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Research Article

Effect of the Domain Geometry on the Solutions to Fractional Brezis-Nirenberg Problem

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In this paper, we consider the Brezis-Nirenberg problem for the nonlocal fractional elliptic equation $\mathscr{A}_\alpha u(x) = N(N-2\alpha)u(x)^p + \varepsilon u(x)$, $x \in \Omega$, $u(x) > 0$, $x \in \Omega$, $u(x) = 0$, $x \in \partial\Omega$, where $0 < \alpha < 1$ is fixed, $p = (N+2\alpha)/(N-2\alpha)$, ε is a small parameter, and Ω is a bounded smooth domain of $\mathbb{R}^N (N \geq 4\alpha)$. $\mathscr{A}_\alpha$ denotes the fractional Laplace operator defined through the spectral decomposition. Under some geometry hypothesis on the domain Ω, we show that all solutions to this problem are least energy solutions.

1. Introduction and Main Results

In the famous paper of Brezis and Nirenberg [1], they studied the following nonlinear critical elliptic partial differential equation:

$$\begin{aligned} -\Delta u(x) &= N(N-2)\,u(x)^{(N+2)/(N-2)} + \varepsilon u(x), \quad x \in \Omega, \\ u(x) &> 0, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{1}$$

where Ω is a bounded domain in $\mathbb{R}^N (N \geq 3)$. They proved that problem (1) has a positive nontrivial solution provided $N \geq 4$ and $\varepsilon \in (0, \lambda_1)$, where λ_1 is the first eigenvalue of $-\Delta$ in Ω. This result was extended by Capozzi et al. [2] for every parameter ε. Rey [3] and Han [4] established the asymptotic behavior of positive solutions to problem (1) by different methods independently.

In this paper, we study the following nonlocal Brezis-Nirenberg problem:

$$\begin{aligned} \mathscr{A}_\alpha u(x) &= N(N-2\alpha)\,u(x)^p + \varepsilon u(x), \quad x \in \Omega, \\ u(x) &> 0, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega, \end{aligned} \tag{2}$$

where $0 < \alpha < 1$ and $4\alpha < N$, $p = (N+2\alpha)/(N-2\alpha)$, Ω is a bounded smooth domain of $\mathbb{R}^N$, and $\mathscr{A}_\alpha$ is the spectral fractional Laplacian defined in terms of the spectra of the $-\Delta$ in Ω; for more details see Section 2.

The qualitative properties of solutions to problem (2), such as existence, nonexistence, and multiplicity results, were widely studied; see [5–11] and references therein. It is well-known that [5] problem (2) has at least one positive solution for each small $\varepsilon > 0$. On the other hand, when Ω is a $C^{1,1}$ domain, by the Pohozaev identity [12, 13], we know that problem (2) does not have any solutions in a star-shaped domain when $\varepsilon = 0$. Consequently, the solutions to problem (2) blow up at some points as $\varepsilon \longrightarrow 0$. Therefore, as $\varepsilon \longrightarrow 0$, there exist subsequences $\varepsilon_i \longrightarrow 0$, u_i, x_i and a point $x_0 \in \Omega$, such that $x_i \longrightarrow x_0$ and $\|u_i(x_i)\| := \|u_i(x_i)\|_{L^\infty} \longrightarrow \infty$ as $i \longrightarrow \infty$. In the following, we mainly consider the solution u_i.

Based on fractional harmonic extension formula of Caffarelli and Silvestre [14] (see Cabré and Tan [15] also), Choi et al. [9] studied the asymptotic behavior of least energy solutions u to problem (2); that is, u satisfies

$$\lim_{\varepsilon \longrightarrow 0} \frac{\int_\Omega \left|\mathscr{A}_\alpha^{1/2} u\right|^2 dx}{\left(\int_\Omega |u|^{p+1}\, dx\right)^{2/(p+1)}} = \mathcal{S}_{N,\alpha}, \tag{3}$$

where $\mathcal{S}_{N,\alpha}$ is the best constant in fractional Sobolev inequality. For some other related results, see [6, 8, 16] and references therein.

The main goal of this paper is to show that, under some hypothesis on the domain Ω, all solutions to problem (2) automatically satisfy (3); that is, all solutions are least energy solutions.

Our main result is the following.

Theorem 1. *Let Ω be a smooth bounded domain of $\mathbb{R}^N(N > 4\alpha)$, which is symmetric with respect to the coordinate hyperplanes $\{x_k = 0\}$ and convex in the x_k−directions for $k = 1; 2 \cdots ; N$. Then all solutions to problem (2) are least energy solutions.*

Remark 2. According to Theorem 1.3 in [9], we know that there exist a point $x_0 \in \Omega$ and a family of solutions to (2), which blow up and concentrate at the point x_0 as $\varepsilon \longrightarrow 0$. Without loss of generality we assume that $x_0 = 0 \in \Omega$ in this paper.

Remark 3. This results are motivated by the work of Cerqueti and Grossi [17] about the classical Brezis-Nirenberg problem

$$\begin{aligned} -\Delta u(x) &= N(N-2)\,u(x)^{(N+2)/(N-2)} + \varepsilon u(x), \\ &\qquad x \in \Omega, \\ u(x) &> 0, \quad x \in \Omega, \\ u(x) &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{4}$$

In [17], they obtained the asymptotical behavior of any solution to the above equation in a neighborhood of the origin. Furthermore, uniqueness and nondegeneracy result for the solutions also obtained.

The paper is organized as follows. Section 2 contains some notations and definitions. Section 3 is concerned with the proof of Theorem 1.

2. Useful Definitions

First of all, in this section we recall some basic properties of the spectral fractional Laplacian.

In this paper, the letter C will denote a positive constant, not necessarily the same everywhere. Let Ω be a smooth bounded domain of $\mathbb{R}^N$. $\lambda_k > 0$, $k = 1, 2 \cdots$, are the eigenvalues of the Dirichlet Laplacian on Ω, and ϕ_k are the corresponding normalized eigenfunctions; namely,

$$\begin{aligned} \Delta\phi_k(x) &= \lambda_k\phi_k(x), \quad x \in \Omega, \\ \phi_k(x) &= 0, \quad x \in \partial\Omega. \end{aligned} \tag{5}$$

Define the fractional Laplacian $\mathscr{A}_\alpha : H_0^\alpha(\Omega) \longrightarrow H_0^{-\alpha}(\Omega)$ as

$$\mathscr{A}_\alpha\left(\sum_{k=1}^{\infty} a_k\phi_k\right) = \sum_{k=1}^{\infty} a_k\lambda_k^\alpha\phi_k, \tag{6}$$

where fractional Sobolev space $H_0^\alpha(\Omega)(0 < \alpha < 1)$ is defined as

$$H_0^\alpha(\Omega) = \left\{ f = \sum_{k=1}^{\infty}\lambda_k\phi_k \in L^2(\Omega) : \sum_{k=1}^{\infty}\lambda_k^2\phi_k^2 < \infty \right\}. \tag{7}$$

It is interesting to note that another very popular "integral" fractional Laplacian is defined as

$$\begin{aligned} (-\Delta)^s u(x) &= P.V. \int_{\mathbb{R}^N} \frac{u(x) - u(y)}{|x-y|^{n+2s}}\,dy \\ &= \lim_{\varepsilon\longrightarrow 0} \int_{\mathbb{R}^N\setminus B_\varepsilon(x)} \frac{u(x)-u(y)}{|x-y|^{n+2s}}\,dy, \end{aligned} \tag{8}$$

up to a normalization constant which will be omitted for brevity. For differences between the spectral fractional Laplacian (6) and the fractional Laplacian (8), see [18–20].

In this paper, we mainly consider some properties of isolated blow-up point.

Definition 4. Suppose that u_i is a solution to problem (2). The point $\overline{x} \in \Omega$ is called a blow-up point of $\{u_i\}$, if there exists a sequence of point $x_i \in \Omega$, such that $x_i \longrightarrow \overline{x}$ and $u_i(x_i) \longrightarrow \infty$.

The concept of an isolated blow-up point was first introduced by Schoen; for more details of the definition of isolated blow-up point, see [17, 21].

Definition 5. Let u_i be a solution to problem (2). A point $\overline{x} \in \Omega$ is an isolated blow-up point of $\{u_i\}$ if there exist $0 < \overline{r} < dist(\overline{x}, \partial\Omega), C > 0$, and a sequence $x_i \longrightarrow \overline{x}$, such that x_i is a local maximum point of u_i, $u_i(x_i) \longrightarrow \infty$, and for any $x \in B_{\overline{r}}(x_i)$

$$u_i(x) \le C\,|x - x_i|^{-2\alpha/(p-1)}. \tag{9}$$

3. Proof of Theorem 1

In this section, we give the proof of Theorem 1, which will be divided into three lemmas.

Firstly, according to Remark 2, we know that $x = 0$ is the blow-up point; in view of Definition 4, there exists a sequence of point $x_i \in \Omega$ and $\{u_i\}$, such that $x_i \longrightarrow 0$ and $u_i(x_i) \longrightarrow \infty$. In the following, the index i is omitted for the sake of simplicity. The following lemma shows that $x = 0$ is an isolated blow-up point.

Lemma 6. *Let $u(x)$ be a solution to problem (2). Then there exists a constant $C = C(N)$ such that*

$$u(x) \le \frac{C}{|x|^{(N-2\alpha)/2}}, \quad x \in \Omega. \tag{10}$$

Proof. Suppose, on the contrary, there exists a $x_j \in \Omega$, such that

$$\begin{aligned} M_j := u(x_j)\,|x_j|^{(N-2\alpha)/2} &= \sup_{x\in\Omega} u(x)\,|x|^{(N-2\alpha)/2} \\ &\longrightarrow \infty. \end{aligned} \tag{11}$$

According to Lemma 3.1 in [9], there is a constant $C > 0$ such that $\sup_{x\in\mathcal{O}(\Omega,r)} u(x) \le C$, where

$$\mathcal{O}(\Omega, r) = \{x \in \Omega : dist(x, \partial\Omega) < r\}, \quad r > 0. \tag{12}$$

This fact implies that $B(x_j, r) \subset \Omega$. Define

$$v_j(x) = u(x_j)^{-1} u\left(u(x_j)^{-2/(N-2\alpha)} x + x_j\right), \quad x \in u(x_j)^{2/(N-2\alpha)} (\Omega - x_j). \tag{13}$$

By Lemma 3.3 in [9], we know that, up to a subsequence, $v_j(x)$ converges to the function v uniformly on any compact set, where

$$v(x) = \left(\frac{\lambda}{1 + \lambda^2 |x - x_0|^2}\right)^{(N-2\alpha)/2}. \tag{14}$$

Obviously, $v(0) = 1$; then for fixed $\lambda \ge 1$, $\lambda = 1 + \lambda^2|x_0|^2$, which implies that $|x_0| \le 1$. Note that $v_i \longrightarrow v$ in $C^1(B(x_0, r))$ for $r > 0$ and x_0 is the extreme point of v. Then there exists a sequence of points $x_i \in B(x_0, r)$, such that $\nabla v_i(x_i) = 0$. Taking into account Remark 2, we find

$$u(x_i)^{-2/(N-2\alpha)} x + x_i = 0. \tag{15}$$

This fact, together with (11), implies that

$$|x_i| = u(x_i)^{2/(N-2\alpha)} |x_i| \longrightarrow \infty, \tag{16}$$

which contradicts the fact that $x_i \in B(x_0, r) \subset \Omega$. This proves the validity of this lemma. □

Lemma 7. *Let $u(x)$ be a solution of problem (2). Then there exist two positive constants C and δ, such that*

$$u(x) \le \frac{C}{\|u(x)\|_{L^\infty} \delta^{N-2\alpha}}, \quad x \in D := \Omega \cap \{|x| > \delta\}. \tag{17}$$

Remark 8. It can be easily seen that $\lim_{i\longrightarrow\infty} u_i(x) = 0$ uniformly for $x \in \Omega \cap \{|x| > \delta\}$.

Proof. Analysis similar to that in the proof of Proposition 4.9 in [21] shows that there exists a positive constant C such that, for any $|x - x_i| \le 1$,

$$u_i(x) \le C u_i^{-1}(x_i) |x - x_i|^{2\alpha-N}. \tag{18}$$

This fact implies that there exist $\delta > 0$ and $C > 0$ such that for any $|x| \le \delta$

$$u_i(x) \le C \|u_i\|^{-1} |x|^{2\alpha-N}. \tag{19}$$

Particularly,

$$u_i(x) \le C \|u_i\|^{-1} \delta^{2\alpha-N}, \quad |x| = \delta. \tag{20}$$

Now we argue by contradiction to show (17) holds. Suppose that there exists a point $x_i \in \overline{D}$, say the maximum point of u_i in $\overline{D}$, such that

$$u_i(x) > C \|u_i(x_i)\|^{-1} \delta^{2\alpha-N} > 0. \tag{21}$$

It can be easily seen that $x_i \in D$. Thus $\nabla u_i(x_i) = 0$. Consequently, $x_i = 0$; this contradicts the fact that $x_i \in D$. □

Remark 9. It is worth pointing out that, in [21], Jin et al. consider the integral fractional Laplacian, but the arguments of Propositions 4.4 and 4.9 in [21] still hold for spectral fractional Laplacian.

Lemma 10. *Let $u_i(x)$ be a solution to problem (2). Then*

$$\lim_{i\longrightarrow\infty} \int_\Omega u_i(x)^{p+1} dx = \int_{\mathbb{R}^N} U(x)^{p+1} dx. \tag{22}$$

Proof. Decompose Ω as $\Omega = \Omega_1 \cup \Omega_2$, where $\Omega_1 = \{x \in \Omega : |x| \le \delta\}$ and $\Omega_2 = \{x \in \Omega : |x| > \delta\}$; δ is the constant which appears in Lemma 7; we get

$$\int_\Omega u_i(x)^{p+1} dx = \int_{\Omega_1} u_i(x)^{p+1} dx + \int_{\Omega_2} u_i(x)^{p+1} dx. \tag{23}$$

By (17), it is obvious that $\int_{\Omega_2} u_i(x)^{p+1} dx \longrightarrow 0$. On the other hand, for any $|y| < \delta u_i(0)^{(1-p)/2\alpha}$, define

$$\overline{u}_i(y) = u_i(0)^{-1} u_i\left(u_i(0)^{(1-p)/2\alpha} y\right). \tag{24}$$

By a similar argument as Proposition 4.4 in [21], we derive that $\overline{u}_i(y) \longrightarrow U(y)$ in $C^2_{loc}(\mathbb{R}^N)$. Therefore, by the dominated convergence theorem, we find

$$\int_{\Omega_1} u_i(x)^{p+1} dx = \int_{\{|y|<\|u_i\|^{(p-1)/2\alpha}\}} \overline{u}_i(y)^{p+1} dy \longrightarrow \int_{\mathbb{R}^N} U(x)^{p+1} dx. \tag{25}$$

This finishes the proof of (22). □

Proof of Theorem 1. Since $u_i(x)$ is the solution to problem (2), therefore

$$\begin{aligned}\int_\Omega \left|\mathcal{A}_\alpha^{1/2} u\right|^2 dx &= \int_\Omega \mathcal{A}_\alpha^{1/2} u \cdot \mathcal{A}_\alpha^{1/2} u \, dx \\ &= N(N-2\alpha) \int_\Omega u^{p+1} dx \\ &\quad + \varepsilon \int_\Omega u^2 dx.\end{aligned} \tag{26}$$

According to the Hölder inequality, we find

$$\int_\Omega u^2 dx \le |\Omega|^{(p-1)/(p+1)} \left(\int_\Omega u^{p+1} dx\right)^{2/(p+1)}. \tag{27}$$

Thus, taking into account (26) and (27), we obtain

$$\begin{aligned}&\frac{\int_\Omega \left|\mathcal{A}_\alpha^{1/2} u\right|^2 dx}{\left(\int_\Omega u^{p+1} dx\right)^{2/(p+1)}} \\ &\quad \le N(N-2\alpha) \left(\int_\Omega u^{p+1} dx\right)^{(p-1)/(p+1)} \\ &\quad + \varepsilon |\Omega|^{(p-1)/(p+1)}.\end{aligned} \tag{28}$$

This fact, combined with (22), shows that (3) holds. This completes the proof of Theorem 1. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors equally contributed in this work.

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