Research Article

On the Generalized Reflective Function of the Three-Degree Polynomial Differential System

Jian Zhou and Shiyin Zhao

School of Literature and Science, Suqian College, Suqian 223800, China

Correspondence should be addressed to Shiyin Zhao; 52660078@qq.com

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1. Introduction

As we know, it is very important to study the properties of the solution of differential system

\[ x' = X(t, x) \]  

(1)

for both the theory and application of an ordinary differential equation. If \( X(t + 2\omega, x) = X(t, x) \) (\( \omega \) is a positive constant), we can use the Poincaré mapping introduced in [1] to study the behavior of the solutions of (1). But it is very difficult to find the Poincaré mapping for many systems which cannot be integrated in quadratures. In the 1980s, the Russian mathematician Mironenko first established the theory of reflective functions (RF) in [2, 3]. Since then a quite new method to study (1) has been found. In recent years, more and more experts and scholars have achieved many good results; refer to [4–9] in this direction.

The aim of this article is to use the theory of the generalized reflective function to study the behavior of the solutions of differential systems. The obtained results will provide a new theoretical basis and criterion to further explain the laws of the movement of objects.

In the present section, we first recall some basic notions and results of the generalized reflective function (GRF), which will be used throughout the rest of this article.

Now consider system (1) with a continuous differentiable right-hand side and with a general solution \( \varphi(t; t_0, x_0) \). For each such system, the GRF of system (1) is defined as

**\[ F(t, x) := \varphi(\alpha(t); t, x), \quad (t, x) \in D \subset R \times R^n, \]**

where \( \alpha(t) \) is a continuous differentiable function such that \( \alpha(\alpha(t)) = t, \alpha(0) = 0 \). Then for any solution \( x(t) \) of (1), we have \( F(t, x(t)) = x(\alpha(t)) \) and \( F(\alpha(t), F(t, x)) \equiv F(0, x) \equiv x \). By the definition in [3], a continuous differentiable vector function \( F(t, x) \) on \( R \times R^n \) is called GRF if and only if it is a solution of the Cauchy problem

**\[ F_t (t, x) + F_x (t, x) X(t, x) = \alpha' (t) X(\alpha(t), F(t, x)), \]**

\[ F(0, x) = x. \]  

(2)

Relation (2) is called a basic relation (BR).

Besides this, suppose that system (1) is \( 2\omega \)–periodic with respect to the variable \( t \), and \( F \) is its GRF; if there exists a number \( \eta \) on \( R \) such that \( \alpha(\eta) = 2\omega + \eta \), then \( T(x) = F(\eta, x) = \varphi(\alpha(\eta); \eta, x) \) is the Poincaré mapping of (1) over the period \( [\eta, \eta + 2\omega] \). So, for any solution \( x(t) \) of (1) defined on \( [\eta, \eta + 2\omega] \), it will be \( 2\omega \)–periodic if and only if \( F(\eta, x) = x \) which is called a basic lemma (BL).

Now, we consider the higher dimensional polynomial differential system

**\[ x' = a_1 + a_2 y + a_3 z \equiv A(t, x, y, z) \]**

**\[ y' = b_1 + b_2 y + b_3 z + b_4 y^2 + b_5 y z + b_6 z^2 \equiv B(t, x, y, z) \]**


\[ z' = c_1 + c_2 y + c_3 z + c_4 y^2 + c_5 y z + c_6 z^2 \]
\[ = C(t, x, y, z), \]  
(3)

where \( a_i = a_i(t, x), b_j = b_j(t, x), \) and \( c_j = c_j(t, x) \) (\( i = 1, 2, 3; \) \( j = 1, 2, ..., 6 \)) are continuously differentiable functions in \( R^2 \), \( a_2^2 + a_3^2 \neq 0 \) (in some deleted neighborhood of \( t = 0 \) and \(|t| \) being small enough but different from zero), and there exists a unique solution for the initial value problem of (3). And suppose that

\[ F(t, x, y, z) \]
\[ = (F_1(t, x, y, z), F_2(t, x, y, z), F_3(t, x, y, z))^T \]  
(4)

is the GRF of (3).

In this article, we discuss the structure of \( F_i(t, x, y, z) \) \( (i = 2, 3) \) when \( F_1(t, x, y, z) = f(t, x) \) and obtain the good results \( F_i(t, x, y, z) = f_{i1}(t, x) + f_{i2}(t, x) y + f_{i3}(t, x) z \) \((i = 2, 3)\) which are useful for the research of the existence of periodic solutions and establishing the sufficient conditions of system (3) with the form of GRF.

In the following, we denote

\[ \bar{a}_i = a_i(\alpha(t), x); \]
\[ \bar{b}_j = b_j(\alpha(t), x); \]
\[ \bar{c}_j = c_j(\alpha(t), x); \]
\[ \alpha = \alpha(t); \]
\[ F_i = F_i(t, x, y, z) \]  
(5)

\( (i = 1, 2, 3; j = 1, 2, ..., 6) \). The notation "\( a_i(t, x) \neq 0 \)" means that, in some deleted neighborhood of \( t = 0 \), \( a_i(t, x) \) is small enough but different from zero.

2. Main Results

Without loss of generality, we suppose that \( f(t, x) = x \). Otherwise, we can take the transformation \( \xi = f(t, x), \eta = y, \zeta = z \).

Now, we consider system (3).

**Lemma 1.** For system (3), let \( F_1 = x, \alpha'(0) = -1 \); then

\[ a_i(0, x) = 0, \quad i = 1, 2, 3. \]  
(6)

**Proof.** Using relation (2), we can get

\[ A(t, x, y, z) - \alpha'(t) A(\alpha, x, F_2, F_3) = 0; \]  
(7)

i.e.,

\[ a_1 - \alpha' \bar{a}_1 + a_2 y + a_3 z - \alpha' \bar{a}_2 F_2 - \alpha' \bar{a}_3 F_3 = 0. \]  
(8)

Putting \( t = 0 \), we have \( a_1(0, x) + a_2(0, x) y + a_3(0, x) z = 0 \), \( \forall x, y, z \), which implies that relation (6) is valid.

In the following discussion, we always assume (6) holds without further mentioning.

**Case 1** \((a_3 \neq 0)\). From relation (8), we get

\[ F_3 = l_1 + l_2 F_2, \]  
(9)

where

\[ l_1 = l_{11} + l_{12} y + l_{13} z, \]
\[ l_{11} = \frac{a_1 - \alpha' \bar{a}_1}{\alpha' \bar{a}_3}, \]  
(10)
\[ l_{12} = \frac{a_2}{\alpha' \bar{a}_3}, \]
\[ l_{13} = \frac{a_3}{\alpha' \bar{a}_3}, \]
\[ l_2 = -\frac{\bar{a}_2}{\alpha' \bar{a}_3}. \]

Differentiating relation (9) with respect to \( t \) implies

\[ \sum_{j=0}^{2} p_j F_j^2 = 0, \]  
(11)

where

\[ P_0 = p_0 y + p_0 y z + p_0 y z^2 + p_0 z^2 + p_0 z^2, \]
\[ p_{01} = \alpha' \left[ \bar{c}_1 - l_{12} \bar{b}_1 + l_{11} (\bar{c}_3 - l_{12} \bar{b}_3) + l_{12}^2 (\bar{c}_6 - l_{12} \bar{b}_6) \right] \]
\[ - (l_{12} t + l_{12} \bar{a}_1 + l_{12} \bar{b}_1 + l_{13} \bar{c}_1), \]
\[ p_{02} = \alpha' \left[ l_{12} (\bar{c}_3 - l_{12} \bar{b}_3) + 2 l_{11} l_{12} (\bar{c}_6 - l_{12} \bar{b}_6) \right] - (l_{12} t) \]
\[ + l_{11} \bar{a}_2 + l_{12} \bar{a}_1 + l_{12} \bar{b}_1 + l_{13} \bar{c}_2, \]
\[ p_{03} = \alpha' \left[ l_{13} (\bar{c}_6 - l_{12} \bar{b}_6) + 2 l_{11} l_{13} (\bar{c}_6 - l_{12} \bar{b}_6) \right] - (l_{13} t) \]
\[ + l_{11} \bar{a}_3 + l_{13} \bar{a}_1 + l_{13} \bar{b}_1 + l_{13} \bar{c}_3, \]
\[ p_{04} = \alpha' l_{12}^2 \left( \bar{c}_6 - l_{12} \bar{b}_6 \right) - (l_{12} t r_a + l_{12} b_4 + l_{13} c_4), \]
\[ p_{05} = 2 \alpha' l_{12} l_{13} (\bar{c}_6 - l_{12} \bar{b}_6) - (l_{12} t \bar{a}_3 + l_{13} \bar{a}_2 + l_{13} \bar{b}_2 \]
\[ + l_{13} \bar{c}_4), \]
\[ p_{06} = \alpha' l_{12}^3 \left( \bar{c}_6 - l_{12} \bar{b}_6 \right) - (l_{13} t \bar{a}_3 + l_{12} b_6 + l_{13} c_6), \]
\[ p_{11} = p_{11} t + p_{12} y + p_{13} z, \]
\[ p_{11} = \alpha' \left[ \bar{c}_2 - l_{12} \bar{b}_2 + l_{12} (\bar{c}_3 - l_{12} \bar{b}_3) + l_{11} (\bar{c}_6 - l_{12} \bar{b}_6) \right] \]
\[ + 2 l_{12} \bar{a}_1 + l_{12} \bar{a}_1 + l_{12} \bar{b}_1 + l_{13} \bar{c}_1, \]
\[ p_{12} = \alpha' \left[ l_{12} (\bar{c}_6 - l_{12} \bar{b}_6) + 2 l_{12} l_{12} (\bar{c}_6 - l_{12} \bar{b}_6) \right] - l_{12} t \bar{a}_2, \]
\[ p_{13} = \alpha' \left[ l_{13} (\bar{c}_6 - l_{12} \bar{b}_6) + 2 l_{12} l_{13} (\bar{c}_6 - l_{12} \bar{b}_6) \right] - l_{12} t \bar{a}_3, \]
\[ p_{12} = \alpha' \left[ \bar{c}_4 - l_{12} \bar{b}_4 + l_{12} (\bar{c}_6 - l_{12} \bar{b}_6) + l_{12}^2 (\bar{c}_6 - l_{12} \bar{b}_6) \right]. \]
Lemma 2. For system (3), suppose $P_2 \neq 0$, $F_1 = x$ and the limit $\lim_{\gamma \to \infty}(p_0 / P_2)$ ($i = 1, 2, 3, 6$) exist; then

$$\lim_{t \to 0} \frac{p_{0i}}{P_2} = 0 \quad (j = 1, 3, 6),$$

$$\lim_{t \to 0} \frac{p_{00} + p_{11}}{P_2} = 0,$$

$$\lim_{t \to 0} \frac{p_{05} + p_{15}}{P_2} = 0,$$

$$\lim_{t \to 0} \frac{p_{04} + p_{12}}{P_2} = -1.$$ (13)

Proof. Using relation (11), we have $\lim_{t \to 0} F_2 = 0$, i.e.

$$\lim_{t \to 0} F_2^2 + \lim_{t \to 0} \frac{P_{11} + P_{12}y + P_{13}z}{P_2} F_2$$

$$+ \lim_{t \to 0} \frac{P_{01} + P_{02}y + P_{03}z + P_{04}y^2 + P_{05}yz + P_{06}z^2}{P_2} = 0,$$ (14)

by $F_2(0, x, y, z) = y$, which implies that the results of Lemma 2 are true.

Theorem 3. For system (3), suppose that $\lim_{t \to 0}(p_{12} / P_2) + 2 > 0$ and all the conditions of Lemmas 1 and 2 are satisfied; then

$$F_i(t, x, y, z) = f_{i0}(t, x) + f_{i1}(t, x) y + f_{i2}(t, x) z$$

(i = 2, 3). (15)

Proof. As $P_2 \neq 0$, by relation (11), we have

$$F_2^2 = \frac{P_1}{P_2} F_2 - \frac{P_0}{P_2},$$

$$F_2^3 = \frac{P_1}{P_2} F_2^2 - \frac{P_0}{P_2} F_2 = \left( \frac{P_1^2}{P_2^2} - \frac{P_0}{P_2} \right) F_2 + \frac{P_0}{P_2} P_1 F_2.$$ (16)

Differentiating relation (11) with respect to $t$, we can obtain

$$DP_0 + (DP_1) F_2 + \alpha' P_1 B(\alpha, x, F_2, F_3) + (DP_2) F_2^2$$

$$+ 2\alpha' P_3 B(\alpha, x, F_2, F_3) F_2 = 0;$$ (17)

Substituting (9) into the above, we get

$$\sum_{i=0}^{3} Q_i F_2^i = 0,$$ (18)

where

$$Q_0 = DP_0 + \alpha' \mu_1 P_1,$$

$$Q_1 = DP_1 + \alpha' \mu_2 P_1 + 2\alpha' \mu_1 P_2,$$

$$Q_2 = DP_2 + \alpha' \mu_3 P_1 + 2\alpha' \mu_2 P_2,$$

$$Q_3 = 2\alpha' \mu_3 P_3;$$ (19)

$$\mu_1 = b_1 + l_1 b_3 + l_1^2 b_6,$$

$$\mu_2 = b_2 + l_2 b_3 + l_2^2 b_6 + l_2 l_1 b_6,$$

$$\mu_3 = b_4 + l_2 b_5 + l_2^2 b_6,$$

$$DP_1 = \frac{\partial P_1}{\partial t} + \frac{\partial P_1}{\partial x} A(t, x, y, z) + \frac{\partial P_1}{\partial y} B(t, x, y, z)$$

$$+ \frac{\partial P_1}{\partial z} C(t, x, y, z) \quad (i = 0, 1, 2).$$

Substituting (16) into (18), we have

$$R_0 + R_1 F_2 = 0,$$ (20)

in which

$$R_0 = Q_0 - Q_2 \frac{P_0}{P_2} + Q_3 \frac{P_1 P_3 P_2}{P_2^2},$$

$$R_1 = Q_1 - Q_2 \frac{P_1}{P_2} - Q_3 \left( \frac{P_1}{P_2} \frac{P_3}{P_2} - \frac{P_0}{P_2} \right).$$ (21)

(1) If $R_1 \neq 0$, we get $F_2 = -R_0 / R_1$ by relation (20). Through the expression of $P_1, Q_1$ ($i = 0, 1, 2; j = 0, 1, 2, 3$), we know that $R_1$ is a quadratic polynomial with respect to $y, z$ and $R_0$ is a cubic polynomial with respect to $y, z$. Substituting $F_2 = -R_0 / R_1$ into (11), we obtain $P_2 R_0^3 = -R_0 (P_1 R_1 - P_2 R_0)$ which implies that $R_1 = R_0$ or $R_2 = P_2$ and $F_2 = \sum_{i+j=0} f_{2i}(t, x) y^i z^j$. Substituting the relations into (11) and equating the coefficients of the same powers of $y$ and $z$, we have $f_{2i}(t, x) = 0, i + j > 1$. Finally, we arrive at

$$F_i(t, x, y, z) = f_{i0}(t, x) + f_{i1}(t, x) y + f_{i2}(t, x) z$$

(i = 2, 3). (22)

(2) If $R_1 = 0$, it follows $R_0 = 0$ from (20). By simple computation, we obtain

$$DP_0 = -\alpha' \mu_1 P_1 + 2\alpha' \mu_2 P_0 - \alpha' \mu_3 P_0 P_1,$$ (23)

$$DP_2 = -2\alpha' \mu_1 + \alpha' \mu_2 P_1 - \alpha' \mu_3 (P_1^2 - \frac{P_0}{P_2})$$ (24)

Let $\Delta = P_1^2 / P_2^2 - 4P_0 / P_2$; we have

$$D\Delta = \frac{\Delta}{P_2^2} = \frac{2^2 P_1 D P_1 - 4 P_1}{P_2^2}$$

$$= 2\alpha' \left( \mu_2 - \mu_3 \frac{P_1}{P_2} \right) \Delta$$ (25)

by (23) and (24).
Since
\[
\Delta = \frac{{P_1^2}}{P_2^2} - \frac{{4P_0}}{{P_2}} = \frac{1}{{P_2^2}} \left( {\left( {P_{11} + P_{12}y + P_{13}z} \right)^2} - 4P_2 \left( {P_{01} + P_{02}y + P_{03}z + P_{04}y^2 + P_{05}y} \right) \right.
\]
\[
+ \left. {P_{06}y^2} \right) = s_1 + s_2y + s_3 + s_4y^2 + s_5yz + s_6z^2
\]
\[
= s_4\left( {y + \frac{s_5}{{2s_4}}z + \frac{s_6}{{2s_4}}} \right)^2 + w(t, x, z),
\]
where
\[
\begin{align*}
  s_1 &= \frac{1}{{P_2^2}} \left( {P_{11}^2 - 4P_2P_{01}} \right), \\
  s_2 &= \frac{1}{{P_2^2}} \left( {2P_{11}P_{12} - 4P_2P_{02}} \right), \\
  s_3 &= \frac{1}{{P_2^2}} \left( {2P_{11}P_{13} - 4P_2P_{03}} \right), \\
  s_4 &= \frac{1}{{P_2^2}} \left( {P_{12}^2 - 4P_2P_{04}} \right), \\
  s_5 &= \frac{1}{{P_2^2}} \left( {2P_{12}P_{13} - 4P_2P_{05}} \right), \\
  s_6 &= \frac{1}{{P_2^2}} \left( {P_{13}^2 - 4P_2P_{06}} \right); \\
  w(t, x, z) &= w_0 + w_1z + w_2z^2.
\end{align*}
\]

Using identity (25), we get
\[
\begin{align*}
  w_i + w_x A \left( {t, x, -\frac{s_5}{{2s_4}}z - \frac{s_2}{{2s_4}}, z} \right) \\
  + w_z C \left( {t, x, -\frac{s_5}{{2s_4}}z - \frac{s_2}{{2s_4}}, z} \right) = 0.
\end{align*}
\]

By the uniqueness of solutions of the initial problem of linear partial differential equations, we have \( w = 0 \). Therefore
\[
\Delta = s_4\left( {y + \frac{s_5}{{2s_4}}z + \frac{s_6}{{2s_4}}} \right)^2.
\]

So, using relation (11), we get
\[
F_2 = \left( {s_5 \frac{1}{{4s_4}} - \frac{P_1}{{2P_2}}} \right) + \left( {\frac{1}{2} \sqrt{s_4}y + \frac{s_5}{{4s_4}} \sqrt{s_4}z} \right) = f_{20}(t, x) + f_{21}(t, x) y + f_{22}(t, x) z.
\]

Finally, we get
\[
F_3 = l_1 + l_2 F_2 = f_{30}(t, x) + f_{31}(t, x) y + f_{32}(t, x) z
\]
by relation (9).

Summarizing the above, the proof is finished. \( \square \)

**Theorem 4.** For system (3), let \( F_i(t, x, y, z) = x, P_1 \neq 0, P_2 = 0, \) and \( a_i(0, x) = 0 \quad (i = 1, 2, 3); \) then
\[
F_2(t, x, y, z) = -\frac{P_0}{P_1}
\]
\[
= -\frac{P_{01} + P_{02}y + P_{03}z + P_{04}y^2 + P_{05}y + P_{06}z^2}{P_{11} + P_{12}y + P_{13}z},
\]
\[
F_3 = l_1 + l_2 F_2.
\]

**Case II** \( (a_3 = 0, a_2 \neq 0). \) From relation (8), we have
\[
F_2 = \delta_1 + \delta_2 y,
\]
where
\[
\delta_1 = \frac{a_1 - a'_1}{a_2}, \quad \delta_2 = \frac{a_2}{a_2}.
\]

Differentiating this identity with respect to \( t \) gives
\[
\sum_{i=0}^{2} V_i F_3^i = 0.
\]
where
\[ V_0 = v_{01} + v_{02}y + v_{03}z + v_{04}y^2 + v_{05}yz + v_{06}z^2, \]
\[ v_{01} = \alpha'(b_1 + \delta_1b_2 + \delta_2z_4) - (\delta_1' + \delta_1'a_1 + \delta_2b_1), \]
\[ v_{02} = \alpha'(b_2 + 2\delta_2d_3), \]
\[ v_{03} = -\delta_1'a_3 - \delta_2b_3, \]
\[ v_{04} = -\delta z_4 - \delta z_6, \]
\[ v_{05} = -\delta z_2 + y_2 + \delta_2z_4, \]
\[ v_{06} = -\delta z_2, \]
\[ V_j = v_{11} + v_{12}y, \]
\[ V_{12} = \alpha(b_3 + d_3b_5), \]
\[ v_{12} = \alpha\delta_2d_5, \]
\[ V_2 = \alpha\delta d_6. \]

Likewise, we have the following conclusions.

**Lemma 5.** Suppose \( F_1 = x, a_3 \equiv 0, a_2 \neq 0, b_k \neq 0, \) and the limits \( \lim_{t \to 0}(v_{01}/b_6) \) \((i = 1, 2, ..., 5) \) and \( \lim_{t \to 0}(a_1/\alpha a_2) \) exist; then
\[
\lim_{t \to 0} \frac{v_{0j}}{\alpha a_2} = 0 \quad (j = 1, 2, 4),
\]
\[
\lim_{t \to 0} \frac{v_{01} + v_{11}}{\alpha a_2} = 0,
\]
\[
\lim_{t \to 0} \frac{v_{02} + v_{12}}{\alpha a_2} = 0,
\]
\[
\lim_{t \to 0} \frac{a_1 - \alpha a_3}{\alpha a_2} = 0,
\]
\[
\lim_{t \to 0} \frac{a_2}{\alpha a_2} = \lim_{t \to 0} \frac{b_k}{\alpha a_2} = 1.
\]

**Theorem 6.** Suppose that all the conditions of Lemmas 1 and 5 are satisfied; then
\[
F_2 = \delta_1(t, x) + \delta_2(t, x) y,
\]
\[
F_3 = f_{31}(t, x) + f_{32}(t, x) y + f_{33}(t, x) z.
\]

**Theorem 7.** Suppose that \( F_1 = x, v_{11} \neq 0, v_{12} \neq 0, b_k = 0, a_k(0, x) = 0 \) \((i = 1, 2); \) then
\[
F_2 = \delta_1(t, x) + \delta_2(t, x) y,
\]
\[
F_3(t, x, y, z) = \frac{V_0}{V_1},
\]
\[
= -\frac{v_{01} + v_{02}y + v_{03}z + v_{04}y^2 + v_{05}yz + v_{06}z^2}{v_{11} + v_{12}y}.
\]

**Theorem 8.** Function \( F(t, x, y, z) = (x, f_{21}y + f_{22}z, g_{21}y + g_{22}z)^T \) is the GRF of system (3), if the following conditions are satisfied:
\[
a_2 - a_3 \alpha f_{21} - \alpha f_{23}g_{21} = 0,
\]
\[
a_3 - a_3 \alpha f_{22} - \alpha f_{23}g_{22} = 0
\]
\[
\left(\begin{array}{c}
(f_{21}f_{22})_b + (f_{22}g_{22})_z c_1 - \alpha' (\bar{x}, \bar{z}) = 0,
(f_{21}f_{22})_g + (f_{22}g_{22})_x a_4
\end{array}\right),
\]
\[
\left(\begin{array}{c}
(f_{21}f_{22})_a + (f_{22}g_{22})_c
\end{array}\right) = 0.
\]

**Theorem 9.** Let the hypotheses of Theorem 8 be satisfied, system (3) is \( 2\omega \)-periodic with respect to \( t \) and there exists a number \( \eta \in R \) such that \( \alpha(\eta) = 2\omega + \eta; \) then all the solutions of system (3) defined on the interval \([\eta, \eta + 2\omega]\) are \( 2\omega \)-periodic if and only if \( f_{21}(2\omega + \eta, x) = 1, f_{22}(2\omega + \eta, x) = 0, g_{21}(2\omega + \eta, x) = 0, \) and \( g_{22}(2\omega + \eta, x) = 1. \)
Proof. By Theorem 8, the Poincaré mapping of system (3) is
\[ T(x, y, z) = F(\alpha(\eta), x, y, z) = (x, f_{21}(2\omega + \eta, x) y + f_{22}(2\omega + \eta, x) z, g_{21}(2\omega + \eta, x) y + g_{22}(2\omega + \eta, x) z)^T. \]

According to the previous introduction, the solutions of system (3) are \(2\pi\)-periodic if and only if \(T(x, y, z) \equiv (x, y, z)^T\), which implies that the assertions of the present theorem hold. The proof is finished. \(\square\)

Example 10. The differential system
\[
\begin{align*}
x' &= (1 - e^{\sin t} (1 - x^3 \sin t)) y - x^2 z \sin t \\
y' &= \frac{1}{2} y \cos t (x^6 \sin^2 t + x^3 - 1) - \frac{1}{2} x^2 z \cos t e^{-\sin t} (1 + \sin t + x^3 \sin^2 t) \\
&\quad + y^2 (3x^2 \sin t - x^5 \sin^2 t) \\
&\quad + yze^{-\sin t} (x^4 \sin^4 t - 2x \sin t) \\
z' &= \frac{1}{2} x^4 y \cos t e^{\sin t} (1 - \sin t + x^3 \sin^2 t) \\
&\quad + (x^6 \sin^2 t + x^3 - 1) + \frac{1}{2} z \cos t (t - x^3 - x^6 \sin^2 t) \\
&\quad - y^2 e^{\sin t} (x^6 \sin^2 t - 4x^3 \sin t) \\
&\quad - yze^{-\sin t} (3x^2 \sin^2 t - x^5 \sin^2 t)
\end{align*}
\]
has GRF \(F(t, x, y, z) = (x, F_2, F_3)^T\), in which
\[
\begin{align*}
F_2(t, x, y, z) &= (1 + \alpha(t) x^3 \sin t) ye^{\sin t} \\
&\quad - \alpha(t) x^2 z \sin t, \\
F_3 &= (1 - \alpha(t) x^3 \sin t) ze^{-\sin t} \\
&\quad + \alpha(t) x^4 y \sin t, \quad \alpha(t) = -t.
\end{align*}
\]
Since this system is a \(2\pi\)-periodic system, and there exists a number \(\eta = -\pi\) in \(R\) such that \(\alpha(\eta) = 2\pi + \eta\), then \(T(x) = F(-\pi, x) = x\); by Theorem 9, all the considered solutions defined on \([-\pi, \pi]\) are \(2\pi\)-periodic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

References
