots, K_{n-1}, L_n) \leq V(K_1, \dots, K_n)$, with equality if and only if K_n and L_n are homothetic. Hence $V(K_1, \dots, K_{n-1}, L_n) = V(K_1, \dots, K_n)$, and K_n and L_n are homothetic; it follows that K_n and L_n must be equal. $\square$

Corollary 24. Let $K, L \in \mathcal{M} \subset \mathcal{K}_o^n$, $0 \leq i < n$, and $\varphi \in \Phi$ be strictly convex, and if either

$$W_{\varphi,i}(Q, K) = W_{\varphi,i}(Q, L), \quad \text{for all } Q \in \mathcal{M}, \quad (126)$$

or

$$\frac{W_{\varphi,i}(K, Q)}{W_i(K)} = \frac{W_{\varphi,i}(L, Q)}{W_i(L)}, \quad \text{for all } Q \in \mathcal{M}, \quad (127)$$

then $K = L$.

Proof. This yields immediately from Theorem 23 and Lemma 12. $\square$

6. Orlicz-Brunn-Minkowski Inequality for Mixed Volumes

Theorem 25 (Orlicz-Brunn-Minkowski inequality). If $K_1, \dots, K_n, L_n \in \mathcal{K}_o^n$, and $\varphi \in \Phi$, then for $\varepsilon > 0$

$$1 \geq \varphi \left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, K_n^{+_{\varphi, \varepsilon} L_n})} \right) + \varepsilon \cdot \varphi \left(\frac{V(K_1, \dots, K_{n-1}, L_n)}{V(K_1, \dots, K_{n-1}, K_n^{+_{\varphi, \varepsilon} L_n})} \right), \quad (128)$$

If φ is strictly convex, equality holds if and only if K_n and L_n are homothetic.

Proof. Suppose $K_n, L_n \in \mathcal{K}_o^n$, and let

$$Q = K_n^{+_{\varphi, \varepsilon} L_n}. \quad (129)$$

From Definition 1, (7), (48), and (51), and Lemma 16, we obtain

$$\begin{aligned} V(K_1, \dots, K_{n-1}, Q) &= \frac{1}{n} \\ &\cdot \int_{S^{n-1}} \varphi \left(\frac{h(K_n, u)}{h(Q, u)}, \frac{h(L_n, u)}{h(Q, u)} \right) h(Q, u) \\ &\cdot dS(K_1, \dots, K_{n-1}; u) = \frac{1}{n} \\ &\cdot \int_{S^{n-1}} \left(\varphi \left(\frac{h(K_n, u)}{h(Q, u)} \right) + \varepsilon \cdot \varphi \left(\frac{h(L_n, u)}{h(Q, u)} \right) \right) \\ &\cdot h(Q, u) dS(K_1, \dots, K_{n-1}; u) \end{aligned}$$

$$\begin{aligned} &= V_\varphi(K_1, \dots, K_n, Q) + \varepsilon \cdot V_\varphi(K_1, \dots, K_{n-1}, L_n, Q) \\ &\geq V(K_1, \dots, K_{n-1}, Q) \left\{ \varphi \left(\frac{V(K_1, \dots, K_n)}{V(K_1, \dots, K_{n-1}, Q)} \right) \right. \\ &\quad \left. + \varepsilon \cdot \varphi \left(\frac{V(K_1, \dots, K_{n-1}, L_n)}{V(K_1, \dots, K_{n-1}, Q)} \right) \right\}. \end{aligned} \quad (130)$$

Taking Q for $K_n^{+_{\varphi} L_n}$ in (130), (128) follows.

From the equality condition of Lemma 16, the equality in (130) holds if and only if K_n and Q are homothetic, and L_n and Q are homothetic. This yields that if φ is strictly convex, the equality in (130) holds if and only if K_n and L_n are homothetic. $\square$

Corollary 26. If $K_1, \dots, K_n, L_n \in \mathcal{K}_o^n$, $0 \leq i \leq n$, $1 \leq r \leq n$, and $\varphi \in \Phi$, then for $\varepsilon > 0$

$$1 \geq \varphi \left(\frac{\prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}}{V(K_1, \dots, K_{n-1}, K_n^{+_{\varphi, \varepsilon} L_n})} \right) + \varepsilon \cdot \varphi \left(\frac{\prod_{i=1, i \neq n}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_{n-1}, L_n)^{1/r}}{V(K_1, \dots, K_{n-1}, K_n^{+_{\varphi, \varepsilon} L_n})} \right). \quad (131)$$

Proof. This follows immediately from Theorem 25 combining the Aleksandrov-Fenchel inequality. $\square$

Corollary 27. If $K_1, \dots, K_n, L_n \in \mathcal{K}_o^n$, and $\varphi \in \Phi$, then for $\varepsilon > 0$

$$1 \geq \varphi \left(\left(\frac{V(K_1) \cdots V(K_n)}{V(K_1, \dots, K_{n-1}, K_n^{+_{\varphi, \varepsilon} L_n})^n} \right)^{1/n} \right) + \varepsilon \cdot \left(\left(\frac{V(K_1) \cdots V(K_{n-1}) V(L_n)}{V(K_1, \dots, K_{n-1}, K_n^{+_{\varphi, \varepsilon} L_n})^n} \right)^{1/n} \right). \quad (132)$$

If φ is strictly convex, equality holds if and only if $K_1, \dots, K_n, L_n$ are all homothetic of each other.

Proof. This inequality (132) follows immediately from (131) with $r = n$.

Next, we discuss the equality condition of (132). The precise equality for the Aleksandrov-Fenchel inequality is unknown in general, but the following inequality with equality is clear:

$$V(K_1, \dots, K_n)^n \geq V(K_1) \cdots V(K_n), \quad (133)$$

with equality if and only if $K_1, \dots, K_n$ are all homothetic of each other. Hence combining the equality conditions of (128) and (133), it follows that the equality in (132) holds if φ is strictly convex and if and only if $K_1, \dots, K_n, L_n$ are all homothetic of each other. $\square$

Corollary 28. If $K, L \in \mathcal{K}_o^n$, $\varphi \in \Phi$, and $0 \leq i < n$, then for $\varepsilon > 0$

$$1 \geq \varphi \left(\left(\frac{W_i(K)}{W_i(K +_{\varphi, \varepsilon} L)} \right)^{1/(n-i)} \right) + \varepsilon \cdot \varphi \left(\left(\frac{W_i(L)}{W_i(K +_{\varphi, \varepsilon} L)} \right)^{1/(n-i)} \right). \quad (134)$$

If φ is strictly convex, equality holds if and only if K and L are homothetic.

Proof. This inequality (134) follows immediately from Corollary 26 with $r = n - i - 1$, $K_1 = \dots = K_{n-i-1} = K +_{\varphi, \varepsilon} L$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = K$, and $L_n = L$.

Next, we discuss the equality condition of (134). On the one hand, putting $r = n - i - 1$, $K_1 = \dots = K_{n-i-1} = K +_{\varphi, \varepsilon} L$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = K$, and $L_n = L$ in Theorem 25, we have

$$1 \geq \varphi \left(\left(\frac{W_i(K)}{W_i(K +_{\varphi, \varepsilon} L)} \right)^{1/(n-i)} \right) + \varepsilon \cdot \varphi \left(\left(\frac{W_i(L)}{W_i(K +_{\varphi, \varepsilon} L)} \right)^{1/(n-i)} \right). \quad (135)$$

On the other hand, putting $K_1 = \dots = K_{n-i-1} = K +_{\varphi, \varepsilon} L$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = K$, and $L_n = L$ in Theorem 25, we have

$$1 \geq \varphi \left(\frac{W_i(K +_{\varphi, \varepsilon} L, K)}{W_i(K +_{\varphi, \varepsilon} L)} \right) + \varepsilon \cdot \varphi \left(\frac{W_i(K +_{\varphi, \varepsilon} L, L)}{W_i(K +_{\varphi, \varepsilon} L)} \right), \quad (136)$$

It is not difficult to see that we derive (135) from (136) by using the following inequalities and with equalities:

$$W_i(K +_{\varphi, \varepsilon} L, K)^{n-i} \geq W_i(K +_{\varphi, \varepsilon} L)^{n-i-1} W_i(K), \quad (137)$$

with equality if K and $K +_{\varphi, \varepsilon} L$ are all homothetic, and

$$W_i(K +_{\varphi, \varepsilon} L, L)^{n-i} \geq W_i(K +_{\varphi, \varepsilon} L)^{n-i-1} W_i(L), \quad (138)$$

with equality if $K +_{\varphi, \varepsilon} L$ and L are all homothetic. Although, the precise equality for the Aleksandrov-Fenchel inequality are unknown in general, but combining the equality conditions of (128), (137), and (138), we obtain that the equality in (134) holds if φ is strictly convex, if and only if K and L are homothetic, K and $K +_{\varphi, \varepsilon} L$ are homothetic, and $K +_{\varphi, \varepsilon} L$ and L are homothetic, this shows that the equality in (134) holds if and only if K and L are homothetic, if φ is strictly convex. $\square$

Theorem 29 (L_p -Brunn-Minkowski inequality for the mixed volumes). If $K_1, \dots, K_n, L_n \in \mathcal{K}_o^n$, $0 \leq i < n$, $1 \leq r \leq n$, and $p \geq 1$, then

$$V(K_1, \dots, K_{n-1}, K_n +_p L_n)^p \geq \prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{p/r} + \prod_{i=1, i \neq n}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_{n-1}, L_n)^{p/r}, \quad (139)$$

Proof. This follows immediately from Corollary 26 with $\varphi(t) = t^p$ and $p \geq 1$. $\square$

Putting $r = n - i - 1$, $K_1 = \dots = K_{n-i-1} = K +_p L$, $K_{n-i} = \dots = K_{n-1} = B$, $K_n = K$, and $L_n = L$ in (139), (139) becomes also Lutwak's L_p -Brunn-Minkowski inequality (47). Putting $r = n$ in (139), we get the following interesting result:

$$V(K_1 \cdots K_{n-1}, K_n +_p L_n)^p \geq (V(K_1) \cdots V(K_n))^{p/n} + (V(K_1) \cdots V(K_{n-1}) V(L_n))^{p/n}. \quad (140)$$

with equality if and only if $K_1, \dots, K_n, L_n$ are all homothetic of each other.

Corollary 30. If $K_1, \dots, K_n \in \mathcal{K}_o^n$, $L_n \in \mathcal{K}_{oo}^n$, $1 \leq r \leq n$, and $\varphi \in \Phi$, then

$$V_\varphi(K_1, \dots, K_n, L_n) \geq V(K_1, \dots, K_{n-1}, L_n) \cdot \varphi \left(\frac{\prod_{i=1}^r V(K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}}{V(K_1, \dots, K_{n-1}, L_n)} \right). \quad (141)$$

Proof. Let

$$K_\varepsilon = L_n +_{\varphi, \varepsilon} K_n. \quad (142)$$

From (8), (69), and (131), we obtain

$$\begin{aligned} \frac{1}{\varphi'_t(1)} \cdot V_\varphi(K_1, \dots, K_n, L_n) &= \frac{d}{d\varepsilon} \Big|_{\varepsilon=0^+} V(K_1, \dots, K_{n-1}, K_\varepsilon) \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{V(K_1, \dots, K_{n-1}, K_\varepsilon) - V(K_1, \dots, K_{n-1}, L_n)}{\varepsilon} \\ &= \lim_{\varepsilon \rightarrow 0^+} \frac{1 - V(K_1, \dots, K_{n-1}, L_n)/V(K_1, \dots, K_{n-1}, K_\varepsilon)}{1 - \varphi(V(K_1, \dots, K_{n-1}, L_n)/V(K_1, \dots, K_{n-1}, K_\varepsilon))} \\ &\quad \cdot \frac{1 - \varphi(V(K_1, \dots, K_{n-1}, L_n)/V(K_1, \dots, K_{n-1}, K_\varepsilon))}{\varepsilon} \\ &\quad \cdot V(K_1, \dots, K_{n-1}, K_\varepsilon) \end{aligned}$$

$$\begin{aligned}
&= \lim_{t \rightarrow 0^+} \frac{1-t}{\varphi(1) - \varphi_1(t)} \\
&\quad \cdot \lim_{\varepsilon \rightarrow 0^+} \frac{1 - \varphi(V(K_1, \dots, K_{n-1}, L_n) / V(K_1, \dots, K_{n-1}, K_\varepsilon))}{\varepsilon} \\
&\quad \cdot \lim_{\varepsilon \rightarrow 0^+} V(K_1, \dots, K_{n-1}, K_\varepsilon) \\
&\geq \frac{1}{\varphi'_l(1)} \cdot \varphi \left(\frac{V(K_1, \dots, K_{n-1}, K_n)}{V(K_1, \dots, K_{n-1}, L_n)} \right) \cdot V(K_1, \dots, K_{n-1}, L_n) \\
&\geq \frac{1}{\varphi'_l(1)} \cdot \varphi \left(\frac{\prod_{i=1}^r (K_i, \dots, K_i, K_{r+1}, \dots, K_n)^{1/r}}{V(K_1, \dots, K_{n-1}, L_n)} \right) \\
&\quad \cdot V(K_1, \dots, K_{n-1}, L_n).
\end{aligned} \tag{143}$$

From (143), (141) easy follows. This proof is complete. $\square$

Finally, it is worth mentioning that we may see that the Orlicz-Aleksandrov-Fenchel inequality for the Orlicz multiple mixed volumes implies Orlicz-Brunn-Minkowski inequality for the mixed volumes, from the proof of Theorem 25. This combines Corollary 30, it yields that the Orlicz-Aleksandrov-Fenchel inequality is equivalent to the Orlicz-Brunn-Minkowski inequality.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that he has no conflicts of interest.

Authors' Contributions

Chang-Jian Zhao provided the questions and gave the proof for the all results.

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