Research Article

Directional Multifractal Analysis in the $L^p$ Setting

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Abstract

The classical H"older regularity is restricted to locally bounded functions and takes only positive values. The local $L^p$ regularity covers unbounded functions and negative values. Nevertheless, it has the same apparent regularity in all directions. In the present work, we study a recent notion of directional local $L^p$ regularity introduced by Jaffard. We provide its characterization by a supremum of a wide range oriented anisotropic Triebel wavelet coefficients and leaders. In addition, we deduce estimates on the Hausdorff dimension of the set of points where the directional local $L^p$ regularity does not exceed a given value. The obtained results are illustrated by some examples of self-affine cascade functions.

1. Introduction

Multifractal analysis describes geometrically and statistically the distribution of pointwise regularities (or singularities) of irregular functions $f$ on $\mathbb{R}^m$. It was first introduced in the context of the statistical study of fully developed turbulence in the mid 80’s [1]. The classical notion of pointwise regularity most commonly used is Hölder regularity. Let $y \in \mathbb{R}^m$ and $\alpha > 0$. Let $B(y, r)$ denote the open ball of radius $r$, centered at $y$. Recall that a function $f$ on $\mathbb{R}^m$ belongs to $C^\alpha(y)$ if there exist constants $C$ and $0 < R < 1$ and a polynomial $P$ of degree at most the integer part $[\alpha]$ of $\alpha$, such that

\[ \forall x \in B(y, R), \quad |f(x) - P(x - y)| \leq C |x - y|^\alpha. \]  

(1)

Clearly, this condition makes sense only if $f$ is locally bounded. It is equivalent to

\[ \forall r \leq R \quad \|f(\cdot) - P(\cdot - y)\|_{L^\infty(B(y, r))} \leq Cr^\alpha. \]  

(2)

It describes how the $L^\infty$ norm of $f$, properly renormalized by subtracting a polynomial, behaves in small neighborhoods of $y$.

The Hölder exponent of $f$ at $y$ is defined as

\[ \alpha (y) = \sup \{\alpha; \ f \in C^\alpha (y)\}. \]  

(3)

The multifractal formalism relates some function spaces norms of $f$ to the Hölder spectrum (which is the Hausdorff dimension of the set of points where $f$ has a given Hölder exponent). The idea of using wavelets in multifractal analysis has been worked out first by Arneodo et al. [2]. Wavelets characterize Hölder regularity and many function spaces.

In many PDE’s, the natural function space setting is $L^p$ for $1 \leq p < \infty$ or a Sobolev space which includes unbounded functions. In such cases, one replaces the $L^\infty$ norm in (2) by the $L^p$ norm. This corresponds to the following weaker condition introduced by Calderón and Zygmund [3].

Definition 1 (let $p \geq 1$). Let $f$ be a function which belongs locally to $L^p(\mathbb{R}^m)$.
If \( \alpha \geq 0 \), one says that \( f \in T^p_\alpha(y) \) if there exist constants \( C \) and \( 0 < R < 1 \) and a polynomial \( P \) of degree at most \( \lfloor \alpha \rfloor \) such that

\[
\forall r \leq R \quad \| f \|_{L^p(B(y,r))} \leq CR^{\alpha+m/p}.
\]

If \( -m/p \leq \alpha < 0 \), one says that \( f \in T^p_\alpha(y) \) if there exist constants \( C \) and \( 0 < R < 1 \) such that

\[
\forall r \leq R \quad \| f \|_{L^p(B(y,r))} \leq CR^{\alpha+m/p}.
\]

The \( p \)-exponent of \( f \) at \( y \) is defined as

\[
\alpha^p(y) = \sup \{ \alpha; f \in T^p_\alpha(y) \}.
\]

Clearly if \( f \in L^p_{\text{loc}}(\mathbb{R}^m) \) then \( f \in T^p_{-m/p}(y) \). The usual Hölder regularity \( C^\alpha(y) \), for \( \alpha > 0 \), corresponds to \( p = \infty \). If \( f \in C^\alpha(y) \), then \( f \in T^p_\alpha(y) \) for all \( p \geq 1 \). If \( f \in T^p_\alpha(y) \) then \( f \in T^p_{\alpha}(y) \) for all \( p' \leq p \).

In [4, 5], Jaffard and Mélot proved that if \( \Omega \) is a domain of \( \mathbb{R}^m \) and \( y \) is on the boundary \( \partial\Omega \) of \( \Omega \), then by taking \( p = 0 \) or \( p = 1 \), the \( T^p_\alpha(y) \) condition for the characteristic function \( \chi_\Omega \) of \( \Omega \) coincides with the bilaterally weak \( \alpha \) accessibility of \( \partial\Omega \) at \( y \). Let \( \alpha \geq 0 \). Recall that \( \Omega \) is called bilaterally weak \( \alpha \) accessible at \( y \) if there exist two constants \( C \) and \( 0 < R < 1 \) such that

\[
\forall r \leq R \quad \min \{ \text{meas}(\Omega \cap B(y,r)), \text{meas}(\Omega \cap B(y,r)) \} \leq CR^{\alpha+m},
\]

where \( \text{meas} \) is the Lebesgue measure. This remark allows performing a multifractal analysis of fractal boundaries [5, 6]. This analysis has many applications in physics, mechanics, or chemistry where many phenomena involve fractal interfaces. More pointwise exponents for classification of the geometry of fractal boundaries were studied by Jaffard and Heuretoux [7]. The relationship of these exponents to local dimension computation was proved by Tricot [8].

In [4, 5], a wavelet characterization of the \( T^p_\alpha(y) \) regularity was obtained. Moreover, an associated multifractal formalism was conjectured: the Hausdorff dimension \( d_\alpha(f)(H) \) of the set of points \( y \), where the \( p \)-exponent \( \alpha^p(y) \) of a function \( f \) is equal to \( H \), may be derived from some global functional norms extracted from \( f \).

Unfortunately, if \( m \geq 2 \) then Definition 1 has the same apparent value in all coordinate directions. However many signals belong to classes of functions with various directional regularity behaviors (see, for instance, [9–17] and the references therein). These behaviors are important for detection of edges, efficient image compression, etc. (see, for instance, [12] and the references therein). Classical (isotropic) wavelets are not optimal for analyzing directional or anisotropic features like edges. Extensions of wavelet bases which can be elongated in particular directions were considered. Candes and Donoho [18] and Mallat [19] (resp., Donoho [20] and Guo and Labate [21]) have used ridgelets and bandelets (resp., wedgelets and shearlets) to detect singularities along lines and hyperplanes (resp., discontinuities along smooth edges). Sampo and Sumetkijakan [16, 22, 23] (resp., Jaffard [24]) have used curvelets and Hart-Smith transform (resp., the anisotropic Gabor-wavelet transform) to detect directional pointwise Hölder singularities.

Many signals present anisotropies quantified through regularity characteristics and features that strongly differ when measured in different directions [13, 14, 24–29]. Several authors were concerned with the problem of obtaining useful decompositions of isotropic and anisotropic function spaces in simple building blocks: atoms, quarks, wavelets, splines (see, for example, Farkas [30], Garrigós et al. [31, 32], Hochmuth [33], and Kamont [34]). Triebel family of anisotropic wavelets yield characterizations of anisotropic Besov spaces [35, 36] and anisotropic Hölder regularity (see Ben Slimane et al. [37–39]).

Let \( e \in \mathbb{R}^m \) be a unit vector. To take into account pointwise directional \( L^p \) behavior in direction \( e \), it is natural to define the local \( L^p \) regularity \( \alpha^{p,e}(y) \) at a point \( y \) in direction \( e \) as the \( p \)-exponent at \( 0 \) of the one variable function \( f_{y,e} : t \mapsto f(y + te) \), that is,

\[
\alpha^{p,e}(y) = \sup \{ \alpha; f_{y,e} \in T^p_\alpha(0) \}.
\]

We wish to derive or estimate the Hausdorff dimension \( d_\alpha(f)(H) \) of the set of points \( y \) where \( \alpha^{p,e}(y) = H \) from some global quantities extracted from the function \( f \) itself and not its traces \( f_{y,e} \) (actually these traces are unknown since corresponding points \( y \) are unknown). One cannot expect directional local \( L^p \) regularity \( \alpha^{p,e}(y) \) to be characterized in terms of the size of the wavelet coefficients by taking wavelets on \( \mathbb{R}^m \) because \( f_{y,e} \) is defined as the trace of \( f \) on a line, which is a set of vanishing measure and wavelets on \( \mathbb{R}^m \) have a support of nonempty interior. Thus one should take into account the values of \( f \) around the line considered. Therefore the definition of directional local \( L^p \) regularity should include such information. However, in the asymptotic of small scales, the values taken into account should be localized more and more sharply around this line. These considerations motivated the following definition of Jaffard [24].

**Definition 2.** Let \( \overline{\alpha} = (\alpha_1, \ldots, \alpha_m) \) be an \( m \)-uple of nonnegative real numbers. The average regularity \( \overline{\alpha} \) is the harmonic mean of the \( \alpha_n \), i.e.,

\[
\overline{\alpha} = \left( \frac{1}{\alpha_1} + \cdots + \frac{1}{\alpha_m} \right)^{-1} = \frac{m}{\sum_{n=1}^{m} \frac{1}{\alpha_n}}.
\]

The anisotropy indices are

\[
v_n = \frac{\overline{\alpha}}{\alpha_n}.
\]

Let \( b = (e_1, \ldots, e_m) \) be an orthonormal basis of \( \mathbb{R}^m \). Denote by \((x_1, \ldots, x_m)\) the coordinates of \( x \) on the basis \( b \).
Set
\[ \mathbf{v} = (v_1, \ldots, v_m) \]
and \(|x|_v = \max\left( |x_1|^{1/v_1}, \ldots, |x_m|^{1/v_m} \right)\). \hfill (11)

Then \(| \cdot |_v\) is a quasi-norm on \(\mathbb{R}^m\); i.e., it satisfies the requirements of a norm except for the triangular inequality which is replaced by the weaker condition
\[ \exists C > 0 \ \forall x, y \in \mathbb{R}^m \]
\[ \|x + y\|_v \leq C (\|x\|_v + \|y\|_v). \] \hfill (12)

It is also continuous and homogeneous in the sense that
\[ \forall r > 0 \ \forall x \in \mathbb{R}^m \]
\[ \|r^\alpha x_1, \ldots, r^\alpha x_m\|_v = r |x|_v. \] \hfill (13)

The corresponding v-ball \(B_v(x, r) = \{ y \in \mathbb{R}^m : |x - y|_v < r \}\) of v-radius \(r\) centered on \(x\) is a rectangle with sides parallel to the axes of coordinates, centered at \(x\) and with side-lengths \(2r\alpha\) in the \(x_i\) direction.

Let \(f : \mathbb{R}^m \rightarrow \mathbb{C}\) be a function which belongs locally to \(L^p\). One says that \(f \in T^{p,\alpha}(y, b)\) if there exist constants \(C, 0 < R < 1\) and a polynomial \(P(x) = \sum_{i=0}^{m} a_i x_i^i = \sum_{i=0}^{m} a_i x_i^i \ldots x_i^m\) of degree
\[ d_j(P) < \alpha \] \hfill (14)
in the sense that
\[ \max \left\{ \sum_{i=1}^{m} i a_i : a_i \neq 0 \right\} < 1 \] \hfill (15)
such that
\[ \forall r < R \]
\[ \left\| f(-P(-y)) \right\|_{L^p(B_v(x, r))} \leq C r^{\sum_{i=1}^{m} a_i}/p. \] \hfill (16)

Remark 3. Actually, \(B_v(x, r)\) replaces the elongated ellipsoid \(N_{k,\alpha}(x, r)\) of axis of lengths \(2r\alpha\), \ldots, \(2r\alpha\), centered on \(x\), considered in [24]. This ellipsoid corresponds to the ball centered at \(x\) with \(\rho_v\) radius \(r\), where \(\rho_v\) is the quasi-norm on \(\mathbb{R}^m\) defined by \(\rho_v(0) = 0\), and for \(x \neq 0\), \(\rho_v(x)\) is the unique value of \(r\) for which \(\sum_{i=1}^{m} (x_i/r^{\alpha})^2 = 1\). Both \(| \cdot |_v\) and \(\rho_v\) are equivalent. Nevertheless, \(| \cdot |_v\) has the advantage of being easier in the computations of distances.

In the isotropic case \(v_n = 1\) for all \(1 \leq n \leq m\), \(\rho_v\) coincides with the Euclidean norm.

We first extend the definition of \(T^{p,\alpha}(y, b)\) for an \(m\)-uple \(\alpha = (\alpha_1, \ldots, \alpha_m)\) of nonpositive real numbers as follows.

Definition 4. Let \(\alpha = (\alpha_1, \ldots, \alpha_m)\) be an \(m\)-uple of nonpositive real numbers. We say that \(f \in T^{p,\alpha}(y, b)\) if there exist constants \(C\) and \(0 < R < 1\) such that
\[ \forall r < R \]
\[ \left\| f \right\|_{L^p(B_v(x, r))} \leq C r^{\sum_{i=1}^{m} \alpha_i}/p. \] \hfill (17)

Contrary to the case \(p = \infty\) studied in [24], there is no partial ordering property for \(1 \leq p < \infty\); i.e., if either \(0 < \alpha_1 \leq \beta_1, \ldots, 0 < \alpha_m \leq \beta_m\) or \(\alpha_1 \leq \beta_1 < 0, \ldots, \alpha_m \leq \beta_m < 0\) then \(f \in T^{p,\alpha}(y, b)\) does not imply that \(f \in T^{p,\beta}(y, b)\). Indeed, the harmonic mean values \(\bar{\alpha}\) and \(\bar{\beta}\) of \(\alpha\) satisfy \(\bar{\alpha} \leq \bar{\beta}\), but the corresponding anisotropy indices \(\mathbf{v} = (v_1, \ldots, v_m)\) and \(\mathbf{u} = (u_1, \ldots, u_m)\) are not necessarily the same. Nevertheless, we will show that the following substitute for the directional local \(L^p\) regularity \(\alpha^{p \bullet}(y)\) given in (8) makes sense (see next section).

Definition 5. Let \(e \in \mathbb{R}^m\) be a unit vector and \(b\) be any orthonormal basis starting with the vector \(e\). Let \(f : \mathbb{R}^m \rightarrow \mathbb{C}\) be a function. Set
\[ E^+(p, y, e) := \left\{ \alpha_1 > 0 : \exists \varepsilon \leq \alpha_1 f \in T^{p, (\alpha_1, -\varepsilon, \ldots, -\varepsilon)}(y, b) \right\}. \] \hfill (18)
and
\[ E^-(p, y, e) := \left\{ \alpha_1 < 0 : \exists \varepsilon \leq -\alpha_1 f \in T^{p, (-\alpha_1, -\varepsilon, \ldots, -\varepsilon)}(y, b) \right\}. \] \hfill (19)

If \(E^+(p, y, e) \neq \emptyset\), define the \(p\)-exponent of \(f\) at \(y\) in direction \(e\) by
\[ \alpha(p, y, e) = \sup E^+(p, y, e). \] \hfill (20)
If \(E^+(p, y, e) = \emptyset\), define the \(p\)-exponent of \(f\) at \(y\) in direction \(e\) by
\[ \alpha(p, y, e) = \sup E^-(p, y, e). \] \hfill (21)

Remark 6. Clearly we can choose any orthonormal basis \(b\) starting with the vector \(e\), in fact the component \(x_1 - y_1\) is the same in any \(b\) and is equal to the inner product of \(x - y\) with \(e\), and all norms of \(\mathbb{R}^{m-1}\) are equivalent.

Definition 7. Let \(e \in \mathbb{R}^m\) be a unit vector. Let \(f : \mathbb{R}^m \rightarrow \mathbb{C}\) be a function. The directional \(L^p\) spectrum (resp., upper \(L^p\) spectrum) of \(f\) in direction \(e\) is the map which associates with each \(H\) the Hausdorff dimension \(d_p(H, e)\) (resp., \(D_p(H, e)\)) of the set of points \(y\) where \(\alpha(p, y, e) = H\) (resp., \(\alpha(p, y, e) \leq H\)).

The paper is organized as follows. In Section 2, we first show that Definition 5 makes sense. Then we prove a criterion of directional pointwise \(p\)-exponent in terms of a supremum on a wide range of oriented anisotropic \(L^p\) regularities. We therefore deduce a first upper bound for the directional \(L^p\) spectra. In Section 3, we establish criteria of directional pointwise \(p\)-exponent in terms of decay condition on Triebel anisotropic wavelet coefficients and wavelet leaders. In Section 4, we obtain an alternative upper bound for the directional \(L^p\) spectra expressed as an infimum of Legendre transforms of some \(p\)-anisotropic scaling functions. Finally, in Section 5, the obtained results are illustrated by some examples of self-affine cascade functions.
2. Directional $L^p$ Regularity Criterion

2.1. Approval of Definition 5. We first show that Definition 5 makes sense.

(i) Suppose that $f \in T^p(\tilde{\alpha},\epsilon)(y, b)$ with $0 < \epsilon < \beta_1$. Let $\alpha_1 < \beta_1$. Set $\epsilon' = (\alpha_1/\beta_1)\epsilon$. Then $\epsilon' < \epsilon$ and $\epsilon' < \alpha_1$. Set $\tilde{\alpha} = (\alpha_1, \epsilon', \ldots, \epsilon')$ and $\tilde{\beta} = (\beta_1, \epsilon, \ldots, \epsilon)$. We will prove that $f \in T^p(\tilde{\alpha}, \epsilon)$; clearly, the harmonic mean values $\alpha$ of $\tilde{\alpha}$ and $\beta$ of $\tilde{\beta}$ satisfy $\alpha \leq \beta$. The corresponding anisotropy indices $\nu = (v_1, \ldots, v_m)$ are the same.

Since $f \in T^p(\tilde{\alpha}, \epsilon)$ then there exist constants $C, 0 < R < 1$ and a polynomial $P(x) = \sum_{k=(i_1, \ldots, i_m) \in \mathbb{N}} a_k x^i$ of degree $d_f(P)$ less than $\beta$ (see (14) and (15)), i.e.,

$$\max \left\{ \frac{i_1}{\beta_1} + \frac{i_2}{\epsilon} + \cdots + \frac{i_m}{\epsilon} : a_i \neq 0 \right\} < 1,$$

such that

$$\forall r \leq R \quad \left\| f(\cdot) - P(\cdot - y) \right\|_{L^p(B_0, \nu, \epsilon)} \leq C R^{\beta + m/p}.$$

(ii) Suppose that $f \in T^p(\tilde{\alpha}, \epsilon)(y, b)$ with $0 < \epsilon < \beta_1$. Let $\alpha_1 < \beta_1$. Set $\epsilon' = (\alpha_1/\beta_1)\epsilon$. Then $\epsilon' < \epsilon$ and $0 < \epsilon' < \alpha_1$. Clearly the harmonic mean values of $\tilde{\alpha}$ and $\tilde{\beta}$ satisfy $\tilde{\alpha} \leq \tilde{\beta} < 0$. The corresponding anisotropy indices $\nu = (v_1, \ldots, v_m)$ are the same. Since $f \in T^p(\tilde{\alpha}, \epsilon)$ then there exist constants $C$ and $0 < R < 1$ such that

$$\forall r \leq R \quad \left\| f(\cdot) \right\|_{L^p(B_0, \nu, \epsilon)} \leq C R^{\beta + m/p} \leq C R^{\tilde{\beta} + m/p}.$$

It follows that $f \in T^p(\tilde{\alpha}, \epsilon)(y, b)$.

2.2. Criterion of Directional Pointwise $p$–Exponent. We will establish a criterion of directional pointwise $p$–exponent in terms of a supremum on a wide range of oriented anisotropic $L^p$ regularities. The latest regularity is reminiscent of [40, 41] where Calderón and Torchinsky on one side and Folland and Stein on the other side have developed a theory of anisotropic $\mathcal{R}_u^p(\mathbb{R}^m)$ spaces using the $p_u$ quasi-norm of Remark 3.

Let $u = (u_1, \ldots, u_m) \in \mathbb{R}^m$ be such that

$$\forall \nu u \eta > 0$$

and

$$\sum_{n=1}^{m} u_n = m.$$  \hfill (26)

We will say that $u$ is an anisotropy vector.

If $I = (i_1, \ldots, i_m) \in \mathbb{N}_0^m$, set $d_u(I) = \sum_{n=1}^{m} u_n i_n$. If $P = \sum_i a_i x^i$, $a_i \in \mathbb{C}$ is a polynomial, define its $u$-homogeneous degree by

$$d_u(P) = \max \{ d_u(I) : a_i \neq 0 \}.$$  \hfill (27)

Definition 8. Let $f : \mathbb{R}^m \rightarrow \mathbb{C}$ be a function which belongs locally to $L^p$. Let $b$ be an orthonormal basis of $\mathbb{R}^m$.

If $\alpha \geq 0$, we say that $f$ belongs to $T^p_u(\alpha, \epsilon)$ if there exist constants $C, 0 < R < 1$ and a polynomial $P$ of $u$-homogeneous degree less than $\alpha$ such that

$$\forall r \leq R \quad \left\| f(\cdot) - P(\cdot - y) \right\|_{L^p(B_0, \nu, \epsilon)} \leq C R^{\alpha + m/p}.$$  \hfill (28)

If $-m/p \leq \alpha < 0$, we say that $f$ belongs to $T^p_u(\alpha, \epsilon)$ if there exist constants $C$ and $0 < R < 1$ such that

$$\forall r \leq R \quad \left\| f(\cdot) \right\|_{L^p(B_0, \nu, \epsilon)} \leq C R^{\alpha + m/p}.$$  \hfill (29)

Definitions 2 and 4 are related to the previous anisotropic regularity in $L^p$ setting. Let $\alpha = (\alpha_1, \ldots, \alpha_m)$ be an $m$-uple of either nonpositive or nonnegative real numbers. Let $\alpha$ (resp., $\nu = (v_1, \ldots, v_m)$) be the corresponding harmonic mean value as in (9) (resp., anisotropy indices as in (10)).

If $\alpha$ is an $m$-uple of nonnegative real numbers then it follows from (29) and Definition 4 that

$$T^p_u(\alpha, \epsilon)(y, b) = T^p_{\alpha, \epsilon}(y, b).$$  \hfill (30)

If $\alpha$ is an $m$-uple of nonnegative real numbers then

$$d_u(I) = \sum_{n=1}^{m} v_n i_n < \alpha \iff \sum_{n=1}^{m} \frac{i_n}{\alpha_n} < 1.$$  \hfill (31)

It follows from (14) and (15) that

$$d_f(P) < \alpha \iff d_f(P) < \frac{1}{\alpha}.$$  \hfill (32)
We deduce that
\[ T_{\alpha}^p (y, b) = T_{\chi_{\alpha}}^p (y, b). \] (33)

The following theorem extends the result obtained for the directional Hölder regularity in [38] to the $L^p$ setting. It has the advantage to cover unbounded functions and negative values.

**Theorem 9** (let $m \geq 2$). Let $e \in \mathbb{R}^m$ be a unit vector and $b$ be any orthonormal basis starting with the vector $e$. Let $\mathcal{E}$ be the set of all anisotropy vectors $u$ with $u_2 = \cdots = u_m$.

Let $f \in L^p_{\text{loc}} (\mathbb{R}^m)$. Then the $p$-exponent of $f$ at $y$ in direction $e$ is given by
\[ \alpha (p, y, e) = \sup_{u \in \mathcal{E}} \left( \frac{\alpha_{p, u} (y, b)}{u_1} \right), \] (34)
where $\alpha_{p, u} (y, b)$ is the so-called anisotropic $p - u$ regularity of $f$ at $y$ oriented in basis $b$, defined as
\[ \alpha_{p, u} (y, b) = \sup \{ \bar{\alpha}; f \in T_{\alpha}^p (y, b) \}. \] (35)

**Proof.** (let $u \in \mathcal{E}$). Assume that $f \in T_{\alpha}^p (y, b)$. Take $\alpha_1 = \bar{\alpha}/u_1$ and for all $2 \leq n \leq m$ take $\alpha_n = \bar{\alpha}/u_2$.
Then $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$.
Clearly, the anisotropy indices $\nu = (v_1, \ldots, v_m)$ of $\bar{\alpha}$ coincide with $u$.

(i) If $\bar{\alpha} < 0$ then
\[ f \in T_{\alpha}^p (y, b) \iff f \in T_{\alpha}^p (y, b). \] (36)

Since $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ with $0 > \alpha_m = \cdots = \alpha_2 \geq \alpha_1$, then it follows from Definition 5 that
\[ \alpha (p, y, e) \geq \alpha_1 = \frac{\bar{\alpha}}{u_1}. \] (37)

Then
\[ \alpha (p, y, e) \geq \frac{1}{u_1} \sup \{ \bar{\alpha}; f \in T_{\alpha}^p (y, b) \}. \] (38)

Taking at the right the supremum over $\mathcal{E}$ yields the lower bound in (34).

(ii) If $\bar{\alpha} > 0$ then
\[ d_u (I) = \sum_{i=1}^{m} u_i i_n < \bar{\alpha} \iff \sum_{i=1}^{m} \frac{i_n}{\alpha_n} < 1. \] (39)
So
\[ d_u (P) < \bar{\alpha} \iff \] (40)

We deduce that (36) holds too.
Since $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$ with $0 < \alpha_2 = \cdots = \alpha_m \leq \alpha_1$, then it follows from Definition 5 that
\[ \alpha (p, y, e) \geq \alpha_1 = \frac{\bar{\alpha}}{u_1}. \] (41)

Then
\[ \alpha (p, y, e) \geq \frac{1}{u_1} \sup \{ \bar{\alpha}; f \in T_{\alpha}^p (y, b) \}. \] (42)

Taking at the right the supremum over $\mathcal{E}$ yields the lower bound in (34).

Let us now prove the upper bound.

(i) Assume that $f \in T_{\alpha}^p (y, b)$ with $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$,
then $\alpha (p, y, e) \leq \alpha_1$. It follows from (30) that
\[ \bar{\alpha} \leq \alpha_{p, u} (y, b). \] (43)

Then
\[ \alpha_1 \leq \frac{1}{\nu_1} \sup \{ \alpha_{p, u} (y, b); u \in \mathcal{E} \}. \] (44)

Thus
\[ \alpha (p, y, e) \leq \sup \{ \frac{\alpha_{p, u} (y, b)}{u_1}; u \in \mathcal{E} \}. \] (45)

Whence Theorem 9 holds.

(ii) The case where $f \in T_{\alpha}^p (y, b)$ with $\bar{\alpha} = (\alpha_1, \ldots, \alpha_m)$
with $\alpha_2 = \cdots = \alpha_m = -\epsilon$ and $0 < \epsilon \leq -\alpha_1$ is similar.

\[ \square \]

2.3. General Upper Bound for the Directional $L^p$ Spectra. Let $\mathcal{E}$ be the set of anisotropy vectors $u$ given in Theorem 9. Let $b$ be an orthonormal basis of $\mathbb{R}^m$. Define the $p - u$-sets of $f$ in basis $b$ by
\[ E_{p, u} (h, b) = \{ x; \alpha_{p, u} (x, b) = h \}. \] (46)
and the upper $p - u$-sets of $f$ in basis $b$ by
\[ E_{p, u}^{\bar{u}, h} = \{ x; \alpha_{p, u} (x, b) = h \}. \] (47)

Theorem 9 yields the following general upper bound for the directional upper $L^p$ spectrum given in Definition 7.

**Corollary 10.** Let $\mathcal{E}$ be as in Theorem 9. Let $e \in \mathbb{R}^m$ be a unit vector and $b$ be any orthonormal basis starting with the vector $e$. If $f \in L^p_{\text{loc}} (\mathbb{R}^m)$ then the directional $L^p$ spectra of $f$ in direction $e$ given in Definition 7 satisfy
\[ d_f (P) < \bar{\alpha}. \] (48)

We have that $d_f (P) \leq D_f (H, e) \leq \inf_{u \in \mathcal{E}} \dim E_{p, u}^{\bar{u}, h}$, where $\dim$ denotes the Hausdorff dimension.
3. Criteria of Directional $L^p$ Regularity in Anisotropic Triebel Wavelet Bases

For each $\beta \in \mathbb{N}$ there are Daubechies [42] real valued compactly supported father and mother wavelets $\psi_F$ and $\psi_M$ in $C_0^\infty(\mathbb{R})$ (in the sense that they have classical continuous derivatives up to order $\beta$) such that $\int_\mathbb{R} \psi_F(x) dx = 1$, the moments $\int_\mathbb{R} x^n \psi_F(x) dx = 0$ for $n = 0, \ldots, \beta$, and the collection $(\psi_F(-k))_{k \in \mathbb{Z}}$ and $(2^{-j/2} \psi_M(2^j \cdot -k))_{j \in \mathbb{N}_0, k \in \mathbb{Z}}$ is an orthonormal basis of $L^2(\mathbb{R})$.

Let $u$ be an anisotropic vector. For $r > 0$, define the $u$-anisotropy map as

$$ r^u x = (r^{\mu_1} x_1, \ldots, r^{\mu_m} x_m). $$

(49)

In [35, 36], Triebel has considered anisotropic multiresolution analysis; consider, for any $j \in \mathbb{N}_0$, the closed subspace $V_{j,u}$ of $L^2(\mathbb{R}^m)$ spanned by the orthonormal basis $(2 \sum_{n \in [j u_m], b} \phi_{j,k,u} k \in \mathbb{Z}^m$, where

$$ \Phi_{j,k,u}(x) = \sum_{n=1}^m \psi_F (2 |ju_n| x_n - k_n). $$

(50)

The sequence $(V_{j,u})_{j \in \mathbb{N}_0}$ is an anisotropic multiresolution analysis of $L^2(\mathbb{R}^m)$ in the sense that

(i) $\forall j \in \mathbb{N}_0$, $V_{j,u} \subset V_{j+1,u}$

(ii) $f(x) \in V_{j,u} \iff f(2^{j} x) \in V_{j+1,u}$

(iii) $\bigcup_{j \in \mathbb{N}_0} V_{j,u} = L^2(\mathbb{R}^m)$.

For $j \in \mathbb{Z}$, let $I_{j,u}$ be the set of pairs $(G, I)$ where $G = (G_1, \ldots, G_m) \in \{F, M\}^m$ such that at least one component $G_n$ is $M$ and $I = (I_1, \ldots, I_m) \in \mathbb{N}_0^m$ where

$$ l_n = \lfloor ju_n \rfloor \quad \text{if} \quad G_n = F, $$

$$ [ju_n] \leq l_n < [(j + 1) u_n] \quad \text{if} \quad G_n = M $$

and

$$ [(j + 1) u_n] > [ju_n]. $$

(51)

and

$$ l_n = \lfloor ju_n \rfloor \quad \text{if} \quad G_n = M $$

and

$$ [(j + 1) u_n] = [ju_n]. $$

(52)

Clearly the cardinality $|I_{j,u}|$ of $I_{j,u}$ is bounded independently of $j$, more precisely

$$ 1 \leq |I_{j,u}| \leq (2^m - 1) \prod_{n=1}^m (2 + u_n). $$

(53)

The following proposition is given in [35, 36] in the case where $b$ is the canonical basis of $\mathbb{R}^m$. It remains valid in the case where $b$ is any orthonormal basis of $\mathbb{R}^m$.

**Proposition 11.** Let $b$ be an orthonormal basis of $\mathbb{R}^m$. Let $(x_1, \ldots, x_m)$ be the coordinates of $x$ in $b$. Set

$$ \Phi_{k,b}(x) = \prod_{i=1}^m \Phi_{k,b}(x_i - k_i), $$

$$ \psi^{(G)}_{j,k,u,b}(x) = \prod_{i=1}^m \psi^{(G)}_{j,k,u,b}(x_i - k_i) $$

(54)

and $|b| = \sum_{i=1}^m l_i$.

The collection of the union of $(\Phi_{k,b})$ for $k \in \mathbb{Z}^m$ and $(2^{\|b\|/2} \psi^{(G)}_{j,k,u,b})$ for $j \in \mathbb{N}_0, (G, I) \in I_{j,u}$ and $k \in \mathbb{Z}^m$ is then an orthonormal basis of $L^2(\mathbb{R}^m)$. Thus any function $f \in L^2(\mathbb{R}^m)$ can be written as

$$ f(x) = \sum_{k \in \mathbb{Z}^m} C_{k,b} \Phi_{k,b}(x) $$

$$ + \sum_{j=0}^\infty \sum_{k \in \mathbb{Z}^m, (G, I) \in I_{j,u}} \sum_{b} c^{(G)}_{j,k,u,b} \psi^{(G)}_{j,k,u,b}(x), $$

(55)

with

$$ C_{k,b} = \int_{\mathbb{R}^m} f(x) \Phi_{k,b}(x) \, dx $$

(56)

and

$$ c^{(G)}_{j,k,u,b} = 2^{\|b\|} \int_{\mathbb{R}^m} f(x) \psi^{(G)}_{j,k,u,b}(x) \, dx. $$

(57)

A straightforward extension is given by the following result.

**Proposition 12.** The collection of $(2^{\|b\|/2} \psi^{(G)}_{j,k,u,b})$ for $j \in \mathbb{Z}$, $(G, I) \in I_{j,u}$ and $k \in \mathbb{Z}^m$ is an orthonormal basis of $L^2(\mathbb{R}^m)$. Thus any function $f \in L^2(\mathbb{R}^m)$ can be written as

$$ f(x) = \sum_{j=0}^\infty \sum_{k \in \mathbb{Z}^m, (G, I) \in I_{j,u}} \sum_{b} c^{(G)}_{j,k,u,b} \psi^{(G)}_{j,k,u,b}(x). $$

(58)

3.1. Wavelet Characterization of the $T_{p,w}^b(y)$ Spaces. Let $u$ be an anisotropic vector. One of the fundamental properties of the anisotropic wavelet bases is that it characterizes $T_{p,w}^b(y)$ spaces. The results are reminiscent of [5] where the authors have considered the isotropic case $u_1 = \cdots = u_m = 1$.

Without any loss of generality, we will present the results in the canonical basis and drop the letter $b$. The proofs adapt all steps and arguments of [5] to the anisotropic setting, using

(i) the properties of the homogeneous norm $|\cdot|_w$,

(ii) anisotropic versions of mean value theorem and Taylor’s theorem with remainder (see [40, 41]),

(iii) anisotropic Triebel wavelet characterization of anisotropic Besov spaces $B_{p,w}^s(\mathbb{R}^m)$ (see [35, 36]).
Proposition 13 (the $u$-mean value theorem). Let $u$ be an anisotropic vector. There exist two positive constants $C$ and $v$ such that for all functions $f$ of class $C^v$ on $\mathbb{R}^m$ and all $x, y \in \mathbb{R}^m$,

$$|f(y) - f(x)| \leq C \sup_{|h| \leq \delta_u} |\partial_a f(x + h)|,$$

(60)

We denote by $\Delta_u$ the additive subsemigroup of $\mathbb{R}$ generated by $0, u_1, u_2, \ldots$ and $\delta_u$. In other words, $\Delta_u$ is the set of all numbers $\delta_u(I)$ as $I$ ranges over $\mathbb{N}^m_0$.

Proposition 14 (the $u$-Taylor inequality). Let $u$ be an anisotropic vector. Put $u_{\min} = \min_{1 \leq i \leq m} u_i$. Suppose $\delta_u \in \Delta_u$, $\delta_u > 0$, and $k = [\delta_u/\Delta_u]$. There are two constants $C_\delta > 0$ and $v > 0$ such that for all functions $f$ of class $C^{(k+1)}$ on $\mathbb{R}^m$ and all $x, y \in \mathbb{R}^m$,

$$|f(y) - P(y - x)| \leq C_\delta \sum_{|I| = k+1, \sum I_i \leq \delta_u} |y - x|^{|I|}$$

$$\cdot \sup_{|h| \leq \delta_u} |\partial^I f(x + h)|,$$

(61)

where $P$ is the so-called $u$-Taylor polynomial of $f$ at $x$ of homogeneous degree $\delta_u$, given by

$$P(y - x) = \sum_{|I| = k+1, \sum I_i \leq \delta_u} \frac{\partial^I f(x)}{I!} (y - x)^I.$$

(62)

Proposition 15. Let $u$ be an anisotropic vector. Let $0 < p, q \leq \infty$ and $s \in \mathbb{R}$. Then $f \in B_s^{a,p}(\mathbb{R}^m)$ if and only if its norm

$$\|f\|_{B_s^{a,p}(\mathbb{R}^m)}$$

$$= \left( \sum_{k \in \mathbb{Z}^m} |C_k|^p \right)^{1/p}$$

(63)

$$+ \left( \sum_{j \in \mathbb{N}_0} \left( \sum_{k \in \mathbb{Z}^m} \sum_{l \geq 1, |l| \leq k} |2^{-j(m/p)} \tilde{\varphi}_{j,k}^{GL}|^p \right) \chi_{j,k} \right)^{1/p}$$

is finite, with the usual modification if $p = \infty$ and/or $q = \infty$. This characterization does not depend on the chosen (smooth enough) wavelets $\psi_F$ and $\psi_M$.

In [35] (section 5.3), a transference method has been proposed with the aim of showing that isotropic and anisotropic Besov spaces are isomorphic together and with the space $b^s_{p,q}$ of all union of sequences $(C_k)_{k \in \mathbb{Z}^m}$ and $(\epsilon_{j,k})_{j \in \mathbb{N}_0, k \in \mathbb{Z}^m}$ such that

$$\left( \sum_{k \in \mathbb{Z}^m} |C_k|^p \right)^{1/p}$$

$$+ \left( \sum_{j \in \mathbb{N}_0} \left( \sum_{k \in \mathbb{Z}^m} |2^{-j(m/p)} \epsilon_{j,k}|^p \right)^{q/p} \right)^{1/q} < \infty,$$

(64)

with the usual modification if $p = \infty$ and/or $q = \infty$. In particular, any embedding theorem for isotropic Besov spaces can be carried over to anisotropic Besov spaces.

Definition 16. Let $u$ be an anisotropic vector. Let $s, s'$ be real number and let $p$ and $q$ be positive real numbers. One says that a function $f$ belongs to $X^{s',p}_{u}(y)$ if there exists $A > 0$ such that

$$\varepsilon_j = 2^{j(s-m/p)} \left( \sum_{|I| \leq 2^j \epsilon_{j,k} \leq A} \sum_{(\epsilon_{j,k}) \in \mathbb{I}} |\chi_{j,k}^{GL}|^p \right)$$

$$\cdot \left( 1 + 2^j |y - 2^{-j} \epsilon_{j,k}|^{s'/p} \right)^{1/p} \in \mathcal{E}^j,$$

(65)

where

$$2^{-j} \epsilon_{j,k} = \left( \frac{k_1}{2^j}, \ldots, \frac{k_m}{2^j} \right).$$

(66)

If $u_1 = \cdots = u_m = 1$ then this space was already introduced by Meyer and Xu [43] in order to study local oscillating behaviors.

Proposition 17. Let $u$ be an anisotropic vector. Let $1 \leq p < \infty$, $0 \leq s \leq \beta$ where $\beta$ is the regularity of the Daubechies wavelets. Let $y \in \mathbb{R}^m$ and $f \in L_p^u(\mathbb{R}^m)$.

(i) If $f \in X^{s,-p,1}_{u}(y)$, then $f \in T_p^u(y)$.

(ii) If $f \in T_{u,s-m/p}(y)$, then there exist $A$ and $C > 0$ such that

$$\forall j$$

$$2^{j(s-m/p)} \sum_{|I| \leq 2^j \epsilon_{j,k} \leq A} \sum_{(\epsilon_{j,k}) \in \mathbb{I}} |\chi_{j,k}^{GL}|^p$$

$$\cdot \left( 1 + 2^j |y - 2^{-j} \epsilon_{j,k}|^{-sp} \right) \leq C.$$
(1) \( i_{u,p}(y) \) is positive and independent of \( A \),
(2) we have always
\[
\alpha_{p,u}(y) \leq i_{u,p}(y) - \frac{m}{p},
\]
(3) if \( f \in B^{\delta,p}_{p,u}(\mathbb{R}^m) \) for some \( \delta > 0 \) then
\[
\alpha_{p,u}(y) = i_{u,p}(y) - \frac{m}{p}.
\]

Both Theorem 9 and Proposition 18 yield the following characterization of the pointwise directional \( p \)-exponent.

**Corollary 19** (let \( m \geq 2 \)). Let \( e \in \mathbb{R}^m \) be a unit vector and \( b \) be any orthonormal basis starting with the vector \( e \). Let \( \mathcal{E} \) be the set of all anisotropy vectors \( u \) such that \( u_2 = \cdots = u_m \).

Let \( f \in L^p_{\mathcal{E}}(\mathbb{R}^m) \). If \( f \in B^{\delta,p}_{p,u}(\mathbb{R}^m) \) for some \( \delta > 0 \) then the \( p \)-exponent of \( f \) at \( y \) in direction \( e \) is given by
\[
\alpha(p, y, e) = \sup_{u \in \mathcal{E}} \left( i_{u,p}(y) - m/p \right).
\]

### 3.2. Wavelet Leaders Characterization

Without any loss of generality, we will present the results in the canonical basis. Let \( u \) be an anisotropic vector. By \( \lambda_u = \lambda_{1,j,u} \) we denote a \( u \)-dyadic rectangle in \( \mathbb{R}^m \) of scale \( j \), which has the form
\[
\lambda_u = \lambda_{1,j,u} = 2^{-j}k + \prod_{i=1}^m \left( 0, 2^{-j}i \right),
\]
where \( 2^{-j}k \) was defined in (66).

Denote by \( \Lambda_{j,u} \) all \( u \)-dyadic rectangles in \( \mathbb{R}^m \) of scale \( j \). For \( \lambda_u \in \Lambda_{j,u} \) set
\[
|c_{\lambda_u}| = \max |c_{\lambda_{j,u}}|,
\]
where the maximum is taken over all indices \( G \) giving the same \( l \) at scale \( j \).

Let \( f \) be a function on \( \mathbb{R}^m \). Define its \( u \) wavelet scaling function as
\[
\eta_u(p) = \lim_{j \to \infty} \frac{\log \left( 2^{-mj} \sum_{\lambda_u \in \Lambda_{j,u}} |c_{\lambda_u}|^p \right)}{\log(2^j)}.
\]

By Proposition 15, this scaling function does not depend on the chosen (smooth enough) wavelets, and
\[
\forall 0 < q \leq \infty \quad \eta_u(p) = \sup \left\{ r : f \in B^{r,p}_{p,u}(\mathbb{R}^m) \right\}.
\]

For \( p > 0 \) such that \( \eta_u(p) > 0 \), the \( u \)-\( p \) wavelet leader of \( f \) at a rectangle \( \lambda_u \in \Lambda_{j,u} \) is defined as
\[
\epsilon_{\lambda_u}^{(p)} = \left( \sum_{j=j_{\lambda_u} - 3}^{\infty} \sum_{\lambda_{j,u} \in \Lambda_{j,u} \cap \lambda_u} |c_{\lambda_{j,u}}|^p 2^{-m(j-i)} \right)^{1/p},
\]
where \( 3\lambda_u \) is the set formed by the rectangle \( \lambda_u \) and all its adjacent rectangles at scale \( j \).

Remark 20. Note that if \( \eta_u(p) > 0 \) then \( f \in L^p_{\lambda_u} \) and if \( \eta_u(p) < 0 \) then \( f \notin L^p_{\lambda_u} \) (see [44]).

If \( x \in \mathbb{R}^m \), denote by \( \lambda_{j,u}(x) \) the unique rectangle at the scale \( j \) that contains \( x \).

As in [45], the third point in Proposition 18 yields the following result.

**Proposition 21.** Let \( u \) be an anisotropic vector. Let \( p \geq 1 \). If \( \eta_u(p) > 0 \) then
\[
\alpha_{p,u}(x) = \liminf_{j \to \infty} \frac{\log \left( \frac{\epsilon_{\lambda_u}^{(p)}}{\lambda_{j,u}(x)} \right)}{\log(2^j)}.
\]

Both Theorem 9 and Proposition 21 yield the following characterization of the pointwise directional \( p \)-exponent.

**Corollary 22** (let \( m \geq 2 \)). Let \( e \in \mathbb{R}^m \) be a unit vector and \( b \) any orthonormal basis starting with the vector \( e \). Let \( \mathcal{E} \) be the set of all anisotropy vectors \( u \) such that \( u_2 = \cdots = u_m \).

Let \( p \geq 1 \). If \( \eta_u(p) > 0 \) then the \( p \)-exponent of \( f \) at \( y \) in direction \( e \) is given by
\[
\alpha(p, y, e) = \sup_{u \in \mathcal{E}} \left( \liminf_{j \to \infty} \frac{\log \left( \frac{\epsilon_{\lambda_u}^{(p)}}{\lambda_{j,u}(x)} \right)}{\log(2^j)} \right).
\]

### 4. Alternative Upper Bound of the Directional \( L^p \) Spectra

In Corollary 10, if \( u \) is an anisotropic vector different from \((1, \ldots, 1)\), then the sets \( E_{p,u} \) are anisotropic but \( \dim \) is isotropic. Computing \( \dim E_{p,u} \) is a very difficult task (see the example of Sierpinski carpet in [46] pages 118–119). To overcome this problem, in [47, 48], we adapted the notion of Hausdorff dimension to the anisotropy \( u \); if \( \Omega \subset \mathbb{R}^d \), define its \( u \)-dimension to be \( \dim u(\Omega) = \sup_{x,y \in \Omega} |x - y|_u \). By replacing in the definition of Hausdorff measure, the usual notion of diameter by the \( u \)-diameter, we easily check (see [49]) that we get the following notion of anisotropic dimension.

**Definition 23.** Let \( \Omega \subset \mathbb{R}^d \), \( \varepsilon > 0 \) and let \( R_{\varepsilon} \) be the set of all coverings \( R = (\Omega_n)_{n \in \mathbb{N}} \) of \( \Omega \) by sets \( \Omega_n \) of \( u \)-diameter \( |\Omega_n|_u \) at most \( \varepsilon \). Let
\[
M^\delta_{\varepsilon,u} (\Omega) = \inf \left\{ \sum_{n \in \mathbb{N}} |\Omega_n|_u^\delta \right\}.
\]

The \( \delta \)-dimensional \( u \)-Hausdorff measure of \( \Omega \) is
\[
M^\delta_u (\Omega) = \lim_{\varepsilon \to 0} M^\delta_{\varepsilon,u} (\Omega).
\]

The \( u \)-Hausdorff dimension of \( \Omega \) is
\[
\dim_u (\Omega) = \inf \left\{ \delta : M^\delta_u (\Omega) = 0 \right\} = \sup \left\{ \delta : M^\delta_u (\Omega) = \infty \right\}.
\]
Note that we get the same value of \( \text{dim}_u(\Omega) \) if we use coverings \( R = (\Omega_n)_{n \in \mathbb{N}} \) of \( \Omega \) by rectangles \( \Omega_n \) with sides parallel to the axes of coordinates and with side-lengths \( 2e^n \) in the \( x_i \)-direction.

In the isotropic case, \(|\Omega|_{(1,1)}\) is equivalent to the Euclidean norm \(|\cdot|\) on \( \mathbb{R}^d \) and \( \text{dim}_u(\Omega) \) coincides with \( \text{dim} \). But if \( u \neq (1, \ldots, 1) \), then \( \text{dim}_u(\Omega) \) does not necessarily coincide with \( \text{dim} \). Actually, if \( u_{\min} = \min_{1 \leq i \leq d} u_i \) and \( u_{\max} = \max_{1 \leq i \leq d} u_i \) there exists \( C \geq 1 \) such that
\[
\forall x \in \mathbb{R}^d \quad \frac{1}{C} \min \{ |x|^{1/u_{\min}}, |x|^{1/u_{\max}} \} \leq |x|_u \quad (83)
\]
and
\[
\frac{1}{C} \min \{ |x|^{1/u_{\min}}, |x|^{1/u_{\max}} \} \leq C \max \{ |x|^{1/b_{\min}}, |x|^{1/b_{\max}} \} .
\]

Definition 24. The \( p - u \) spectrum is defined as
\[
\mathcal{S}_{p,u}(H) = \text{dim} \{ x \in \mathbb{R}^m : \alpha_{p,u}(x) = H \} .
\]

The \( p - u \) upper-spectrum is defined as
\[
\mathcal{D}_{p,u}(H) = \text{dim} \{ x \in \mathbb{R}^m : \alpha_{p,u}(x) \leq H \} .
\]

The restricted \( p - u \) wavelet leader of \( f \) at a rectangle \( \lambda_u \in \Lambda_{j,u} \) is defined as
\[
\tilde{\xi}^{(p)}_{\lambda_u} = \left( \sum_{j=j^{m_{\lambda_u}} \in \Lambda_{j,u} \setminus \Lambda_{j-1,u}} |c_{\lambda_u}|^p 2^{-m_{\lambda_u} j^p} \right)^{1/p} .
\]

The \( p - u \) structure function is defined as
\[
\tilde{S}^{(p)}(q,j) = 2^{-mj} \sum_{\lambda_u \in \Lambda_{j,u}} \left( \tilde{\xi}^{(p)}_{\lambda_u} \right)^q .
\]

The \( p - u \) wavelet leader scaling function is given by
\[
\xi^{(p)}_u(q) = \lim_{j \to \infty} \log \tilde{S}^{(p)}(q,j) / \log(2^{-r}) .
\]

We can easily adapt the (isotropic) general upper bound obtained in \([45]\) or \([50]\) to the anisotropic setting.

**Proposition 25** (let \( p \geq 1 \)). If \( \eta_u(p) > 0 \) then
\[
\forall h \quad \mathcal{D}_{p,u}(h) \leq \inf_{q \in \mathbb{R}} \left( d + hq - \tilde{\xi}^{(p)}_u(q) \right) .
\]

Consequently, we obtain the following result.

**Theorem 26** (let \( m \geq 2 \)). Let \( e \in \mathbb{R}^m \) be a unit vector and \( b \) an orthonormal basis starting with the vector \( e \). Let \( \mathcal{E} \) be the set of all anisotropy vectors \( u \) such that \( u_2 = \cdots = u_m \).

Let \( p \geq 1 \). Then the directional \( L^p \) spectra of \( f \) in direction \( e \) given in Definition 7 satisfy
\[
\forall H \quad d_p(H,e) \leq D_p(H,e)
\]
\[
\leq \inf_{u \in \mathcal{E}} \inf_{H \in \mathbb{R}} \left( u_H q - \tilde{\xi}^{(p)}_u(q) + m \right) .
\]

Proof. By (84)
\[
D_p(H,e) \leq \inf_{u \in \mathcal{E}} \inf_{H \in \mathbb{R}} D_{p,u}(u,H) .
\]

Thus (90) yields (91). \( \square \)

### 5. Examples of Affine Cascade Functions

We will apply Theorem 26 for some examples of self-affine cascade functions. Let \( s \) and \( t \) be two nonnegative integers. We divide the unit square \( \mathcal{R} = [0,1]^2 \) into a uniform grid of rectangles of sides \( 1/t \) and \( 1/s \). Choose \( A \subset \{0,1, \ldots, s-1\} \times \{0,1, \ldots, t-1\} \). For \( \omega = (a,b) \in A \), the contraction \( S_{\omega}(x_1,x_2) = (x_1/s + a/s,x_2/t + b/t) \) maps the unit square \( \mathcal{R} \) into the rectangle
\[
\mathcal{R}_\omega = \left[ \frac{a}{s} , \frac{a+1}{s} \right] \times \left[ \frac{b}{t}, \frac{b+1}{t} \right] .
\]

If \( G \) is a subset of \( \mathbb{R}^m \), we define the mapping \( S \) by
\[
S(G) = \bigcup_{\omega \in A} S_{\omega}(G) .
\]

The Sierpinski carpet \( K \) (see \([28, 29, 48]\)) and references therein) is the unique nonempty compact set satisfying
\[
K = \bigcap_{n \in \mathbb{N}} S^n(\mathcal{R}) .
\]

Let \( \gamma_{\omega} \) be nonnegative scalars. Put \( g(x) = \psi_M(x_1)\psi_M(x_2) \) where \( \psi_M \) is a mother (smooth enough) wavelet. The Sierpinski cascade function adapted to the subdivision \( A \) satisfies the self-affine equation
\[
\forall x \in \mathcal{R} \quad F(x) = \sum_{\omega \in A} \gamma_{\omega} F(S^{-1}_{\omega}(x)) + g(x) .
\]

**Proposition 27** (let \( 1 \leq p < \infty \)). If \( \sum_{\omega \in A} \gamma_{\omega}^p < st \), then the series
\[
F(x) = \sum_{\omega \in A} \gamma_{\omega} F(S^{-1}_{\omega}(x)) + g(x)
\]

is the unique solution in \( L^p(\mathcal{R}) \) for equation (96).

If moreover, \( s = 2^s \) and \( t = 2^t \) with \( S \) and \( T \) two nonnegative integers such that \( S < T \) and \( S + T \) is even and
\((e_1, e_2)\) is the canonical basis in \(\mathbb{R}^2\), then the directional \(L^p\) spectra of \(F\) in directions \(e_1\) and \(e_2\), respectively, satisfy
\[
\forall H \quad d_p(H, e_1) \leq D_p(H, e_1) \leq \inf_{q \in \mathbb{R}} \left( Hq - \frac{\log \sum_{\omega \in A} Y_{\omega}^q}{\log s} \right)
\]
and
\[
\forall H \quad d_p(H, e_2) \leq D_p(H, e_2) \leq \inf_{q \in \mathbb{R}} \left( \frac{T}{S} Hq - \frac{\log \sum_{\omega \in A} Y_{\omega}^q}{\log s} \right).
\]

**Proof.** Clearly series (97) satisfies (96). Since \(\sum_{\omega \in A} Y_{\omega}^q < st\), then assumption \((\sum_{\omega \in A} Y_{\omega})^p < st\) implies that \(F\) is the unique solution in \(L^p(\mathbb{R})\).

Clearly if \(\omega_0 = (a, b)\) then
\[
g \left( S_{\omega_0}^{-1} \cdots S_{e_1}^{-1} (x) \right) = \psi_M \left( s^n x_1 - s^{n-1} a_1 - \cdots - s a_{n-1} - a_n \right) \cdot \psi_M \left( t^n x_2 - t^{n-1} b_1 - \cdots - t b_{n-1} - b_n \right).
\]

Let \(S\) and \(T\) be two positive integers such that \(S < T\) and \(S + T\) is even. Assume that \(s = 2^k\) and \(t = 2^j\). Put
\[
u_1 = \frac{2S}{S + T} \quad \text{and} \quad \nu_2 = \frac{2T}{S + T}.
\]

Clearly \(u = (\nu_1, \nu_2)\) is an anisotropic vector with \(\nu_1 < \nu_2\),
\[
s^n = 2^{\nu_1 n} \quad \text{and} \quad t^n = 2^{\nu_2 n}.
\]
with
\[
j = \frac{n (S + T)}{2}.
\]

By a straightforward computation, the \(u\) wavelet scaling function defined in (75) is given by
\[
\eta_u (p) = 2 - \frac{\log S}{S + T} \sum_{\omega \in A} Y_{\omega}^q.
\]
Since \((\sum_{\omega \in A} Y_{\omega})^p < st\), then \(\sum_{\omega \in A} Y_{\omega}^p < st\). It follows that \(\eta_u (p) > 0\).

The restricted \(p - u\) wavelet leader of \(F\) at a rectangle \(\lambda_u \in \Lambda_{j, u}\) (defined in (87)) satisfies
\[
\left( \tilde{e}_{\lambda_u}^{(p)} \right) = \sum_{n' \geq n} (st)^{|n' - n|} \left( \sum_{\omega \in A} Y_{\omega}^p \right)^{n' - n} \left( \gamma_{\omega_1} \cdots \gamma_{\omega_n} \right)^p.
\]

Since \(\sum_{\omega \in A} Y_{\omega}^p < st\), then
\[
\tilde{e}_{\lambda_u}^{(p)} \sim \gamma_{\omega_1} \cdots \gamma_{\omega_n},
\]
and
\[
\tilde{e}_u^{(p)} (q) = 2 - \frac{2 \log S}{S + T} \sum_{\omega \in A} Y_{\omega}^q.
\]

If \((e_1, e_2)\) is the canonical basis in \(\mathbb{R}^2\) then Theorem 26 yields
\[
\forall H \quad d_p(H, e_1) \leq D_p(H, e_1) \leq \inf_{u_1 \in \mathbb{R}} \left( u_1 Hq - \tilde{e}_u^{(p)} (q) + 2 \right)
\]
and
\[
\forall H \quad d_p(H, e_2) \leq D_p(H, e_2) \leq \inf_{u_1 \in \mathbb{R}} \left( u_2 Hq - \tilde{e}_u^{(p)} (q) + 2 \right).
\]

Using (102) and (108), we deduce results (98) and (99). \(\Box\)

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.

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### References


