Research Article

On Fourth-Order Elliptic Equations of Kirchhoff Type with Dependence on the Gradient and the Laplacian

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We consider a nonlocal fourth-order elliptic equation of Kirchhoff type with dependence on the gradient and Laplacian

\[ \Delta^2 u - (a + b \int \Omega |\nabla u|^2 \, dx) \Delta u = f(x, u, \nabla u, \Delta u), \text{ in } \Omega, \]

\[ u = 0, \]

\[ \Delta u = 0, \text{ on } \partial \Omega, \]

where \( a \) and \( b \) are positive constants. We will show that there exists \( b^* > 0 \) such that the problem has a nontrivial solution for \( 0 < b < b^* \) through an iterative method based on the mountain pass lemma and truncation method developed by De Figueiredo et al., 2004.

1. Introduction

This paper concerns with the existence of solutions of the fourth-order Kirchhoff type elliptic equation as follows:

\[ \Delta^2 u - \left( a + b \int \Omega |\nabla u|^2 \, dx \right) \Delta u = f(x, u, \nabla u, \Delta u), \text{ in } \Omega, \]

\[ u = 0, \]

\[ \Delta u = 0, \text{ on } \partial \Omega, \]

where \( \Omega \) is a bounded and smooth domain in \( \mathbb{R}^N \) (\( N \geq 5 \)), \( a \), \( b \) are positive constants, and \( f : \Omega \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is locally Lipschitz continuous.

The fourth-order elliptic equation

\[ \Delta^2 u - a \Delta u = f(x, u), \text{ in } \Omega, \]

\[ u = 0, \]

\[ \Delta u = 0, \text{ on } \partial \Omega, \]

arises in the study of traveling waves in suspension bridges, which has been extensively investigated in recent years, such as [1–6]. To our attention, some authors paid more attention to a more general biharmonic elliptic problem

\[ \Delta^2 u + q \Delta u + \alpha(x) u = f(x, u, \Delta u, \nabla u), \text{ in } \Omega, \]

\[ u(x) = 0, \]

\[ \Delta u(x) = 0, \text{ on } \partial \Omega. \]

For this problem, due to the presence of \( \Delta u \) and \( \nabla u \) in \( f \), it is not variational. To overcome this difficulty, in [5], Wang deals with this problem via the upper and lower solutions and monotone iterative methods; in [7], the authors apply a technique developed by De Figueiredo et al. [8, 9] in studying a second-order elliptic problem involving the gradient, which “freezes” the gradient, and use truncation on the nonlinearity \( f \). Thus the new problem becomes variational and an iterative scheme of the mountain pass “approximated” solutions is built.

In addition, the nonlocal fourth-order equation

\[ \Delta^2 u - M \left( \int \Omega |\nabla u|^2 \, dx \right) \Delta u = f(x, u), \text{ in } \Omega, \]

\[ u = 0, \]

is considered here.
\[
\Delta u = 0, \quad \text{on } \partial \Omega \tag{4}
\]

has also been studied by many authors. We refer the readers to [10–20]. Particularly, Wang et al. studied the following fourth-order equation of Kirchhoff type equation
\[
\Delta^2 u - \lambda \left( a + b \int_{\Omega} |\nabla u|^2 \, dx \right) \Delta u = f(x, u), \quad \text{in } \Omega, \quad u = 0, \quad \Delta u = 0, \quad \text{on } \partial \Omega, \tag{5}
\]

where \( \lambda \) is a positive parameter. The authors showed that there exists \( \lambda^* \) such that the fourth-order elliptic equation has a nontrivial solution for \( 0 < \lambda < \lambda^* \) by using the mountain pass iterative techniques and the truncation method.

Motivated by these works, to study problem (1), we combine the famous mountain pass lemma with a technique developed by De Figueiredo et al. [8], which “freezes” the gradient and the Laplacian variable and use truncation on the nonlinearity of \( f \). For convenience, we recall a definition and restate the mountain pass theorem.

**Definition 1.** Let \( X \) be a real Banach space and \( I : X \to R \) a \( C^1 \)-functional. A sequence \( \{u_n\} \) in \( X \) is a (PS)-sequence for \( I \) if \( I(u_n) \to C \) for some constant \( C \geq 0 \) as \( n \to \infty \), while \( \langle I'(u_n), u_n \rangle \to 0 \) as \( n \to \infty \). We say that the functional \( I \) satisfies the (PS)-condition if any (PS)-sequence for \( I \) has a convergent subsequence.

**Theorem A** (mountain pass lemma). Let \( E \) be a real Banach space; \( I \in C^1(E, R) \) satisfying (PS)-condition. Suppose the following:

1. There exist \( \rho > 0, \alpha > 0 \) such that
   \[
   I|_{B_{\rho}} \geq I(0) + \alpha, \tag{6}
   \]
   where \( B_{\rho} = \{u \in E \mid \|u\| \leq \rho \} \).
2. There is \( e \in E \) and \( \|e\| > \rho \) such that
   \[
   I(e) \leq I(0). \tag{7}
   \]

Then \( I(u) \) has a critical value \( c \) which can be characterized as
\[
C = \inf_{\gamma \in \Gamma} \max_{\gamma(1)} I(u), \tag{8}
\]
where \( \Gamma = \{ \gamma \in C([0, 1], E) \mid \gamma(0) = 0, \gamma(1) = e \} \).

**2. The Main Result**

**Theorem 2.** Assume that the function \( f \) satisfies the following conditions:

\((f_1)\) \( f : \Omega \times R \times R^N \times R \to R \) is locally Lipschitz continuous, and there exist \( d_1 > 0, 1 < p < (N + 4)/(N - 4) \) which satisfy
\[
r = r_1 + r_2 < 2 \quad \text{such that}
\]
\[
|f(x, t, \xi_1, \xi_2)| \leq d_1 (1 + |t|^p) (1 + |\xi_1|)^{r_1} (1 + |\xi_2|)^{r_2}, \tag{9}
\]
for all \( (x, t, \xi_1, \xi_2) \in \Omega \times R^{N+2} \).
\[(f_2)\] \( \lim_{t \to -\infty} |f(x, t, \xi_1, \xi_2)|/t = 0 \) uniformly with respect to \( x \in \Omega, \xi_1 \in R^N \) and \( \xi_2 \in R \).
\[(f_3)\] There exist \( \Theta > 2 \) and \( t_1 > 0 \) such that
\[
0 < \Theta F(x, t, \xi_1, \xi_2) \leq tf(x, t, \xi_1, \xi_2), \tag{10}
\]
\[\forall |t| \geq t_1, x \in \Omega, (\xi_1, \xi_2) \in R^{N+1}, \]
where \( F(x, t, \xi_1, \xi_2) = \int_1^t f(x, s, \xi_1, \xi_2) \, ds \).
\[(f_4)\] There exist positive constants \( \rho_i > 0 \) (\( i = 1, 2, 3 \)) depending on \( a, b, \Theta, \) and \( d_1 \) such that \( L_{\rho_i} (i = 1, 2, 3) \) satisfy
\[
\left( \tau_2^2 L_{\rho_1} + \tau_1 \tau_2 L_{\rho_2} + \tau_1 \tau_3 L_{\rho_3} \right) < \min \{1, a\}, \tag{11}
\]
where
\[
L_{\rho_1} = \sup_{x \in \Omega, |t'|, |t''| \leq \rho_1, |\xi_1| \leq \rho_2, |\xi_2| \leq \rho_3} \left\{ \frac{|f(x, t', \xi_1, \xi_2) - f(x, t'', \xi_1, \xi_2)|}{|t' - t''|} \right\}, \tag{12}
\]
\[
L_{\rho_2} = \sup_{x \in \Omega, |t| \leq \rho_1, |\xi_1'| \leq \rho_2, |\xi_2'| \leq \rho_3} \left\{ \frac{|f(x, t, \xi_1, \xi_2') - f(x, t, \xi_1, \xi_2)|}{|\xi_2'|} \right\}, \tag{13}
\]
\[
L_{\rho_3} = \sup_{x \in \Omega, |t| \leq \rho_1, |\xi_1| \leq \rho_2, |\xi_1'| \leq \rho_3} \left\{ \frac{|f(x, t, \xi_1, \xi_2') - f(x, t, \xi_1, \xi_2)|}{|\xi_2'|} \right\}, \tag{14}
\]
and \( \tau_i (i = 1, 2, 3) \) are the optimal constants of the following inequalities:
\[
\left( \int_{\Omega} |u|^p \, dx \right)^{1/2} \leq \tau_1 \|u\|, \tag{15}
\]
\[
\left( \int_{\Omega} |\nabla u|^p \, dx \right)^{1/2} \leq \tau_2 \|u\|, \tag{16}
\]
\[
\left( \int_{\Omega} |u|^p \, dx \right)^{1/2} \leq \tau_3 \|u\|, \tag{17}
\]
where \( \|\cdot\| \) is the norm of the Hilbert space \( X = H^1(\Omega) \cap H_0^1(\Omega) \) defined by
\[
\|u\|^2 = \int_{\Omega} |\Delta u|^2 \, dx + \int_{\Omega} |\nabla u|^2 \, dx. \tag{18}
\]

Then there exists \( b^* > 0 \) such that (1) has at least a non-trivial solution for \( 0 < b < b^* \).
For each \( \omega \in X \) and \( R > 0 \), we study the following “truncate” and “freezed” problem
\[
\Delta^2 u^R_\omega - \Phi_R(V\omega) \Delta u^R_\omega = f_R(x, u^R_\omega, V\omega, \Delta\omega), \quad \text{in } \Omega,
\]
\[
u^R_\omega = 0, \quad \Delta u^R_\omega = 0,
\]on \( \partial\Omega \),
where
\[
f_R(x, t, \nu\omega, \Delta\omega) = f(x, t, \nu\omega\phi_R(\nu\omega), \Delta\omega\phi_R(\Delta\omega)),
\]
\[
\Phi_R(\nu\omega) = a + b \int_\Omega |\nu\omega\phi_R(\nu\omega)|^2 \, dx,
\]
\[
\phi_R(\xi) \in C^1(\mathbb{R}, \mathbb{R}) \text{ satisfies } |\phi_R| \leq 1, \quad \text{and}
\]
\[
\phi_R(\xi) = \begin{cases} 1, & \text{if } |\xi| \leq R, \\ 0, & \text{if } |\xi| \geq R + 1. \end{cases}
\]
The associated functional \( J^R_\omega : X \to \mathbb{R} \) is defined as
\[
J^R_\omega(u^R_\omega) = \frac{1}{2} \int_\Omega \Delta u^R_\omega^2 \, dx + \frac{1}{2} \Phi_R(V\omega) \int_\Omega |\nabla u^R_\omega|^2 \, dx
- \int_\Omega F_R(x, u^R_\omega, V\omega, \Delta\omega) \, dx,
\]
where
\[
F_R(x, t, \nu\omega, \Delta\omega) = \int_0^t f_R(x, s, \nu\omega, \Delta\omega) \, ds.
\]

Lemma 3. Let \( R > 0 \) and \( \omega \in X \) be fixed. Then
(1) there exist constants \( \rho > 0 \), \( \alpha > 0 \) such that \( J^R_\omega(v) \big|_{B_R} \geq \alpha \) with \( B_\rho = \{v \in X : ||v|| \leq \rho\} \);
(2) for fixed \( v \) with \( ||v|| = 1 \), \( J^R_\omega(tv) \to -\infty \) as \( t \to +\infty \).

Proof. On one hand, by \( (f_j) \), for any \( \epsilon > 0 \), there exists a constant \( \sigma > 0 \) such that, for \( |t| < \sigma \), one has
\[
F_R(x, t, \nu\omega, \Delta\omega) \leq \frac{1}{2} \epsilon \sigma^2.
\]
On the other hand, if \( |t| > \sigma \), from \( (f_j) \) it follows that there exists \( C_1 > 0 \) such that
\[
F_R(x, t, \nu\omega, \Delta\omega) \leq C_1 |t|^{p+1} (R + 2)^\gamma.
\]
Then, from (20), (21), and the Sobolev inequality, we have
\[
J^R_\omega(v) = \frac{1}{2} \int_\Omega |\Delta v|^2 \, dx + \frac{1}{2} \Phi_R(V\omega) \int_\Omega |\nabla v|^2 \, dx
- \int_\Omega F_R(x, V\omega, \Delta\omega) \, dx
\geq \frac{1}{2} \int_\Omega |\Delta v|^2 \, dx + \frac{1}{2} \Phi_R(V\omega) \int_\Omega |\nabla v|^2 \, dx
- \frac{\epsilon}{2} \int_\Omega |v|^2 \, dx - C_1 (R + 2)^\gamma \int_\Omega |v|^{p+1} \, dx
\geq \left( \frac{\min \{1, \alpha\}}{2} - \frac{\epsilon C}{2} \right) ||v||^2 - C (R + 2)^\gamma ||v||^{p+1}
\]
for some positive constant \( C \). Therefore, for sufficiently small \( \epsilon > 0 \), we can choose \( \rho > 0 \) and \( \alpha > 0 \) such that the first result of Lemma 3 holds.

Now, we show that \( (f_j) \) implies that there exist \( a_2, a_3 > 0 \) such that
\[
F_R(x, t, \xi_1, \xi_2) \geq a_2 |t|^{\theta} - a_3,
\]
\[
\forall x \in \Omega, \ t \in \mathbb{R}, \xi_1 \in \mathbb{R}^N, \xi_2 \in \mathbb{R}.
\]
In fact, from \( (f_j) \), we have \( f(x, t, \xi_1, \xi_2)/F(x, t, \xi_1, \xi_2) \geq \theta/t \), for any \( |t| \geq t_1 \). Being integral from \( t_1 \) to \( t \), we get
\[
F(x, t, \xi_1, \xi_2) - \ln F(x, t_1, \xi_1, \xi_2) \geq \theta (\ln t - \ln t_1);
\]
namely,
\[
F(x, t, \xi_1, \xi_2) \geq \frac{F(x, t_1, \xi_1, \xi_2) t_1^\theta}{t^\theta}, \quad \forall |t| \geq t_1.
\]
Then
\[
F_R(x, t, \xi_1, \xi_2) = F(x, t, \xi_1 \phi_R(\xi_1), \xi_2 \phi_R(\xi_2)) \geq \frac{|t|^{\theta}}{t_1^\theta} F(x, t_1, \xi_1 \phi_R(\xi_1), \xi_2 \phi_R(\xi_2)).
\]
Let
\[
a_2 = \frac{1}{\int_{R \geq \xi \in \mathbb{R}^{N+1} x \in \xi = R} \min F(x, t_1, \xi_1 \phi_R(\xi_1), \xi_2 \phi_R(\xi_2))
= \frac{1}{\int_{R \geq \xi \in \mathbb{R}^{N+1} x \in \xi = R+1} \min F(x, t_1, \xi_1, \xi_2) > 0,
\]
\[
a_3 = \max_{x \in \Omega, |t| \geq t_1, \xi_1 \in \mathbb{R}, \xi_2 \in \mathbb{R+1} F(x, t_1, \xi_1, \xi_2) > 0,
\]
and then inequality (23) holds.

Taking an arbitrary \( v \in X \) with \( ||v|| = 1 \), then from (23), we get
\[
J^R_\omega(tv) = \frac{1}{2} \int_\Omega |\Delta vt|^2 \, dx + \frac{1}{2} \Phi_R(V\omega) \int_\Omega |\nabla vt|^2 \, dx
- \int_\Omega F_R(x, t, V\omega, \Delta\omega) \, dx
\leq \frac{1}{2} \int_\Omega |\Delta vt|^2 \, dx + \frac{1}{2} \Phi_R(V\omega) \int_\Omega |\nabla vt|^2 \, dx
- a_2 t^{\theta} \int_\Omega |v|^\theta \, dx + a_3 |\Omega|
\leq \max \left[ 1, \Phi_R(V\omega) \right] \frac{1}{2} ||v||^2 - a_2 t^{\theta} \int_\Omega |v|^\theta \, dx
+ a_3 |\Omega| \longrightarrow -\infty, \quad \text{as } t \longrightarrow +\infty,
\]
which implies that the second result of Lemma 3 holds.

Lemma 4. Let \( R > 0 \) and \( \omega \in X \) be fixed. Then the functional \( J^R_\omega(\cdot) \) satisfies the (PS)-condition.
Proof. Let \( \{u_n\} \subset X \) be a \((PS)\)-sequence; namely,
\[
\lim_{n \to \infty} j^R_\omega(u_n) = C,
\]
\[
\lim_{n \to \infty} \langle j^R_\omega(u_n), u_n \rangle = 0
\]
as \( n \to \infty \).

From the standard processes, we only need to prove that \( \{u_n\} \) is bounded in \( X \). On a contradiction, suppose that \( \|u_n\| \to +\infty \); then, from \((f_3)\), we obtain
\[
j^R_\omega(u_n) - \frac{1}{\Theta} \langle j^R_\omega(u_n), u_n \rangle = \frac{1}{2} \int_\Omega |\Delta u_n|^2 \, dx + \frac{1}{2} \Phi_R(\nabla u_n) \int_\Omega |\nabla u_n|^2 \, dx \\
- \int_\Omega F_R(x, u_n, \nabla \omega, \Delta \omega) \, dx - \frac{1}{\Theta} \int_\Omega |\Delta u_n|^2 \, dx \\
- \frac{1}{\Theta} \Phi_R(\nabla u_n) \int_\Omega |\nabla u_n|^2 \, dx + \frac{1}{\Theta} \Phi_R(\nabla \omega) \int_\Omega |\nabla u_n|^2 \, dx \\
- \int_\Omega f_R(x, u_n, \nabla \omega, \Delta \omega) u_n \, dx = \left( \frac{1}{2} - \frac{1}{\Theta} \right) \min \{1, a\} \|u_n\|^2 + \int_\Omega \left[ \frac{1}{\Theta} f_R(x, u_n, \nabla \omega, \Delta \omega) u_n \\
- F_R(x, u_n, \nabla \omega, \Delta \omega) \right] \, dx \geq \left( \frac{1}{2} - \frac{1}{\Theta} \right) \min \{1, a\} \|u_n\|^2.
\]

On the other hand, from \((29)\) we know that
\[
j^R_\omega(u_n) - \frac{1}{\Theta} \langle j^R_\omega(u_n), u_n \rangle \leq C + C \|u_n\|.
\]
Then, from the above inequalities, we get
\[
\left( \frac{1}{2} - \frac{1}{\Theta} \right) \min \{1, a\} \|u_n\|^2 \leq C + C \|u_n\|, \tag{32}
\]
which contradicts with \( \|u_n\| \to +\infty \). Therefore the sequence \( \{u_n\} \) is bounded in \( X \). \( \square \)

**Lemma 5.** For any \( R > 0 \) and \( \omega \in X \), problem \((15)\) has a nontrivial weak solution.

**Proof.** By Theorem A, Lemmas 3, and 4, the result holds. \( \square \)

**Lemma 6.** Let \( R > 0 \) be fixed. Then there exist positive constants \( \gamma_1 \) and \( \gamma_2 = \gamma_2(R) \), independent of \( \omega \), such that
\[
\gamma_1 \leq \|u^R_\omega\| \leq \gamma_2
\]
for every solution \( u^R_\omega \) obtained in Lemma 5.

Proof. Firstly, since \( J^R_\omega(u^R_\omega) \leq \max_{t \geq 0} J^R_\omega(tv) \), from \((23)\) it follows that
\[
J^R_\omega(tv) \leq \frac{\max \{1, \Phi_R\}}{2} \|v\|^2 - a_2 t^\theta \int_\Omega |v|^\theta \, dx \\
+ a_3 |\Omega| \tag{34}
\]
\[
\leq \frac{\max \{1, [a + b (R + 1)]^2 |\Omega|\}}{2} t^2 \\
- a_2 t^\theta \int_\Omega |v|^\theta \, dx + a_3 |\Omega|.
\]
As \( \Theta > 2 \), we can get a \( C > 0 \) such that \( J^R_\omega(u^R_\omega) \leq C \); that is,
\[
\frac{1}{2} \int_\Omega |\Delta u^R_\omega|^2 \, dx + \frac{1}{2} \Phi_R(\nabla \omega) \int_\Omega |\nabla u^R_\omega|^2 \, dx \\
\leq C + \int_\Omega F_R(x, u^R_\omega, \nabla \omega, \Delta \omega) \, dx.
\]
Define \( D = \{ x \in \Omega : |u^R_\omega| > t_1 \} \), where \( t_1 \) is defined in \((f_3)\). Then we get
\[
\frac{1}{2} \int_\Omega |\Delta u^R_\omega|^2 \, dx + \frac{1}{2} \Phi_R(\nabla \omega) \int_\Omega |\nabla u^R_\omega|^2 \, dx \\
\leq C + \int_\Omega F_R(x, u^R_\omega, \nabla \omega, \Delta \omega) \, dx \\
= C + \int_{\Omega \setminus D} F_R(x, u^R_\omega, \nabla \omega, \Delta \omega) \, dx \\
+ \int_{D} F_R(x, u^R_\omega, \nabla \omega, \Delta \omega) \, dx \tag{36}
\]
\[
\leq C + a_1 \left( t_1 + \frac{|t_1|^{p+1}}{p+1} \right) \ominus (R + 2) T |\Omega \setminus D| \\
+ \frac{1}{\Theta} \int_\Omega |\Delta u^R_\omega|^2 \, dx + \frac{1}{\Theta} \Phi_R(\nabla \omega) \int_\Omega |\nabla u^R_\omega|^2 \, dx.
\]
Furthermore, we have
\[
\left( \frac{1}{2} - \frac{1}{\Theta} \right) \min \{1, a\} \|u^R_\omega\|^2 \\
\leq C + a_1 \left( t_1 + \frac{|t_1|^{p+1}}{p+1} \right) \ominus (R + 2) T |\Omega \setminus D| \tag{37}
\]
\[
\leq C (R + 2) T,
\]
where \( C \) is independence of \( b, R > 0 \), and \( \omega \in H \). Therefore, \( \|u^R_\omega\| \leq \gamma_2 \), for some \( \gamma_2 = \gamma_2(R) > 0 \).

Secondly, from \((f_1)\) and \((f_3)\), given \( \epsilon > 0 \), there exists \( C_\epsilon > 0 \) such that
\[
f_R(x, u^R_\omega, \nabla \omega, \Delta \omega) \leq \epsilon \|u^R_\omega\| + C_\epsilon (R + 2) T \|u^R_\omega\|^p. \tag{38}
\]
Since \( J^R_\omega(u^R_\omega) = 0 \), it is easy to obtain that
\[
\min \{1, a\} \|u^R_\omega\|^2 \leq C_\epsilon \|u^R_\omega\|^2 + C_3 C_\epsilon \|u^R_\omega\|^{p+1}. \tag{39}
\]
for some constants $C_2, C_3 \geq 0$. Therefore, there exists $\gamma_1 > 0$ such that $\|u_n^R\| \geq \gamma_1$. \hfill \square

**Lemma 7** (see [7]). **Let** $R$ **be fixed, and choose** $\omega \in C^{k\alpha}(\Omega)$ **for** $\alpha \in (0, 1)$. **If** $u_n^R \in X$ **is a weak solution of problem (15), then** $u_n^R \in C^{k\beta}(\Omega)$ **for some** $\beta \in (0, 1)$, **and** $\Delta (u_n^R)(x) = 0$ **if** $x \in \partial \Omega$.

**Lemma 8.** **There exist three constants** $\eta_i > 0$ **(i = 1, 2, 3), independent of** $b, \omega$, **and** $R$, **such that**

$$
\begin{align*}
\|u_n^R\|_{C^0} &\leq \eta_1 (R + 2)^{\gamma/2}, \\
\|\nabla u_n^R\|_{C^0} &\leq \eta_2 (R + 2)^{\gamma/2}, \\
\|\Delta u_n^R\|_{C^0} &\leq \eta_3 (R + 2)^{\gamma/2}.
\end{align*}
$$

In addition, there exists $\overline{R}$ **such that**

$$
\eta_i (R + 2)^{\gamma/2} \leq \overline{R} \quad (i = 1, 2, 3).
$$

**Proof.** From (37) and the proof of Lemma 6, there exists $C > 0$, independent of $R > 0$ and $\omega \in X$, **such that**

$$
\|u_n^R\| \leq C (R + 2)^{\gamma/2}.
$$

Then by Lemma 7 and the Sobolev embedding theorem, the inequalities in the lemma are as follows. In addition, since $r/2 < 1$ **and** $\lim_{s \to \infty} (s + 2)^{\gamma/2} / s = 0$, there exists a sufficiently large $\overline{R} > 0$ **such that** $\theta_i (R + 2)^{\gamma/2} \leq \overline{R}$. \hfill \square

Now let $u_n^R$ **(n = 1, 2, ...) be the weak solution of the following problem:**

$$
\begin{align*}
\Delta^2 u_n^R - \Phi_R \left( \nabla u_{n-1}^R \right) \Delta u_n^R &= f_R \left( x, u_n^R, \nabla u_{n-1}^R, \Delta u_{n-1}^R \right), \\
\text{in } \Omega, \\
i_n^R &= 0, \\
\Delta u_n^R &= 0, \\
on \partial \Omega
\end{align*}
$$

with $\omega = u_{n-1}^R$, **where** $u_{n-1}^R$ **was found in Lemma 5 and** $R = \overline{R}$ **obtained in Lemma 8. From Lemmas 6–8, we have** $u_n^R \in C^2(\Omega)$ **satisfying** $\|u_n^R\| \geq \gamma_1$ **and**

$$
\|u_n^R\|_{C^0}, \|\nabla u_n^R\|_{C^0}, \|\Delta u_n^R\|_{C^0} \leq \overline{R}.
$$

Thus

$$
\Phi_R \left( \nabla u_{n-1}^R \right) = a + b \int_{\Omega} \left| \nabla u_{n-1}^R \right|^2 dx,
$$

$$
f_R \left( x, u_n^R, \nabla u_{n-1}^R, \Delta u_{n-1}^R \right) = f \left( x, u_n^R, \nabla u_{n-1}^R, \Delta u_{n-1}^R \right).
$$

**Lemma 9.** **Assume that** $(f_2)$ **holds. Let**

$$
\rho_1 = \inf \left\{ \sigma_1 : \|u_n^R\|_{C^0} \leq \sigma_1, \forall n \in N^+ \right\} > 0,
$$

$$
\rho_2 = \inf \left\{ \sigma_2 : \|\nabla u_n^R\|_{C^0} \leq \sigma_2, \forall n \in N^+ \right\} > 0,
$$

$$
\rho_3 = \inf \left\{ \sigma_3 : \|\Delta u_n^R\|_{C^0} \leq \sigma_3, \forall n \in N^+ \right\} > 0.
$$

Then $\{u_n^R\}$ **strongly converges in** $X$.

**Proof.** Let $u_{n+1}^R$ **and** $u_n^R$ **be the weak solutions of (43) with** $\omega = u_{n+1}^R$ **and** $\omega = u_{n-1}^R$, **respectively. Then, multiplying (43) by $u_{n+1}^R - u_n^R$, we obtain**

$$
\min \{1, a\} \|u_{n+1}^R - u_n^R\|^2 \leq \int_{\Omega} \left[ f \left( x, u_{n+1}^R, \nabla u_{n+1}^R, \Delta u_{n+1}^R \right) - f \left( x, u_n^R, \nabla u_n^R, \Delta u_n^R \right) \right] (u_{n+1}^R - u_n^R) dx
$$

$$
+ b \left( \int_{\Omega} |\nabla u_{n+1}^R|^2 dx \right) \int_{\Omega} \Delta u_{n+1}^R \cdot \Delta u_{n+1}^R dx
$$

$$
- b \left( \int_{\Omega} |\nabla u_n^R|^2 dx \right) \int_{\Omega} \Delta u_n^R \cdot \Delta u_n^R dx.
$$

Furthermore, by $(f_1), (f_2), and the Hölder inequality, we have

$$
\int_{\Omega} \left[ \left( f \left( x, u_{n+1}^R, \nabla u_{n+1}^R, \Delta u_{n+1}^R \right) - f \left( x, u_n^R, \nabla u_n^R, \Delta u_n^R \right) \right) (u_{n+1}^R - u_n^R) dx
$$

$$
= \int_{\Omega} \left( f \left( x, u_{n+1}^R, \nabla u_n^R, \Delta u_{n+1}^R \right) - f \left( x, u_n^R, \nabla u_n^R, \Delta u_n^R \right) \right) (u_{n+1}^R - u_n^R) dx
$$

$$
+ \int_{\Omega} \left( f \left( x, u_n^R, \nabla u_{n+1}^R, \Delta u_{n+1}^R \right) - f \left( x, u_n^R, \nabla u_n^R, \Delta u_n^R \right) \right) (u_{n+1}^R - u_n^R) dx
$$

$$
+ \int_{\Omega} \left( f \left( x, u_n^R, \nabla u_n^R, \Delta u_{n+1}^R \right) - f \left( x, u_n^R, \nabla u_n^R, \Delta u_n^R \right) \right) (u_{n+1}^R - u_n^R) dx
$$

$$
\leq 2 \rho_1 \|u_{n+1}^R - u_n^R\|^2 + 2 \rho_2 \|u_{n+1}^R - u_n^R\|^2 + 2 \rho_3 \|u_{n+1}^R - u_n^R\|^2
$$

$$
\leq 2 \rho_1 \|u_{n+1}^R - u_n^R\|^2 + 2 \rho_2 \|u_{n+1}^R - u_n^R\|^2 + 2 \rho_3 \|u_{n+1}^R - u_n^R\|^2
$$

$$
\leq 2 \rho_1 \|u_{n+1}^R - u_n^R\|^2 + 2 \rho_2 \|u_{n+1}^R - u_n^R\|^2 + 2 \rho_3 \|u_{n+1}^R - u_n^R\|^2
$$

$$
\leq b \rho_1 \|u_{n+1}^R - u_n^R\|^2 + b \rho_2 \|u_{n+1}^R - u_n^R\|^2 + b \rho_3 \|u_{n+1}^R - u_n^R\|^2
$$

$$
\leq 2 b \rho_1 \rho_2 \rho_3 \|u_{n+1}^R - u_n^R\|^2.
$$
Now, choosing $c = 1/2$ in (47), we obtain

\[
\min \{1, a\} \|u_{n+1}^R - u_n^R\|^2 \\
\leq \tau_1^2 L_{p_1} \|u_{n+1}^R - u_n^R\|^2 + \tau_1 \tau_2 L_{p_2} \|u_{n+1}^R - u_n^R\| + \|u_{n+1}^R - u_n^R\| + 2\beta p_2 \tau_1 \tau_2 |\Omega| \|u_n^R - u_n^R\|.
\]

Hence, by (47) and (48), we get

\[
\min \{1, a\} \|u_{n+1}^R - u_n^R\|^2 \\
\leq \tau_1^2 L_{p_1} \|u_{n+1}^R - u_n^R\|^2 + \tau_1 \tau_2 L_{p_2} \|u_{n+1}^R - u_n^R\| + \|u_{n+1}^R - u_n^R\| + 2\beta p_2 \tau_1 \tau_2 |\Omega| \|u_n^R - u_n^R\|.
\]

\[
\|u_{n+1}^R - u_n^R\| \\
\leq \frac{\tau_1 \tau_2 L_{p_2} + \tau_1 \tau_2 L_{p_2} + 2\beta p_2 \tau_1 \tau_2 |\Omega|}{\min \{1, a\} - L_{p_1} \tau_1^2} \|u_n^R - u_n^R\|. 
\]

Now, choosing $b^* = (\min\{1, a\} - L_{p_1} \tau_1^2)/2\rho_2 \tau_1 \tau_2 |\Omega|$, then

\[
\tau_1 \tau_2 L_{p_2} + \tau_1 \tau_2 L_{p_2} + 2\beta p_2 \tau_1 \tau_2 |\Omega| < 1,
\]

for $0 < b < b^*$.

Therefore, $\{u_n^R\}$ converges strongly in $X$.

**Proof of Theorem 2.** Firstly, from Lemma 6, we get $\|u_n^R\| \geq \nu_1 > 0$, and $\|u_n^R\|_{C^2}, \|\nabla u_n^R\|_{C^2}, \|\Delta u_n^R\|_{C^2}$ are uniformly bounded. Secondly, set $v_n = \Delta u_n^R$; then

\[
\Delta v_n = h(x) \\
= f(x, u_n^R, \nabla u_n^R, \Delta u_n^R) \\
+ (a + b) \int_{\Omega} |\nabla u_n^R|^2 \, dx \Delta u_n^R.
\]

Since $\|h\|_{C^0} \leq C$, for some positive constant $C$, by the Schauder theorem, there exists a constant $C_0 > 0$ such that $\|v_n\|_{C^0} \leq C_0$; that is, $\|u_n^R\|_{C^0} \leq C_0$. Furthermore, by the Arzela-Ascoli theorem and Lemma 9, the sequence $\{u_n^R\}$ satisfies

\[
\frac{\partial^{j}}{\partial x_i^j} u_n^R(x) \rightarrow \frac{\partial^{j}}{\partial x_i^j} u^R(x), \quad \text{as } n \rightarrow \infty
\]

uniformly in $\Omega$ for $j = 0, 1, 2, 3, 4$ and $i = 1, \ldots, N$. Finally, passing to the limit in (43), we obtain that $u^R(x)$ is a classical solution of (1).

**Example 10.** Consider the following problem:

\[
\Delta^2 u - (a + b) \int_{\Omega} |\nabla u|^2 \, dx \Delta u = f(x, u, \nabla u, \Delta u),
\]

in $\Omega$, $u = 0$, $\Delta u = 0$, on $\partial \Omega$, \hspace{1in} (54)

where $f(x, t, \xi_1, \xi_2) = \alpha(x)|t|t(1 + |\xi_1|)^{1/2}(1 + |\xi_2|)^{3/4} + \beta(x)t^3$ and $\alpha(x)$ and $\beta(x)$ are positive and continuous functions. It is easy to verify that $f(x, t, \xi_1, \xi_2)$ satisfies all the conditions of $(f_1)$–$(f_4)$.

\section{3. Conclusion}

The paper considers a class of fourth-order elliptic equations of Kirchhoff type with dependence on the gradient and Laplacian. The existence of a nontrivial solution of (1) is established when we choose appropriate $b^*$ such that $0 < b < b^*$. The paper generalized the conclusions in [7, 14] and weakened the condition in [7]. In the following research work, we will also consider problem (1), but we just truncate the right side of the equation, and the left of the equation remains the same.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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**References**


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