Research Article

Differential Harnack Estimates for a Semilinear Parabolic System

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In this paper, we prove differential Harnack inequalities for positive solutions of a semilinear parabolic system on hyperbolic space. We use the inequalities to construct classical Harnack estimates by integrating along space-time.

1. Introduction

In this paper, we study the following problem:

\[ f_t = \Delta f + e^{\mu t} g^p, \quad \mathbb{R}^n \times (0, +\infty), \]

\[ g_t = \Delta g + e^{\nu t} f^q, \quad \mathbb{R}^n \times (0, +\infty), \]

\[ f(x, 0) = f_0, \quad g(x, 0) = g_0, \quad \mathbb{R}^n, \tag{1} \]

where \( p, q, \mu, \nu \) are positive constants.

P. Li and S.-T. Yau in [1] were first pioneers to the study of differential Harnack inequalities which were brought to general parabolic geometric flows by R. Hamilton (see [2]). Using these inequalities, one can derive ancient solutions, bounds on gradient Ricci solitons, Holder continuity. Differential Harnack inequalities are important aspects of properties of partial differential equations. Paper [3] described differential Harnack inequalities to the initial value problem of a semilinear parabolic equation when the semilinear term is \( e^{\mu t} f^p, \mu > 0 \). There have been numerous interesting results on the properties of solutions of partial differential equations, such as existence of solutions [4–17], nonexistence and blow-up of solutions [18–22], and asymptotic behaviors of solutions [23–28].

Let \((f(x, t), g(x, t))\) be positive smooth solutions to (1) and \((u, v) = (\log f, \log g)\). The main object of our study is the following Harnack quantities:

\[ H_1 \equiv \alpha \Delta u + \beta_1 |\nabla u|^2 + ce^{\alpha t+p-\alpha u} + \psi_1 (t) + \phi_1 (x), \]

\[ H_2 \equiv \alpha \Delta v + \beta_2 |\nabla v|^2 + ce^{\alpha t+q-\alpha v} + \psi_2 (t) + \phi_2 (x), \tag{2} \]

where \( \alpha, \beta_i, c \in \mathbb{R}, \alpha > \max\{\beta_1, \beta_2\} \) and \( \psi_i, \phi_i \) will be chosen suitably \( i = 1, 2 \). We will derive our differential Harnack estimate.

**Theorem 1.** Let \((f(x, t), g(x, t))\) be positive classical solutions to (1), and \((u, v) = (\log f, \log g)\). If \( \alpha, \beta_1, k_1, c \) satisfy

\[ \alpha > \max\{\beta_1, \beta_2\} \geq 0, \]

\[ k_1 - p k_2 \geq 0, \]

\[ k_2 - q k_1 \geq 0, \]

\[ k_1 \geq \frac{n\alpha^2}{2 (\alpha - \beta_1)} > 0, \]

\[ - \beta_2 + ap + c (1 - p) \geq 0, \]

\[ - \beta_1 + aq + c (1 - q) \geq 0, \]

\[ \frac{4}{n\alpha^2} \beta_1 c + 1 - \frac{p^2 (\alpha - \beta_1)}{-p \beta_2 + ap^2 + cp (1 - p)} \geq 0, \]

\[ \frac{4}{n\alpha^2} \beta_2 c + 1 - \frac{p^2 (\alpha - \beta_2)}{-q \beta_1 + aq^2 + cq (1 - q)} \geq 0, \]
Theorem 1 which describes differential Harnack estimate.

\begin{equation}
H_1 = \alpha \Delta u + \beta_1 |\nabla u|^2 + ce^{dt+p\nu-u} + \frac{k_1}{t} \geq 0,
\end{equation}

\begin{equation}
H_2 = \alpha \Delta v + \beta_2 |\nabla v|^2 + ce^{dt+p\nu-v} + \frac{k_2}{t} \geq 0,
\end{equation}

for all \( t \).

The paper is organized as follows. In Section 2 we prove Theorem 1 which describes differential Harnack estimate. There are applications of Theorem 1 in Section 3.

2. Harnack Estimate

In this section, we shall first obtain our differential Harnack inequalities, relying on the parabolic maximum principle.

Lemma 2. Suppose \((f(x,t), g(x,t))\) are positive solutions to (1) and \((u,v) = (\log f, \log g)\) and \((H_1, H_2)\) are defined as in (2). Assume that \(\alpha, \beta_1, \beta_2, \) and \(c\) satisfy

\[ \alpha > \max \{\beta_1, \beta_2\} \geq 0, \]

\[ -\beta_1 + \alpha p + c (1-p) \geq 0, \]

\[ -\beta_1 + \alpha q + c (1-q) \geq 0, \]

\[ \frac{4}{n \alpha^2} \beta_1 c + 1 - \frac{p^2 (\alpha - \beta_1)}{-p \beta_2 + \alpha p^2 + cp (1-p)} \geq 0, \]

\[ \frac{4}{n \alpha^2} \beta_2 c + 1 - \frac{q^2 (\alpha - \beta_2)}{-q \beta_1 + \alpha q^2 + cq (1-q)} \geq 0. \]

Then we have

\[ \frac{\partial}{\partial t} H_1 \geq \Delta H_1 + 2V H_1 \cdot \nabla u + h_{11} H_1 \]

\[ + e^{dt+p\nu-u} (p H_2 - H_1) + h_{12}, \]

\[ \frac{\partial}{\partial t} H_2 \geq \Delta H_2 + 2V H_2 \cdot \nabla v + h_{21} H_2 \]

\[ + e^{dt+p\nu-v} (q H_1 - H_2) + h_{22}, \]

where

\[ h_{11} = 2 (\alpha - \beta_1) - \frac{1}{n \alpha^2} \left\{ H_1 - 2 \left( \beta_1 |\nabla u|^2 + ce^{dt+p\nu-u} + \psi_1 + \phi_1 \right) \right\}, \]

\[ h_{12} = \frac{\partial}{\partial t} \psi_1 - \Delta \phi_1 + \frac{2}{n \alpha^2} (\alpha - \beta_1) (\psi_1 + \phi_1)^2 \]

\[ - \frac{n \alpha^2 |\nabla \phi_1|^2}{4 \beta_1 (\alpha - \beta_1) (\psi_1 + \phi_1)} \]

\[ + e^{dt+p\nu-u} (c \mu + \psi_1 + \phi_1 - p (\psi_2 + \phi_2)), \]

\[ h_{21} = 2 (\alpha - \beta_2) - \frac{1}{n \alpha^2} \left\{ H_2 - 2 \left( \beta_2 |\nabla v|^2 + ce^{dt+p\nu-v} + \psi_2 + \phi_2 \right) \right\}, \]

\[ h_{22} = \frac{\partial}{\partial t} \psi_2 - \Delta \phi_2 + \frac{2}{n \alpha^2} (\alpha - \beta_2) (\psi_2 + \phi_2)^2 \]

\[ - \frac{n \alpha^2 |\nabla \phi_2|^2}{4 \beta_2 (\alpha - \beta_2) (\psi_2 + \phi_2)} \]

\[ + e^{dt+p\nu-v} (c \nu + \psi_2 + \phi_2 - q (\psi_1 + \phi_1)). \]

Proof. Substituting \((f, g) = (e^\alpha, e^\nu)\) into (1), we have

\[ u_t = \Delta u + |\nabla u|^2 + e^{dt+p\nu-u}, \]

\[ v_t = \Delta v + |\nabla v|^2 + e^{dt+p\nu-v}. \]

Using the above equations, we have

\[ (\partial_t - \Delta) \Delta u \]

\[ = 2 |\nabla u|^2 + 2 \nabla u \cdot \nabla \Delta u \]

\[ + e^{dt+p\nu-u} \left\{ (p \Delta v - \Delta u + |p \nabla v - \nabla u|^2) \right\}, \]

\[ (\partial_t - \Delta) \left( |\nabla u|^2 \right) \]

\[ = 2 \nabla u \cdot \nabla |\nabla u|^2 - 2 |\nabla |\nabla u|^2 + 2 e^{dt+p\nu-u} \nabla u \cdot (p \nabla v - \nabla u). \]

Furthermore, applying (2) and Cauchy-Schwarz inequality

\[ |\nabla u|^2 \geq (1/n) (\Delta u)^2 \]

yields

\[ \partial_t H_1 - \Delta H_1 - 2V H_1 \cdot \nabla u = 2 (\alpha - \beta_1) |\nabla u|^2 \]

\[ + e^{dt+p\nu-u} (p H_2 - H_1) \]

\[ + e^{dt+p\nu-u} \left\{ (\alpha p^2 - p \beta_2 + cp (1-p)) |\nabla v|^2 \right\}, \]

\[ - 2p (\alpha - \beta_1) \nabla u \cdot \nabla v + e^{dt+p\nu-u} (\alpha - \beta_1) |\nabla u|^2 \]

\[ + e^{dt+p\nu-u} (c \mu + \psi_1 + \phi_1 - p (\psi_2 + \phi_2)) + \partial_t \psi_1 \]

\[ - \Delta \phi_1 - 2 \nabla u \cdot \nabla \phi_1 \geq 2 (\alpha - \beta_1) \frac{1}{n \alpha^2} H_1 \left\{ H_1 \right\} \]

\[ - 2 \left( \beta_1 |\nabla u|^2 + ce^{dt+p\nu-u} + \psi_1 + \phi_1 \right) + 2 (\alpha - \beta_1) \]

\[ \frac{1}{n \alpha^2} \left\{ \beta_1 |\nabla u|^2 + 2 |\nabla u|^2 (ce^{dt+p\nu-u} + \psi_1 + \phi_1) \right\} \]

\[ + 2 (\alpha - \beta_1) \frac{1}{n \alpha^2} \left( ce^{dt+p\nu-u} + \psi_1 + \phi_1 \right)^2 \]

\[ + e^{dt+p\nu-u} \left\{ (\alpha p - \beta_2 + c (1-p)) |\nabla v|^2 \right\}, \]

\[ - 2p (\alpha - \beta_1) \nabla u \cdot \nabla v + e^{dt+p\nu-u} (\alpha - \beta_1) |\nabla u|^2 \]

\[ + e^{dt+p\nu-u} (c \mu + \psi_1 + \phi_1 - p (\psi_2 + \phi_2)) + \partial_t \psi_1 \]

\[ - \Delta \phi_1 - 2 \nabla u \cdot \nabla \phi_1 \]

Furthermore, applying (7) and Cauchy-Schwarz inequality

\[ |\nabla u|^2 \geq (1/n) (\Delta u)^2 \]

yields

\[ \partial_t H_2 - \Delta H_2 - 2V H_2 \cdot \nabla v = 2 (\alpha - \beta_2) |\nabla v|^2 \]

\[ + e^{dt+p\nu-v} (p H_1 - H_2) \]

\[ + e^{dt+p\nu-v} \left\{ (\alpha p^2 - p \beta_2 + cp (1-p)) |\nabla v|^2 \right\}, \]

\[ - 2p (\alpha - \beta_1) \nabla u \cdot \nabla v + e^{dt+p\nu-v} (\alpha - \beta_2) |\nabla v|^2 \]

\[ + e^{dt+p\nu-v} (c \mu + \psi_1 + \phi_1 - p (\psi_2 + \phi_2)) + \partial_t \psi_2 \]

\[ - \Delta \phi_2 - 2 \nabla v \cdot \nabla \phi_2 \geq 2 (\alpha - \beta_2) \frac{1}{n \alpha^2} H_2 \left\{ H_2 \right\} \]

\[ - 2 \left( \beta_2 |\nabla v|^2 + ce^{dt+p\nu-v} + \psi_2 + \phi_2 \right) + 2 (\alpha - \beta_2) \]

\[ \frac{1}{n \alpha^2} \left\{ \beta_2 |\nabla v|^2 + 2 |\nabla v|^2 (ce^{dt+p\nu-v} + \psi_2 + \phi_2) \right\} \]

\[ + 2 (\alpha - \beta_2) \frac{1}{n \alpha^2} \left( ce^{dt+p\nu-v} + \psi_2 + \phi_2 \right)^2 \]

\[ + e^{dt+p\nu-v} \left\{ (\alpha p - \beta_2 + c (1-p)) |\nabla v|^2 \right\}, \]

\[ - 2p (\alpha - \beta_1) \nabla u \cdot \nabla v + e^{dt+p\nu-v} (\alpha - \beta_2) |\nabla v|^2 \]

\[ + e^{dt+p\nu-v} (c \mu + \psi_1 + \phi_1 - p (\psi_2 + \phi_2)) + \partial_t \psi_2 \]

\[ - \Delta \phi_2 - 2 \nabla v \cdot \nabla \phi_2 \]
If \( \alpha p - \beta_2 + c(1-p) \geq 0 \), the above inequality is
\[
\geq 2 (\alpha - \beta_1) \frac{1}{na^2} \cdot H_1 \left\{ H_1 - 2 \left( \beta_1 |\nabla u|^2 + e^{\alpha t + p\nu - u} + \psi_1 + \phi_1 \right) \right\} + e^{\alpha t + p\nu - u} \left( p H_2 - H_1 \right) + \frac{4 \beta_2}{na^2} (\alpha - \beta_1) (\psi_1 + \phi_1) \cdot |\nabla u|^2 + 2 (\alpha - \beta_1) \cdot \frac{1}{na^2} \left\{ \beta_1^2 |\nabla u|^4 + (e^{\alpha t + p\nu - u} + \psi_1 + \phi_1)^2 \right\} + \left( \frac{4}{na^2} \beta \left( 1 - \frac{\rho}{\alpha - \beta_2} \right) \right) (\alpha - \beta_1) \cdot e^{\alpha t + p\nu - u} |\nabla u|^2 + e^{\alpha t + p\nu - u} \left( c \mu + \psi_1 + \phi_1 - \rho \left( \psi_2 + \phi_2 \right) \right) + \delta \psi_1 - \Delta \phi_1 - 2 \nu u \cdot \nabla \psi_1.
\]
First note the following inequality:
\[
\geq \frac{4 \beta_2}{na^2} (\alpha - \beta_1) (\psi_1 + \phi_1) |\nabla u|^2 - 2 \nu u \cdot \nabla \psi_1.
\]
If \( \alpha > \beta_1 \) and \( 4/na^2 \beta \left( 1 - \frac{\rho}{\alpha - \beta_2} \right) \leq 0 \), the above inequality is
\[
\geq 2 (\alpha - \beta_1) \frac{1}{na^2} \cdot H_1 \left\{ H_1 - 2 \left( \beta_1 |\nabla u|^2 + e^{\alpha t + p\nu - u} + \psi_1 + \phi_1 \right) \right\} + e^{\alpha t + p\nu - u} \left( p H_2 - H_1 \right) - \frac{na^2}{4 \beta_1 (\alpha - \beta_1)} (\psi_1 + \phi_1)^2 + e^{\alpha t + p\nu - u} \left( c \mu + \psi_1 + \phi_1 - \rho \left( \psi_2 + \phi_2 \right) \right) + \delta \psi_1 - \Delta \phi_1 + 2 \frac{na^2}{\beta_1} (\alpha - \beta_1) (\psi_1 + \phi_1)^2.
\]
For \( H_2 \), we have similar results.
\[
\partial_t H_2 - \Delta H_2 - 2 \nabla H_2 \cdot \nabla \nu \geq 2 (\alpha - \beta_2) \frac{1}{na^2} H_2 \left\{ H_2 - 2 \left( \beta_2 |\nabla u|^2 + e^{\alpha t + p\nu - u} + \psi_2 + \phi_2 \right) \right\} + 2 (\alpha - \beta_2) \frac{1}{na^2} \beta \left\{ \beta_2 |\nabla v|^4 + 2 |\nabla v|^2 \left( e^{\alpha t + p\nu - u} + \psi_2 + \phi_2 \right) \right\} + 2 (\alpha - \beta_2) \frac{na^2}{\beta_2} \left( e^{\alpha t + p\nu - u} + \psi_2 + \phi_2 \right)^2 + e^{\alpha t + p\nu - u} \left( q H_1 - H_2 \right) + e^{\alpha t + p\nu - u} \left\{ q \left( -\beta_1 + \alpha q + c (1-q) \right) |\nabla u|^2 - 2 q \left( \alpha - \beta_2 \right) \nabla u \cdot \nabla v \right\} + e^{\alpha t + p\nu - u} \left( \alpha - \beta_2 \right) |\nabla u|^2 + e^{\alpha t + p\nu - u} \left( c \nu + \psi_2 + \phi_2 - q \left( \psi_1 + \phi_1 \right) \right) + \delta \psi_2 - \Delta \phi_2 - 2 \nu v \cdot \nabla \phi_2.
\]
If \( \alpha q - \beta_1 + c(1-q) \geq 0 \), the above inequality is
\[
\geq 2 (\alpha - \beta_2) \frac{1}{na^2} \cdot H_2 \left\{ H_2 - 2 \left( \beta_2 |\nabla v|^2 + e^{\alpha t + p\nu - u} + \psi_2 + \phi_2 \right) \right\} + e^{\alpha t + p\nu - u} \left( q H_1 - H_2 \right) - \frac{na^2}{4 \beta_2 (\alpha - \beta_2)} (\psi_2 + \phi_2)^2 + e^{\alpha t + p\nu - u} \left( c \nu + \psi_2 + \phi_2 - q \left( \psi_1 + \phi_1 \right) \right) + \delta \psi_2 - \Delta \phi_2 - 2 \nu v \cdot \nabla \phi_2.
\]
If \( (4/na^2) \beta \left( 1 - \frac{\rho}{\alpha - \beta_2} \right) + 1 - q^2 (\alpha - \beta_2) /(-q \beta_1 + \alpha q^2 + c q (1-q)) \geq 0 \), the above inequality is
\[
\geq 2 (\alpha - \beta_2) \frac{1}{na^2} \cdot H_2 \left\{ H_2 - 2 \left( \beta_2 |\nabla v|^2 + e^{\alpha t + p\nu - u} + \psi_2 + \phi_2 \right) \right\} + e^{\alpha t + p\nu - u} \left( q H_1 - H_2 \right) - \frac{na^2}{4 \beta_2 (\alpha - \beta_2)} (\psi_2 + \phi_2)^2 + e^{\alpha t + p\nu - u} \left( c \nu + \psi_2 + \phi_2 - q \left( \psi_1 + \phi_1 \right) \right) + \delta \psi_2 - \Delta \phi_2 + 2 \frac{na^2}{\beta_2} (\alpha - \beta_2) (\psi_2 + \phi_2)^2.
\]
This completes the proof of Lemma 2.

Next, we need to compute specific \( \psi(t) \) and \( \phi(x) \) that guarantee \( h_{ij} > 0 \) for the maximum principle to be applicable where \( i = 1,2 \).

**Lemma 3.** Assume \( k_1 - pk_2 \geq 0 \), \( k_2 - qk_1 \geq 0 \), \( l_1 - pl_2 \geq 0 \), \( l_2 - ql_1 \geq 0 \) and
\[
k_i \geq \frac{na^2}{2 (\alpha - \beta_i)} > 0, \quad i = 1,2.
\]
If \( \psi_i(t) = k_i/t, \phi_i(x) = \sum_{m=1}^{n} l_i / (x_m - a_m)^2 + l_i / (b_m - x_m)^2, \)
i = 1, 2 for \( x \in \Omega = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n], \) then for some \( x_0 \in \Omega, \) \( h_{12} |x-x_0| > 0 \) and \( h_{22} |x-x_0| > 0. \)

Proof. From \( k_1 - pk_2 \geq 0, k_2 - qk_1 \geq 0 \) and \( l_1 - pl_2 \geq 0, \)
\( l_2 - ql_1 \geq 0, \) we get \( c\mu + \psi_1 + \phi_1 - p(\psi_2 + \phi_2) \geq 0 \) and \( c\nu + \psi_2 + \phi_2 - q(\psi_1 + \phi_1) \geq 0. \)

Now applying Lemma 2 yields
\[
\begin{align*}
  h_{12} &\geq \frac{\partial}{\partial t} \psi_1 - \Delta \phi_1 + \frac{2n\alpha^2}{\alpha - \beta_1}(\psi_1 + \phi_1)^2 \\
  &\quad - \frac{n\alpha^2 |\nabla \phi_1|^2}{4\beta_1 \left(\alpha - \beta_1\right)^2}, \\
  h_{22} &\geq \frac{\partial}{\partial t} \psi_2 - \Delta \phi_2 + \frac{2n\alpha^2}{\alpha - \beta_2}(\psi_2 + \phi_2)^2 \\
  &\quad - \frac{n\alpha^2 |\nabla \phi_2|^2}{4\beta_2 \left(\alpha - \beta_2\right)^2}.
\end{align*}
\]

(18)

By the definitions of \( \psi_i \) and \( \phi_i, \) we obtain
\[
\begin{align*}
  \Delta \phi_i &\sum_{m=1}^{n} \left( \frac{6l_i}{(x_m - a_m)^2} + \frac{6l_i}{(b_m - x_m)^2} \right), \\
  |\nabla \phi_i|^2 &\sum_{m=1}^{n} \left( \frac{-2l_i}{(x_m - a_m)^2} + \frac{2l_i}{(b_m - x_m)^2} \right)^2, \\
  &\sum_{m=1}^{n} \left( \frac{2\nabla}{(x_m - a_m)^2} + \frac{2\nabla}{(b_m - x_m)^2} \right)^2.
\end{align*}
\]

(19)

(20)

If
\[
\begin{align*}
  k_i &\geq \frac{n\alpha^2}{2 \left(\alpha - \beta_1\right)} > 0, \\
  l_i &\geq \frac{n\alpha^2}{2 \left(\alpha - \beta_1\right)} \left(6 + \frac{\alpha^2}{\beta_1 \left(\alpha - \beta_1\right)}\right), \quad i = 1, 2,
\end{align*}
\]

then \( h_{12} > 0 \) and \( h_{22} > 0. \)

This proves Lemma 3.

Proof of Theorem 1. Choose
\[
\begin{align*}
  \psi_i(t) &= \frac{k_i}{t}, \\
  \phi_i(x) &= \sum_{m=1}^{n} \left( \frac{l_i}{(x_m - a_m)^2} + \frac{l_i}{(b_m - x_m)^2} \right), \\
  &\sum_{m=1}^{n} \left( \frac{l_i}{(x_m - a_m)^2} + \frac{l_i}{(b_m - x_m)^2} \right),
\end{align*}
\]
i = 1, 2 for \( x \in \Omega = [a_1, b_1] \times [a_2, b_2] \times \ldots \times [a_n, b_n]. \)

Note that
\[
\lim_{t \to 0} H_i = \infty = \lim_{x_m \to a_m} H_i = \lim_{x_m \to b_m} H_i, \\
  m = 1, 2, \ldots, n, \quad i = 1, 2.
\]

Assume that there exists a first time \( t_0 \) and point \( x_0 \in \Omega \) where
\( H_1(x_0, t_0) = 0 \) and \( H_2(x_0, t_0) > 0. \) At \( (x_0, t_0), \) we have
\[
\begin{align*}
  \forall &\overline{H}_1 = 0, \\
  \Delta &\overline{H}_1 \geq 0, \\
  \partial_{\overline{t}} &\overline{H}_1 \leq 0.
\end{align*}
\]

Lemma 3 implies that
\[
\begin{align*}
  \overline{H}_1 - \Delta \overline{H}_1 &\geq -\frac{\partial}{\partial t} \psi_1 \cdot \nabla u - h_{11} H_1 \\
  &\geq h_{12} > 0
\end{align*}
\]

This is a contradiction. Assume that there exist a first time \( t_0 \) and point \( x_0 \in \Omega \) where \( H_2(x_0, t_0) = 0 \) and \( H_1(x_0, t_0) > 0. \) At \( (x_0, t_0), \) we have
\[
\begin{align*}
  \forall &\overline{H}_2 = 0, \\
  \Delta &\overline{H}_2 \geq 0, \\
  \partial_{\overline{t}} &\overline{H}_2 \leq 0.
\end{align*}
\]

Lemma 3 implies that
\[
\begin{align*}
  \partial_{\overline{t}} \overline{H}_2 - \Delta \overline{H}_2 &\geq -\frac{\partial}{\partial t} \psi_2 \cdot \nabla v - h_{22} H_2 \\
  &\geq h_{22} > 0
\end{align*}
\]

This is a contradiction. Furthermore, \( H_1(x_0, t_0) = 0 \) and \( H_2(x_0, t_0) = 0 \) cause the same contradiction as \( H_1(x_0, t_0) = 0 \) and \( H_2(x_0, t_0) > 0. \) Thus \( H_1(x, t) > 0 \) and \( H_2(x, t) > 0 \) for all \( x, t > 0. \)

Taking \( \Omega \to \mathbb{R}^n \) which obtains \( \phi_1 \to 0 \) then gives the desired result.

\[ \square \]

3. Applications

In this section, we shall give an application of Theorem 1. We integrate along space-time to derive a classical Harnack inequality.

3.1. Classical Harnack Inequality: In this subsection, we integrate our differential Harnack inequality of Theorem 1 along space-time to derive a classical Harnack inequality.

Proposition 4. Let \( (f(x, t), g(x, t)) \) be positive classical solutions to (1) and \( (u(x, t), v(x, t)) = (\log f, \log g). \) Suppose that \( x_1, x_2 \in \mathbb{R}^n \) and \( t_2 > t_1 > 0. \) Assume further that \( \alpha > 2 \max(\beta_1, \beta_2) \), \( \alpha \geq c \) and \( k_l = n\alpha^2/(2(\alpha - \beta_1)), \) \( i = 1, 2. \) Then we have
\[
\begin{align*}
  f(x_1, t_1) &\leq f(x_2, t_2) \left( \frac{t_2}{t_1} \right)^n \exp \left( \frac{|x_2 - x_1|^2}{2(t_2 - t_1)} \right), \\
  g(x_1, t_1) &\leq g(x_2, t_2) \left( \frac{t_2}{t_1} \right)^n \exp \left( \frac{|x_2 - x_1|^2}{2(t_2 - t_1)} \right).
\end{align*}
\]

(28)

Proof. Define the one-variable functions \( \psi_1 : [t_1, t_2] \to \mathbb{R} \) as
\[
\psi_1(t) = u(y(t), t),
\]

where \( y(t) = \frac{t_2}{t_1} y(t_1). \)

\[ \square \]
for any $C^1$ path $y : [t_1, t_2] \to \mathbb{R}^n$ such that $y(t_1) = x_1, y(t_2) = x_2$.

Applying Theorem 1, we have

$$\Delta u \geq -\alpha^{-1}\left(\beta_1 |\nabla u|^2 + ce^{\mu t + pv-u} + \frac{k_1}{t}\right).$$

(30)

It yields that

$$\frac{d}{dt} \omega_1(t) = u_t + \nabla u \cdot \frac{dy}{dt}$$

$$= \Delta u + |\nabla u|^2 + e^{\mu t + pv-u} + \nabla u \cdot \frac{dy}{dt}$$

$$\geq -\alpha^{-1}\left(\beta_1 |\nabla u|^2 + ce^{\mu t + pv-u} + \frac{k_1}{t}\right) + |\nabla u|^2$$

$$+ e^{\mu t + pv-u} + \nabla u \cdot \frac{dy}{dt}$$

$$\geq \left(\frac{1}{2} - \frac{\beta_1}{\alpha}\right)|\nabla u|^2 + \left(1 - \frac{c}{\alpha}\right)e^{\mu t + pv-u} - \frac{k_1}{\alpha t}$$

$$- \frac{1}{2} \left|\frac{dy}{dt}\right|^2 \geq \frac{1}{2} \left|\frac{dy}{dt}\right|^2 - \frac{n}{t},$$

where $\alpha \geq 0, \alpha \geq 2\beta_1, \alpha \geq c$ and $k_1 = \frac{n\alpha^2}{2(\alpha - \beta_1)}$. Similarly,

$$\frac{d}{dt} \omega_2(t) \geq -\frac{1}{2} \left|\frac{dy}{dt}\right|^2 - \frac{n}{t}.$$  

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for $\alpha \geq 0, \alpha \geq 2\beta_2, \alpha \geq c$ and $k_2 = \frac{n\alpha^2}{2(\alpha - \beta_2)}$.

By

$$\int_{t_1}^{t_2} \left|\frac{dy}{dt}\right|^2 dt \geq \frac{|x_2 - x_1|^2}{t_2 - t_1},$$

$$\int_{t_1}^{t_2} -\frac{d}{dt} \omega_i(t) \leq \inf_{y \in \Omega} \int_{t_1}^{t_2} \left(\frac{1}{2} \left|\frac{dy}{dt}\right|^2 + \frac{n}{t}\right) dt, \quad i = 1, 2,$$

applying $(u, v) = (\log f, \log g)$ gives Proposition 4. \qed

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that there are no conflicts of interest regarding the publication of this paper.

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**References**


