Research Article

Nontrivial Solutions for a Higher Order Nonlinear Fractional Boundary Value Problem Involving Riemann-Liouville Fractional Derivatives

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In this paper using topological degree we study the existence of nontrivial solutions for a higher order nonlinear fractional boundary value problem involving Riemann-Liouville fractional derivatives. Here, the nonlinearity can be sign-changing and can also depend on the derivatives of unknown functions.

1. Introduction

In this paper, we investigate the existence of nontrivial solutions for the higher order nonlinear fractional boundary value problem involving Riemann-Liouville fractional derivatives:

$$\begin{aligned} D_{0+}^{\alpha} u(t) \\ = -f\left(t, u(t), D_{0+}^{\beta_1} u(t), D_{0+}^{\beta_2} u(t), \dots, D_{0+}^{\beta_{n-1}} u(t)\right), \end{aligned} \quad (1)$$

$$0 < t < 1,$$

$$u(0) = u'(0) = \dots = u^{(n-2)}(0) = D_{0+}^{\beta} u(1) = 0,$$

where $n-1 < \alpha \leq n$, $i-1 < \beta_i \leq i$, $\alpha - \beta_{n-1} > \alpha - \beta > 1$, $D_{0+}^{\alpha}, D_{0+}^{\beta}, D_{0+}^{\beta_i}$ are the Riemann-Liouville fractional derivatives, $i = 1, 2, \dots, n-1$, and the function $f : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous; here $\mathbb{R} := (-\infty, +\infty)$.

There are a large number of papers in the literature studying fractional differential equations because of applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, and control theory;

we refer the reader to [1–35] and the references therein. In [1], using fixed point index theory in cones, the authors studied the existence and multiplicity of positive solutions for the nonlocal fractional differential equation boundary value problem

$$\begin{aligned} D_{0+}^{\alpha} u(t) + q(t) f(t, u(t)) &= 0, \\ 0 < t < 1, \quad n-1 < \alpha \leq n, \\ u(0) = u'(0) = \dots = u^{(n-2)}(0) &= 0, \end{aligned} \quad (2)$$

$$u(1) = \int_0^1 u(s) dA(s),$$

where $f(t, u)$ is a nonnegative continuous function on $[0, 1] \times \mathbb{R}^+$ and satisfies the conditions:

(H1) _{Wang} sublinear growth case: $\liminf_{u \rightarrow 0^+} (f(t, u)/u) > \lambda_1$, $\limsup_{u \rightarrow +\infty} (f(t, u)/u) < \lambda_1$, uniformly on $t \in [0, 1]$,

(H2) _{Wang} superlinear growth case: $\liminf_{u \rightarrow +\infty} (f(t, u)/u) > \lambda_1$, $\limsup_{u \rightarrow 0^+} (f(t, u)/u) < \lambda_1$, uniformly on $t \in [0, 1]$,

where λ_1 is the first eigenvalue for the problem:

$$\begin{aligned} D_{0+}^\alpha u(t) + \lambda q(t) u(t) &= 0, \quad 0 < t < 1, \\ u(0) = u'(0) &= \dots = u^{(n-2)}(0) = 0 \\ u(1) &= \int_0^1 u(s) dA(s). \end{aligned} \quad (3)$$

Conditions similar to $(H1)_{\text{Wang}}$ and $(H2)_{\text{Wang}}$ can be found in [2–9]. In [2], the authors studied the existence of positive solutions for the nonlinear fractional differential equation:

$$\begin{aligned} -D_{0+}^\alpha y(t) &= f(t, y(t), y'(t), \dots, y^{(n-2)}(t)), \\ 0 < t < 1, \\ y(0) = y'(0) &= \dots = y^{(n-2)}(0) = 0, \\ y^{(n-2)}(1) &= \lambda [y^{(n-2)}], \end{aligned} \quad (4)$$

where $\lambda[y^{(n-2)}] = \int_0^1 y^{(n-2)}(t) dA(t)$ is a linear functional on $C[0, 1]$ given by a Stieltjes integral. The nonlinearity $f(t, z_1, z_2, \dots, z_{n-1})$ satisfies the conditions:

$$\begin{aligned} \liminf_{z_1 \rightarrow 0, \dots, z_{n-1} \rightarrow 0} \frac{f(t, z_1, z_2, \dots, z_{n-1})}{z_1 + \dots + z_{n-1}} &> \lambda_1, \\ \limsup_{z_1 + \dots + z_{n-1} \rightarrow +\infty, z_{n-1} \rightarrow +\infty} \frac{f(t, z_1, z_2, \dots, z_{n-1})}{z_{n-1}} &< \lambda_1, \\ \text{uniformly on } t \in [0, 1], \end{aligned} \quad (5)$$

and

$$\begin{aligned} \liminf_{z_1 \rightarrow 0, \dots, z_{n-1} \rightarrow 0} \frac{f(t, z_1, z_2, \dots, z_{n-1})}{z_{n-1}} &< \lambda_1, \\ \limsup_{z_1 + \dots + z_{n-1} \rightarrow +\infty} \frac{f(t, z_1, z_2, \dots, z_{n-1})}{z_1 + \dots + z_{n-1}} &> \lambda_1, \\ \text{uniformly on } t \in [0, 1]. \end{aligned} \quad (6)$$

In [10–14], the authors used Lipschitz conditions to study the existence and uniqueness of solutions for various fractional differential equations; for example, in [10], the author used the u_0 -positive operator method to obtain the unique solution for the fractional boundary value problem:

$$\begin{aligned} D_t^p x(t) + p(t) f(t, x(t)) + q(t) &= 0, \quad t \in (0, 1), \\ x(0) = x'(0) &= 0, \\ x(1) &= 0, \end{aligned} \quad (7)$$

where f is a Lipschitz continuous function, with the Lipschitz constant associated with the first eigenvalue for the relevant operator. In [15], the authors used the mixed monotone

method to obtain a unique positive solution for the fractional boundary value problem:

$$\begin{aligned} -D_{0+}^\alpha y(t) &= q(t) f(t, y(t), y'(t), \dots, y^{(n-2)}(t)), \\ 0 < t < 1, \\ y(0) = y'(0) &= \dots = y^{(n-2)}(0) = y^{(n-2)}(1) = 0, \end{aligned} \quad (8)$$

where the nonlinearity $f = g + h$ and g, h have different monotone properties. In [16, 17], the authors extended the methods in [15] to fractional p -Laplacian equations and established some appropriate iterative sequences which converge to their solutions.

There are also some papers in the literature devoted to sign-changing nonlinearities; we refer the reader to [7–14, 18, 19, 22, 23, 26–33, 35]. For example, in [18] using coincidence degree theory, the author investigated the existence of non-trivial solutions for the coupled system of nonlinear fractional differential equations:

$$\begin{aligned} D_{0+}^\alpha u(t) &= f(t, v(t), D_{0+}^{\beta-2} v(t), D_{0+}^{\beta-1} v(t)), \\ 0 < t < 1, \\ D_{0+}^\beta v(t) &= g(t, u(t), D_{0+}^{\alpha-2} u(t), D_{0+}^{\alpha-1} u(t)), \\ 0 < t < 1, \\ u(0) &= 0, \\ D_{0+}^{\alpha-1} u(0) &= D_{0+}^{\alpha-1} u(\eta), \\ u(1) &= \sum_{i=1}^{m_1} \alpha_i u(\eta_i), \\ v(0) &= 0, \\ D_{0+}^{\beta-1} v(0) &= D_{0+}^{\beta-1} v(\xi), \\ v(1) &= \sum_{i=1}^{m_2} \beta_i v(\xi_i), \end{aligned} \quad (9)$$

where $f, g : [0, 1] \times \mathbb{R}^3 \rightarrow \mathbb{R}$ satisfy Carathéodory conditions.

Motivated by the aforementioned papers and some integer order equations [36–39], in this paper we use topological degree to study the existence of nontrivial solutions for the higher order nonlinear fractional boundary value problem (1). Our nonlinearity f can be sign-changing and can also depend on the derivatives of unknown functions. Moreover, our conditions improve the conditions in (5) and (6); see (H4)–(H5) in Section 3.

2. Preliminaries

We present some basic material involving Riemann-Liouville fractional derivatives; for details we refer the reader to the books [40–42].

Definition 1 (see [40–42]). The Riemann-Liouville fractional derivative of order $\alpha > 0$ of a continuous function $f : (0, +\infty) \rightarrow (-\infty, +\infty)$ is given by

$$D_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt} \right)^n \int_0^t (t-s)^{n-\alpha-1} f(s) ds, \quad (10)$$

where $n = [\alpha] + 1$, $[\alpha]$ denotes the integer part of number α , provided that the right side is pointwise defined on $(0, +\infty)$.

Definition 2 (see [40–42]). The Riemann-Liouville fractional integral of order $\alpha > 0$ of a function $f : (0, +\infty) \rightarrow (-\infty, +\infty)$ is given by

$$I_{0+}^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad (11)$$

provided that the right side is pointwise defined on $(0, +\infty)$.

Lemma 3 (see [3, Lemma 2.2]). Let $\alpha > 0$, then for $u, D_{0+}^{\alpha} u \in C(0, 1) \cap L(0, 1)$, we have

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \dots + c_N t^{\alpha-N}, \quad (12)$$

for some $c_i \in \mathbb{R}$, $i = 1, 2, \dots, N$,

where N is the smallest integer greater than or equal to α .

In what follows, we present Green's function associated with problem (1). The following lemma is from [3]; for completeness we provide the proof.

Lemma 4 (see [3, Lemma 2.3]). Let $v(t) = D_{0+}^{\beta_{n-1}} u(t)$. Then (1) can be transformed into the following nonlinear integrodifferential equation:

$$\begin{aligned} D_{0+}^{\alpha-\beta_{n-1}} v(t) &= -f(t, I_{0+}^{\beta_{n-1}} v(t), I_{0+}^{\beta_{n-1}-\beta_1} v(t), \dots, \\ &\quad I_{0+}^{\beta_{n-1}-\beta_{n-2}} v(t), v(t)), \quad 0 < t < 1, \\ I_{0+}^{\beta_{n-1}-n+2} v(0) &= D_{0+}^{\beta-\beta_{n-1}} v(1) = 0. \end{aligned} \quad (13)$$

Consequently, we obtain that (13) is equivalent to the following Hammerstein type integral equation:

$$\begin{aligned} v(t) &= \int_0^1 K_1(t, s) f(s, I_{0+}^{\beta_{n-1}} v(s), I_{0+}^{\beta_{n-1}-\beta_1} v(s), \dots, \\ &\quad I_{0+}^{\beta_{n-1}-\beta_{n-2}} v(s), v(s)) ds, \end{aligned} \quad (14)$$

where

$$K_1(t, s) = \frac{1}{\Gamma(\alpha - \beta_{n-1})} \begin{cases} t^{\alpha-\beta_{n-1}-1} (1-s)^{\alpha-\beta-1} - (t-s)^{\alpha-\beta_{n-1}-1}, & 0 \leq s \leq t \leq 1, \\ t^{\alpha-\beta_{n-1}-1} (1-s)^{\alpha-\beta-1}, & 0 \leq t \leq s \leq 1. \end{cases} \quad (15)$$

Proof. Let $v(t) = D_{0+}^{\beta_{n-1}} u(t)$. Then from [4, equations (2.11)], we have

$$\begin{aligned} D_{0+}^{\alpha} u(t) &= D_{0+}^{\alpha-\beta_{n-1}} v(t), \\ D_{0+}^{\beta_1} u(t) &= D_{0+}^{\beta_{n-1}-\beta_1} v(t), \\ D_{0+}^{\beta_2} u(t) &= D_{0+}^{\beta_{n-1}-\beta_2} v(t), \\ &\dots, \\ D_{0+}^{\beta_{n-2}} u(t) &= D_{0+}^{\beta_{n-1}-\beta_{n-2}} v(t), \\ D_{0+}^{\beta_{n-1}} u(t) &= v(t). \end{aligned} \quad (16)$$

Also we obtain $u^{(n-2)}(t) = I_{0+}^{\beta_{n-1}-n+2} v(t)$, and $D_{0+}^{\beta} u(t) = D_{0+}^{\beta-\beta_{n-1}} v(t)$. Hence, by $u(t) = I_{0+}^{\beta_{n-1}} v(t)$, we have (13). Consequently, Lemma 3 implies that (13) can be reduced to an equivalent integral equation:

$$\begin{aligned} v(t) &= -I_{0+}^{\alpha-\beta_{n-1}} f(t, I_{0+}^{\beta_{n-1}} v(t), I_{0+}^{\beta_{n-1}-\beta_1} v(t), \dots, \\ &\quad I_{0+}^{\beta_{n-1}-\beta_{n-2}} v(t), v(t)) + C_1 t^{\alpha-\beta_{n-1}-1} + C_2 t^{\alpha-\beta_{n-1}-2}, \end{aligned} \quad (17)$$

for some $C_i \in \mathbb{R}$, $i = 1, 2$. From the condition $I_{0+}^{\beta_{n-1}-n+2} v(0) = 0$, we have $C_2 = 0$. Then we obtain

$$\begin{aligned} D_{0+}^{\beta-\beta_{n-1}} v(t) &= -\frac{1}{\Gamma(\alpha-\beta)} \int_0^t (t-s)^{\alpha-\beta-1} \tilde{f}(s) ds \\ &\quad + C_1 \frac{\Gamma(\alpha-\beta_{n-1})}{\Gamma(\alpha-\beta)} t^{\alpha-\beta-1}, \end{aligned} \quad (18)$$

where we use $\tilde{f}(t)$ to replace $f(t, \cdot, \dots, \cdot)$. Then $D_{0+}^{\beta-\beta_{n-1}} v(1) = 0$ solves the above equation, and we obtain

$$C_1 = \frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 (1-s)^{\alpha-\beta-1} \tilde{f}(s) ds. \quad (19)$$

Therefore, we have

$$\begin{aligned} v(t) &= -I_{0+}^{\alpha-\beta_{n-1}} \tilde{f}(t) \\ &\quad + \frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 t^{\alpha-\beta_{n-1}-1} (1-s)^{\alpha-\beta-1} \tilde{f}(s) ds \\ &= -\frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^t (t-s)^{\alpha-\beta_{n-1}-1} \tilde{f}(s) ds \\ &\quad + \frac{1}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 t^{\alpha-\beta_{n-1}-1} (1-s)^{\alpha-\beta-1} \tilde{f}(s) ds \\ &= \int_0^1 K_1(t, s) \tilde{f}(s) ds. \end{aligned} \quad (20)$$

This completes the proof. $\square$

Lemma 5 (see [3, Lemma 2.4]). *The function K_1 has the following properties:*

(i) $K_1 : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^+$ is continuous,

(ii) for all $(t, s) \in [0, 1] \times [0, 1]$, $t^{\alpha-\beta_{n-1}-1} K_1(1, s) \leq K_1(t, s) \leq K_1(1, s)$, where $K_1(1, s) = ((1-s)^{\alpha-\beta-1} - (1-s)^{\alpha-\beta_{n-1}-1})/\Gamma(\alpha-\beta_{n-1})$.

Let $E := C[0, 1]$, $\|v\| := \max_{t \in [0, 1]} |v(t)|$, $P := \{v \in E : v(t) \geq 0, \forall t \in [0, 1]\}$, $P_0 = \{v \in P : v(t) \geq t^{\alpha-\beta_{n-1}-1} \|v\|, \forall t \in [0, 1]\}$. Then $(E, \|\cdot\|)$ is a real Banach space, and P, P_0 are cones on E . From Lemma 4, we can define an operator $A : E \rightarrow E$ as follows:

$$(Av)(t) = \int_0^1 K_1(t, s) f(s, I_{0+}^{\beta_{n-1}} v(s), I_{0+}^{\beta_{n-1}-\beta_1} v(s), \dots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v(s), v(s)) ds, \quad \text{for } v \in E, \quad (21)$$

where K_1 is denoted by (15). Note that our functions K_1, f are continuous, so the operator A is a completely continuous operator. Moreover if there is a $\bar{v} \in E$ as a fixed point of A , i.e., (13) has solution $\bar{v}$, then from $\bar{v}(t) = D_{0+}^{\beta_{n-1}} \bar{u}(t)$, we have that $\bar{u}$ is a solution for (1). Therefore we study the existence of fixed points of the operator A . For n nonnegative real numbers $\mu_1, \mu_2, \dots, \mu_n$ which are not all zero, let

$$(L_{\mu_1, \mu_2, \dots, \mu_n} v)(t) = \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) v(s) ds, \quad \text{for } v \in E, \quad (22)$$

where $K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) = \mu_n K_n(t, s) + \mu_{n-1} K_{n-1}(t, s) + \dots + \mu_1 K_1(t, s)$ for $(t, s) \in [0, 1] \times [0, 1]$, and $K_i (i = 2, \dots, n)$ are

$$\begin{aligned} & \begin{bmatrix} K_2(t, s) \\ K_3(t, s) \\ \vdots \\ K_{n-1}(t, s) \\ K_n(t, s) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{\Gamma(\beta_{n-1} - \beta_{n-2})} \int_0^\tau K_1(t, \tau) (\tau - s)^{\beta_{n-1} - \beta_{n-2} - 1} d\tau \\ \frac{1}{\Gamma(\beta_{n-1} - \beta_{n-3})} \int_0^\tau K_1(t, \tau) (\tau - s)^{\beta_{n-1} - \beta_{n-3} - 1} d\tau \\ \vdots \\ \frac{1}{\Gamma(\beta_{n-1} - \beta_1)} \int_0^\tau K_1(t, \tau) (\tau - s)^{\beta_{n-1} - \beta_1 - 1} d\tau \\ \frac{1}{\Gamma(\beta_{n-1})} \int_0^\tau K_1(t, \tau) (\tau - s)^{\beta_{n-1} - 1} d\tau \end{bmatrix}, \quad (23) \\ & \text{for } t, s \in [0, 1]. \end{aligned}$$

Let $r(L_{\mu_1, \mu_2, \dots, \mu_n})$ denote the spectral radius of $L_{\mu_1, \mu_2, \dots, \mu_n}$, and from Gelfand's theorem we can prove $r(L_{\mu_1, \mu_2, \dots, \mu_n}) > 0$ (the proof is standard; see [6, Lemma 5]).

Define an operator $A_1 : E \rightarrow E$ as follows:

$$(A_1 z)(t) = \int_0^1 K_1(t, s) z(s) ds, \quad \text{for } z \in E. \quad (24)$$

Then from Lemma 5(ii), we have

$$A_1(P) \subset P_0. \quad (25)$$

Lemma 6 (see [43, Theorem 19.3]). *Let P be a reproducing cone in a real Banach space E and let $L : E \rightarrow E$ be a compact linear operator with $L(P) \subset P$. Let $r(L)$ be the spectral radius of L . If $r(L) > 0$, then there exists $\varphi \in P \setminus \{0\}$ such that $L\varphi = r(L)\varphi$.*

Therefore, from Lemma 6 there exists $\varphi_{\mu_1, \mu_2, \dots, \mu_n} \in P \setminus \{0\}$ such that

$$L_{\mu_1, \mu_2, \dots, \mu_n} \varphi_{\mu_1, \mu_2, \dots, \mu_n} = r(L_{\mu_1, \mu_2, \dots, \mu_n}) \varphi_{\mu_1, \mu_2, \dots, \mu_n}. \quad (26)$$

Hence, we obtain

$$\begin{aligned} \varphi_{\mu_1, \mu_2, \dots, \mu_n}(t) &= \frac{1}{r(L_{\mu_1, \mu_2, \dots, \mu_n})} (L_{\mu_1, \mu_2, \dots, \mu_n} \varphi_{\mu_1, \mu_2, \dots, \mu_n})(t) \\ &= \frac{1}{r(L_{\mu_1, \mu_2, \dots, \mu_n})} \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) \varphi_{\mu_1, \mu_2, \dots, \mu_n}(s) ds \\ &= \frac{1}{r(L_{\mu_1, \mu_2, \dots, \mu_n})} \int_0^1 [\mu_n K_n(t, s) + \mu_{n-1} K_{n-1}(t, s) \\ &\quad + \dots + \mu_1 K_1(t, s)] \varphi_{\mu_1, \mu_2, \dots, \mu_n}(s) ds, \quad t \in [0, 1]. \end{aligned} \quad (27)$$

Using Lemma 5(ii) and the definitions of $K_i (i = 1, 2, \dots, n)$, we have

$$\begin{aligned} \|\varphi_{\mu_1, \mu_2, \dots, \mu_n}\| &\leq \frac{1}{r(L_{\mu_1, \mu_2, \dots, \mu_n})} \int_0^1 [\mu_n K_n(1, s) \\ &\quad + \mu_{n-1} K_{n-1}(1, s) + \dots + \mu_1 K_1(1, s)] \\ &\quad \cdot \varphi_{\mu_1, \mu_2, \dots, \mu_n}(s) ds, \end{aligned} \quad (28)$$

and

$$\begin{aligned} \varphi_{\mu_1, \mu_2, \dots, \mu_n}(t) &\geq \frac{t^{\alpha-\beta_{n-1}-1}}{r(L_{\mu_1, \mu_2, \dots, \mu_n})} \int_0^1 [\mu_n K_n(1, s) \\ &\quad + \mu_{n-1} K_{n-1}(1, s) + \dots + \mu_1 K_1(1, s)] \\ &\quad \cdot \varphi_{\mu_1, \mu_2, \dots, \mu_n}(s) ds \geq t^{\alpha-\beta_{n-1}-1} \|\varphi_{\mu_1, \mu_2, \dots, \mu_n}\|. \end{aligned} \quad (29)$$

This implies

$$\varphi_{\mu_1, \mu_2, \dots, \mu_n} \in P_0. \quad (30)$$

Lemma 7 (see [44]). *Let E be a Banach space and Ω a bounded open set in E . Suppose that $A : \Omega \rightarrow E$ is a continuous compact operator. If there exists $u_0 \in E \setminus \{0\}$ such that*

$$u - Au \neq \mu u_0, \quad \forall u \in \partial\Omega, \quad \mu \geq 0, \quad (31)$$

then the topological degree $\deg(I - A, \Omega, 0) = 0$.

Lemma 8 (see [44]). Let E be a Banach space and Ω a bounded open set in E with $0 \in \Omega$. Suppose that $A : \Omega \rightarrow E$ is a continuous compact operator. If

$$Au \neq \mu u, \quad \forall u \in \partial\Omega, \quad \mu \geq 1, \quad (32)$$

then the topological degree $\deg(I - A, \Omega, 0) = 1$.

3. Main Results

Let $\mu_1, \mu_2, \dots, \mu_n; \tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n$ be two families of nonnegative real numbers which are not all zero. Now we list our assumptions for f as follows:

(H1) $f : [0, 1] \times \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous,

(H2) there exist $b(t), c(t) \in C[0, 1]$ with $b(t), c(t) \geq 0$ ($\neq 0$) and $K(x_n, x_{n-1}, \dots, x_1) \in C[\mathbb{R}^n, \mathbb{R}^+]$ such that

$$\begin{aligned} f(t, x_n, x_{n-1}, \dots, x_1) &\geq -b(t) \\ &- c(t) K(x_n, x_{n-1}, \dots, x_1), \end{aligned} \quad (33)$$

$$\forall x_i \in \mathbb{R}, \quad t \in [0, 1], \quad i = 1, 2, \dots, n,$$

(H3) $\lim_{\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| \rightarrow +\infty} (K(x_n, x_{n-1}, \dots, x_1) / (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|)) = 0$,

(H4) $\liminf_{\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| \rightarrow +\infty} (f(t, x_n, x_{n-1}, \dots, x_1) / (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|)) > \lambda_{\mu_1, \mu_2, \dots, \mu_n}$, uniformly for $t \in [0, 1]$,

(H5) $\limsup_{\tilde{\mu}_n|x_n| + \tilde{\mu}_{n-1}|x_{n-1}| + \dots + \tilde{\mu}_1|x_1| \rightarrow 0} (|f(t, x_n, x_{n-1}, \dots, x_1)| / (\tilde{\mu}_n|x_n| + \tilde{\mu}_{n-1}|x_{n-1}| + \dots + \tilde{\mu}_1|x_1|)) < \lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}$, uniformly for $t \in [0, 1]$, where

$$\begin{aligned} \lambda_{\mu_1, \mu_2, \dots, \mu_n} &= r^{-1} (L_{\mu_1, \mu_2, \dots, \mu_n}), \\ \lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} &= r^{-1} (L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}). \end{aligned} \quad (34)$$

Theorem 9. Assume that (H1)-(H5) are satisfied. Then (1) has at least one nontrivial solution.

Proof. From (H4), there exist $\varepsilon_0 > 0$ and $X_0 > 0$ such that

$$\begin{aligned} f(t, x_n, x_{n-1}, \dots, x_1) &\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0) \\ &\cdot (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|), \end{aligned} \quad (35)$$

for all $t \in [0, 1], \mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| > X_0$.

For any given ε with $\varepsilon_0 - \|\varepsilon\| > 0$, and using (H3) there exists $X_1 > X_0$ such that

$$\begin{aligned} K(x_n, x_{n-1}, \dots, x_1) &\leq \varepsilon (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|), \end{aligned} \quad (36)$$

$$\forall \mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| > X_1.$$

Hence, (H2), (35), and (36) enable us to obtain

$$\begin{aligned} f(t, x_n, x_{n-1}, \dots, x_1) &\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0) \\ &\cdot (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|) - b(t) \\ &- c(t) K(x_n, x_{n-1}, \dots, x_1) \geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0) \\ &\cdot (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|) - b(t) \\ &- \varepsilon c(t) (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|) \\ &\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon c(t)) \\ &\cdot (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|) - b(t) \\ &\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \\ &\cdot (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|) - \|b\|, \end{aligned} \quad (37)$$

for all $t \in [0, 1], \mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| > X_1$. Let

$$\begin{aligned} C_{X_1} &= (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) X_1 \\ &+ \max_{0 \leq t \leq 1, \mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| \leq X_1} |f(t, x_n, x_{n-1}, \dots, \\ &x_1)|, \end{aligned} \quad (38)$$

$$K^* = \max_{\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1| \leq X_1} K(x_n, x_{n-1}, \dots, x_1).$$

Then we obtain

$$\begin{aligned} f(t, x_n, x_{n-1}, \dots, x_1) &\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \\ &\cdot (\mu_n|x_n| + \mu_{n-1}|x_{n-1}| + \dots + \mu_1|x_1|) - b(t) \\ &- C_{X_1}, \end{aligned} \quad (39)$$

for all $(t, x_n, x_{n-1}, \dots, x_1) \in [0, 1] \times \mathbb{R}^n$. Note that ε can be chosen arbitrarily small, and let

$$\begin{aligned} R &> \max \left\{ \frac{\|b\| + \|c\| K^* + C_{X_1}}{(\alpha - \beta) (\Gamma(\alpha - \beta_{n-1}) - \varepsilon M_{\mu_1, \mu_2, \dots, \mu_n} \|c\|)}, \right. \\ &\left. \frac{(\lambda_{\mu_1, \mu_2, \dots, \mu_n} + 2\varepsilon_0 - 2\|c\| \varepsilon) (\|b\| + \|c\| K^* + C_{X_1})}{(\alpha - \beta) \Gamma(\alpha - \beta_{n-1}) (\varepsilon_0 - \|c\| \varepsilon - (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + 2\varepsilon_0 - 2\|c\| \varepsilon) (\varepsilon \|c\| M_{\mu_1, \mu_2, \dots, \mu_n} / \Gamma(\alpha - \beta_{n-1})))} \right\}, \end{aligned} \quad (40)$$

where $M_{\mu_1, \mu_2, \dots, \mu_n} = \int_0^1 (1-s)^{\alpha-\beta-1} ((\mu_n/\Gamma(\beta_{n-1})) \int_0^s (s-\tau)^{\beta_{n-1}-1} d\tau + (\mu_{n-1}/\Gamma(\beta_{n-1}-\beta_1)) \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} d\tau + \dots + \mu_1) ds$.

Now we claim

$$v - Av \neq \mu \varphi_{\mu_1, \mu_2, \dots, \mu_n}, \quad \forall v \in \partial B_R, \quad \mu \geq 0, \quad (41)$$

where $\varphi_{\mu_1, \mu_2, \dots, \mu_n}$ is the positive eigenfunction of $L_{\mu_1, \mu_2, \dots, \mu_n}$ corresponding to the eigenvalue $\lambda_{\mu_1, \mu_2, \dots, \mu_n}$, i.e., $\varphi_{\mu_1, \mu_2, \dots, \mu_n} = \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n} \varphi_{\mu_1, \mu_2, \dots, \mu_n}$, and from (30), $\varphi_{\mu_1, \mu_2, \dots, \mu_n} \in P_0$.

Suppose (41) is false. Then there exist $v_0 \in \partial B_R$ and $\mu_0 > 0$ such that

$$v_0 - Av_0 = \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n}. \quad (42)$$

Let

$$\begin{aligned} \tilde{v}(t) = \int_0^1 K_1(t, s) [b(s) + c(s) K(I_{0+}^{\beta_{n-1}} v_0(s), \\ I_{0+}^{\beta_{n-1}-\beta_1} v_0(s), \dots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v_0(s), v_0(s)) + C_{X_1}] ds, \quad (43) \\ \forall t \in [0, 1]. \end{aligned}$$

Then we have

$$\begin{aligned} \tilde{v}(t) &\leq \int_0^1 K_1(t, s) [b(s) + c(s) [\varepsilon (\mu_n |I_{0+}^{\beta_{n-1}} v_0(s)| \\ &+ \mu_{n-1} |I_{0+}^{\beta_{n-1}-\beta_1} v_0(s)| + \dots + \mu_1 |v_0(s)|) + K^*] \\ &+ C_{X_1}] ds \\ &\leq \frac{t^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 (1 \\ &-s)^{\alpha-\beta-1} \left[b(s) + c(s) \left[\varepsilon \left(\frac{\mu_n}{\Gamma(\beta_{n-1})} \right. \right. \right. \\ &\cdot \int_0^s (s-\tau)^{\beta_{n-1}-1} |v_0(\tau)| d\tau + \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \\ &\cdot \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} |v_0(\tau)| d\tau + \dots + \mu_1 |v_0(s)| \Big) \\ &\left. \left. \left. + K^* \right] + C_{X_1} \right] ds \\ &\leq \frac{t^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 (1 \\ &-s)^{\alpha-\beta-1} \left[b(s) + c(s) \left[\varepsilon \left(\frac{\mu_n}{\Gamma(\beta_{n-1})} \right. \right. \right. \\ &\cdot \int_0^s (s-\tau)^{\beta_{n-1}-1} d\tau + \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \\ &\cdot \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} d\tau + \dots + \mu_1 \Big) \\ &\left. \left. \left. + K^* \right] + C_{X_1} \right] ds \\ &\leq \frac{t^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 (1 \\ &-s)^{\alpha-\beta-1} \left[b(s) + c(s) \left[\varepsilon \left(\frac{\mu_n}{\Gamma(\beta_{n-1})} \right. \right. \right. \\ &\cdot \int_0^s (s-\tau)^{\beta_{n-1}-1} d\tau + \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \end{aligned}$$

$$\begin{aligned} &\cdot \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} d\tau + \dots + \mu_1 \Big) \\ &+ K^* \Big] ds \leq \frac{t^{\alpha-\beta_{n-1}-1} (\|b\| + \|c\| K^* + C_{X_1})}{(\alpha-\beta) \Gamma(\alpha-\beta_{n-1})} \end{aligned}$$

$$\begin{aligned} &+ \frac{t^{\alpha-\beta_{n-1}-1}}{\Gamma(\alpha-\beta_{n-1})} \varepsilon \|c\| \int_0^1 (1 \\ &-s)^{\alpha-\beta-1} \left(\frac{\mu_n}{\Gamma(\beta_{n-1})} \int_0^s (s-\tau)^{\beta_{n-1}-1} d\tau \right. \\ &+ \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} d\tau + \dots + \mu_1 \Big) \\ &\cdot \|v_0\| ds \leq \frac{\|b\| + \|c\| K^* + C_{X_1}}{(\alpha-\beta) \Gamma(\alpha-\beta_{n-1})} \end{aligned}$$

$$\begin{aligned} &+ \frac{\varepsilon \|c\| R}{\Gamma(\alpha-\beta_{n-1})} \int_0^1 (1 \\ &-s)^{\alpha-\beta-1} \left(\frac{\mu_n}{\Gamma(\beta_{n-1})} \int_0^s (s-\tau)^{\beta_{n-1}-1} d\tau \right. \\ &+ \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} d\tau + \dots \\ &+ \mu_1 \Big) ds. \quad (44) \end{aligned}$$

Consequently, we have

$$\|\tilde{v}\| \leq \frac{\|b\| + \|c\| K^* + C_{X_1}}{(\alpha-\beta) \Gamma(\alpha-\beta_{n-1})} + \frac{\varepsilon \|c\| R}{\Gamma(\alpha-\beta_{n-1})} M_{\mu_1, \mu_2, \dots, \mu_n} \quad (45)$$

$< R$.

Note from (43) and (25), $\tilde{v} \in P_0$, and then $\varphi_{\mu_1, \mu_2, \dots, \mu_n} \in P_0$ implies that

$$\begin{aligned} v_0(t) + \tilde{v}(t) &= Av_0(t) + \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n}(t) + \tilde{v}(t) \\ &= \int_0^1 K_1(t, s) [f(s, I_{0+}^{\beta_{n-1}} v_0(s), I_{0+}^{\beta_{n-1}-\beta_1} v_0(s), \dots, \\ &I_{0+}^{\beta_{n-1}-\beta_{n-2}} v_0(s), v_0(s)) + b(s) + c(s) K(I_{0+}^{\beta_{n-1}} v_0(s), \quad (46) \\ &I_{0+}^{\beta_{n-1}-\beta_1} v_0(s), \dots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v_0(s), v_0(s)) + C_{X_1}] ds \\ &+ \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n}(t). \end{aligned}$$

Then we have

$$v_0 + \tilde{v} \in P_0. \quad (47)$$

Therefore, (26), (39), and (43) enable us to obtain

$$\begin{aligned}
Av_0(t) + \tilde{v}(t) &= \int_0^1 K_1(t, s) f(s, I_{0+}^{\beta_{n-1}} v_0(s), \\
&\quad I_{0+}^{\beta_{n-1}-\beta_1} v_0(s), \dots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v_0(s), v_0(s)) ds \\
&\quad + \int_0^1 K_1(t, s) [b(s) + c(s) K(I_{0+}^{\beta_{n-1}} v_0(s), \\
&\quad I_{0+}^{\beta_{n-1}-\beta_1} v_0(s), \dots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v_0(s), v_0(s)) + C_{X_1}] ds \\
&\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \int_0^1 K_1(t, s) \\
&\quad \cdot [\mu_n |I_{0+}^{\beta_{n-1}} v_0(s)| + \mu_{n-1} |I_{0+}^{\beta_{n-1}-\beta_1} v_0(s)| + \dots \\
&\quad + \mu_1 |v_0(s)|] ds - \int_0^1 K_1(t, s) [b(s) + C_{X_1}] ds \\
&\quad + \int_0^1 K_1(t, s) [b(s) + c(s) K(I_{0+}^{\beta_{n-1}} v_0(s), \\
&\quad I_{0+}^{\beta_{n-1}-\beta_1} v_0(s), \dots, I_{0+}^{\beta_{n-1}-\beta_{n-2}} v_0(s), v_0(s)) + C_{X_1}] ds \\
&\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \int_0^1 K_1(t, s) \\
&\quad \cdot \left[\frac{\mu_n}{\Gamma(\beta_{n-1})} \left| \int_0^s (s-\tau)^{\beta_{n-1}-1} v_0(\tau) d\tau \right| \right. \\
&\quad \left. + \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \left| \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} v_0(\tau) d\tau \right| + \dots \right. \\
&\quad \left. + \mu_1 |v_0(s)| \right] ds \geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \\
&\quad \cdot \left| \frac{\mu_n}{\Gamma(\beta_{n-1})} \int_0^1 K_1(t, s) \int_0^s (s-\tau)^{\beta_{n-1}-1} v_0(\tau) d\tau ds \right. \\
&\quad \left. + \frac{\mu_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \int_0^1 K_1(t, s) \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} \right. \\
&\quad \left. \cdot v_0(\tau) d\tau ds + \dots + \mu_1 \int_0^1 K_1(t, s) v_0(s) ds \right| \\
&\geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \left| \mu_n \int_0^1 K_n(t, \tau) v_0(\tau) d\tau \right. \\
&\quad \left. + \mu_{n-1} \int_0^1 K_{n-1}(t, \tau) v_0(\tau) d\tau + \dots + \mu_1 \int_0^1 K_1(t, \right. \\
&\quad \left. s) v_0(s) ds \right| = (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \varepsilon \|c\|) \\
&\quad \cdot \left| \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) v_0(s) ds \right| \geq (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 \\
&\quad - \varepsilon \|c\|) \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) v_0(s) ds.
\end{aligned} \tag{48}$$

Consequently, we obtain

$$\begin{aligned}
&(\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \|c\| \varepsilon) \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) v_0(s) ds \\
&= \lambda_{\mu_1, \mu_2, \dots, \mu_n} \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) (v_0(s) + \tilde{v}(s)) ds \\
&\quad + (\varepsilon_0 - \|c\| \varepsilon) \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) v_0(s) ds \\
&\quad - \lambda_{\mu_1, \mu_2, \dots, \mu_n} \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) \tilde{v}(s) ds \\
&= \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n}(v_0 + \tilde{v})(t) + (\varepsilon_0 - \|c\| \varepsilon) \\
&\quad \cdot \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) (v_0(s) + \tilde{v}(s)) ds \\
&\quad - (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \|c\| \varepsilon) \\
&\quad \cdot \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) \tilde{v}(s) ds.
\end{aligned} \tag{49}$$

From (47), $v_0(t) + \tilde{v}(t) \geq t^{\alpha-\beta_{n-1}-1} \|v_0 + \tilde{v}\| \geq t^{\alpha-\beta_{n-1}-1} (\|v_0\| - \|\tilde{v}\|)$, $t \in [0, 1]$, and from (44), we have

$$\begin{aligned}
&(\varepsilon_0 - \|c\| \varepsilon) \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) (v_0(s) + \tilde{v}(s)) ds \\
&\quad - (\lambda_{\mu_1, \mu_2, \dots, \mu_n} + \varepsilon_0 - \|c\| \varepsilon) \\
&\quad \cdot \int_0^1 K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) \tilde{v}(s) ds \geq (\varepsilon_0 - \|c\| \varepsilon) (R \\
&\quad - \|\tilde{v}\|) \int_0^1 s^{\alpha-\beta_{n-1}-1} K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) ds - (\lambda_{\mu_1, \mu_2, \dots, \mu_n} \\
&\quad + \varepsilon_0 - \|c\| \varepsilon) \times \left[\frac{\|b\| + \|c\| K^* + C_{X_1}}{(\alpha - \beta) \Gamma(\alpha - \beta_{n-1})} \right. \\
&\quad \left. + \frac{\varepsilon \|c\| R}{\Gamma(\alpha - \beta_{n-1})} M_{\mu_1, \mu_2, \dots, \mu_n} \right] \\
&\quad \cdot \int_0^1 s^{\alpha-\beta_{n-1}-1} K_{\mu_1, \mu_2, \dots, \mu_n}(t, s) ds \geq 0.
\end{aligned} \tag{50}$$

Now from (48), (49), and (50), we find

$$\begin{aligned}
Av_0(t) + \tilde{v}(t) &\geq \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n}(v_0 + \tilde{v})(t), \\
&\quad t \in [0, 1].
\end{aligned} \tag{51}$$

Therefore, using (42) and (51), we have

$$\begin{aligned}
v_0 + \tilde{v} &= Av_0 + \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n} + \tilde{v} \\
&\geq \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n}(v_0 + \tilde{v}) + \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n} \\
&\geq \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n}.
\end{aligned} \tag{52}$$

Define $\mu^* = \sup\{\mu > 0 : \nu_0 + \tilde{\nu} \geq \mu \varphi_{\mu_1, \mu_2, \dots, \mu_n}\}$. Then we have $\mu^* \geq \mu_0$ and $\nu_0 + \tilde{\nu} \geq \mu^* \varphi_{\mu_1, \mu_2, \dots, \mu_n}$. From $\varphi_{\mu_1, \mu_2, \dots, \mu_n} = \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n} \varphi_{\mu_1, \mu_2, \dots, \mu_n}$, we obtain

$$\begin{aligned} & \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n} (\nu_0 + \tilde{\nu}) \\ & \geq \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n} \mu^* \varphi_{\mu_1, \mu_2, \dots, \mu_n} = \mu^* \varphi_{\mu_1, \mu_2, \dots, \mu_n}. \end{aligned} \quad (53)$$

Hence

$$\begin{aligned} \nu_0 + \tilde{\nu} & \geq \lambda_{\mu_1, \mu_2, \dots, \mu_n} L_{\mu_1, \mu_2, \dots, \mu_n} (\nu_0 + \tilde{\nu}) + \mu_0 \varphi_{\mu_1, \mu_2, \dots, \mu_n} \\ & \geq (\mu_0 + \mu^*) \varphi_{\mu_1, \mu_2, \dots, \mu_n}, \end{aligned} \quad (54)$$

which contradicts the definition of μ^* . Therefore, (41) holds, and from Lemma 7, we obtain

$$\deg(I - A, B_R, 0) = 0. \quad (55)$$

From (H5), there exist $0 < \varepsilon_1 < \lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}$ and $0 < r < R$ such that

$$\begin{aligned} & |f(t, x_n, x_{n-1}, \dots, x_1)| \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \\ & \cdot (\tilde{\mu}_n |x_n| + \tilde{\mu}_{n-1} |x_{n-1}| + \dots + \tilde{\mu}_1 |x_1|), \end{aligned} \quad (56)$$

for all $x_i \in \mathbb{R}$, $i = 1, 2, \dots, n$, $t \in [0, 1]$ with $0 \leq \tilde{\mu}_n |x_n| + \tilde{\mu}_{n-1} |x_{n-1}| + \dots + \tilde{\mu}_1 |x_1| < r$. Consequently, we obtain

$$\begin{aligned} |(Av)(t)| & \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \int_0^1 K_1(t, s) (\tilde{\mu}_n |I_{0+}^{\beta_{n-1}} \nu(s)| \\ & + \tilde{\mu}_{n-1} |I_{0+}^{\beta_{n-1}-\beta_1} \nu(s)| + \dots + \tilde{\mu}_1 |\nu(s)|) ds \\ & \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \int_0^1 K_1(t, s) \\ & \cdot \left(\frac{\tilde{\mu}_n}{\Gamma(\beta_{n-1})} \int_0^s (s-\tau)^{\beta_{n-1}-1} |\nu(\tau)| d\tau \right. \\ & + \frac{\tilde{\mu}_{n-1}}{\Gamma(\beta_{n-1}-\beta_1)} \int_0^s (s-\tau)^{\beta_{n-1}-\beta_1-1} |\nu(\tau)| d\tau + \dots \\ & + \tilde{\mu}_1 |\nu(s)| \Big) ds = (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \\ & \cdot \left[\tilde{\mu}_n \int_0^1 K_n(t, \tau) |\nu(\tau)| d\tau \right. \\ & + \tilde{\mu}_{n-1} \int_0^1 K_{n-1}(t, \tau) |\nu(\tau)| d\tau + \dots \\ & + \tilde{\mu}_1 \int_0^1 K_1(t, s) |\nu(s)| ds \Big] = (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \\ & \cdot \int_0^1 K_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}(t, s) |\nu(s)| ds = (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \\ & \cdot (L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} |\nu|)(t), \quad \forall t \in [0, 1], \quad \nu \in E, \quad \|\nu\| \leq r. \end{aligned} \quad (57)$$

Now for this r , we claim

$$Av \neq \lambda v, \quad \forall v \in \partial B_r, \quad \lambda \geq 1. \quad (58)$$

Suppose the contrary. Then there exist $\nu_0 \in \partial B_r$ and $\lambda_0 \geq 1$ such that $A\nu_0 = \lambda_0 \nu_0$. Let $\omega(t) = |\nu_0(t)|$. Then $\omega \in \partial B_r \cap P$ and

$$\begin{aligned} \omega & \leq \frac{1}{\lambda_0} (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} \omega \\ & \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} \omega. \end{aligned} \quad (59)$$

By induction, we have $\omega \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1)^m L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}^m \omega$, for $m = 1, 2, \dots$. As a result, we have

$$\begin{aligned} \|\omega\| & \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1)^m \|L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}^m \omega\|, \\ 1 & \leq (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1)^m \|L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}^m\|. \end{aligned} \quad (60)$$

Therefore, by Gelfand's theorem, we have

$$\begin{aligned} & (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) r (L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}) \\ & = (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) \lim_{m \rightarrow \infty} \sqrt[m]{\|L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}^m\|} \geq 1. \end{aligned} \quad (61)$$

This contradicts

$$\begin{aligned} & (\lambda_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n} - \varepsilon_1) r (L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}) = 1 - \varepsilon_1 r (L_{\tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_n}) \\ & < 1. \end{aligned} \quad (62)$$

Hence (58) holds, and from Lemma 8, we have

$$\deg(I - A, B_r, 0) = 1. \quad (63)$$

Now (55) and (63) together imply that

$$\begin{aligned} & \deg(I - A, B_R \setminus \overline{B_r}, 0) \\ & = \deg(I - A, B_R, 0) - \deg(I - A, B_r, 0) = -1. \end{aligned} \quad (64)$$

Therefore the operator A has at least one fixed point in $B_R \setminus \overline{B_r}$. Equivalently, (1) has at least one nontrivial solution. This completes the proof. $\square$

Example 10. For convenience in our example, we let $\mu_i = \tilde{\mu}_i$, $i = 1, 2, \dots, n$. Let

$$f(t, x_n, x_{n-1}, \dots, x_1) = \begin{cases} \sum_{i=1}^m (-1)^i \rho_i - (1+t^2) \left| \sum_{j=1}^n \mu_j x_j \right|^{1/3} \ln \left(\left| \sum_{j=1}^n \mu_j x_j \right| + 1 \right) - (1+t^2) \ln 2, & \sum_{j=1}^n \mu_j x_j \in (-\infty, -1], \\ \sum_{i=1}^m \rho_i \left(\sum_{j=1}^n \mu_j x_j \right)^i, & \sum_{j=1}^n \mu_j x_j \in [-1, +\infty), \end{cases} \quad (65)$$

where $m \in \mathbb{N}_+$, $0 < \rho_1 < \lambda_{\mu_1, \mu_2, \dots, \mu_n}$, and $\rho_i \geq 0$, $i = 2, 3, \dots, m$, with $\sum_{i=1}^m (-1)^i \rho_i \geq 0$. Let $b(t) = (1+t^2) \ln 2$, $c(t) = 1+t^2$, and $K(x_n, x_{n-1}, \dots, x_1) =$

$|\sum_{j=1}^n \mu_j x_j|^{1/3} \ln(|\sum_{j=1}^n \mu_j x_j| + 1)$. We now show that conditions (H1)-(H5) are satisfied. Indeed, for the function K , we obtain

$$\begin{aligned} & \lim_{\mu_n |x_n| + \mu_{n-1} |x_{n-1}| + \dots + \mu_1 |x_1| \rightarrow +\infty} \frac{|\sum_{j=1}^n \mu_j x_j|^{1/3} \ln(|\sum_{j=1}^n \mu_j x_j| + 1)}{\mu_n |x_n| + \mu_{n-1} |x_{n-1}| + \dots + \mu_1 |x_1|} \\ & \leq \lim_{\mu_n |x_n| + \mu_{n-1} |x_{n-1}| + \dots + \mu_1 |x_1| \rightarrow +\infty} \frac{(\sum_{j=1}^n |\mu_j x_j|)^{1/3} \ln(\sum_{j=1}^n |\mu_j x_j| + 1)}{\sum_{j=1}^n \mu_j |x_j|} = 0, \end{aligned} \quad (66)$$

and hence, (H3) holds. For the function f , it is clear that (H1)-(H2) hold. Also note we have

$$\begin{aligned} & \liminf_{\mu_n |x_n| + \mu_{n-1} |x_{n-1}| + \dots + \mu_1 |x_1| \rightarrow +\infty} \frac{\sum_{i=1}^m \rho_i \left(\sum_{j=1}^n \mu_j x_j \right)^i}{\sum_{j=1}^n \mu_j |x_j|} \\ & = +\infty, \end{aligned} \quad (67)$$

and

$$\begin{aligned} & \limsup_{\mu_n |x_n| + \mu_{n-1} |x_{n-1}| + \dots + \mu_1 |x_1| \rightarrow 0} \frac{\left| \sum_{i=1}^m \rho_i \left(\sum_{j=1}^n \mu_j x_j \right)^i \right|}{\sum_{j=1}^n \mu_j |x_j|} = \rho_1 \\ & < \lambda_{\mu_1, \mu_2, \dots, \mu_n}. \end{aligned} \quad (68)$$

Then (H4)-(H5) hold. Therefore, all the conditions in Theorem 9 are satisfied, and thus (1) has at least one nontrivial solution.

Now, we consider a special case of this example. Let $n = 2$, $\mu_1 = 1$, $\mu_2 = 1$, and consider the operators

$$\begin{aligned} (L_{\mu_1} v)(t) &= \int_0^1 \mu_1 K_1(t, s) v(s) ds, \\ (L_{\mu_2} v)(t) &= \int_0^1 \mu_2 K_2(t, s) v(s) ds, \end{aligned} \quad (69)$$

for $v \in E$.

Then for all $m \in \mathbb{N}$, from Lemma 5(ii), we have

$$\begin{aligned} \|L_{\mu_1}^m\| &= \max_{t \in [0,1]} \underbrace{\int_0^1 \dots \int_0^1}_{m} K_1(t, s_1) \\ & \quad \cdot K_1(s_1, s_2) \dots K_1(s_{m-1}, s_m) ds_1 ds_2 \dots ds_m \\ & \leq \underbrace{\int_0^1 \dots \int_0^1}_{m} K_1(1, s_1) \\ & \quad \cdot K_1(1, s_2) \dots K_1(1, s_m) ds_1 ds_2 \dots ds_m \\ & = \left(\int_0^1 K_1(1, s) ds \right)^m, \end{aligned} \quad (70)$$

and so

$$\lim_{m \rightarrow \infty} \sqrt[m]{\|L_{\mu_1}^m\|} \leq \int_0^1 K_1(1, s) ds. \quad (71)$$

For all $m \in \mathbb{N}$, from Lemma 5(ii), we have

$$\begin{aligned} \|L_{\mu_1}^m\| &= \max_{t \in [0,1]} \underbrace{\int_0^1 \dots \int_0^1}_{m} K_1(t, s_1) K_1(s_1, s_2) \dots K_1(s_{m-1}, s_m) ds_1 ds_2 \dots ds_m \\ & \geq \max_{t \in [0,1]} \underbrace{\int_0^1 \dots \int_0^1}_{m} t^{\alpha-\beta_1-1} K_1(1, s_1) s_1^{\alpha-\beta_1-1} K_1(1, s_2) \dots s_{m-1}^{\alpha-\beta_1-1} K_1(1, s_m) ds_1 ds_2 \dots ds_m \\ & = \int_0^1 K_1(1, s) ds \left(\int_0^1 K_1(1, s) s^{\alpha-\beta_1-1} ds \right)^{m-1}, \end{aligned} \quad (72)$$

and so

$$\lim_{m \rightarrow \infty} \sqrt[m]{\|L_{\mu_1}^m\|} \geq \int_0^1 K_1(1, s) s^{\alpha-\beta_1-1} ds. \quad (73)$$

From Gelfand's theorem, we obtain that the spectral radius of L_{μ_1} , denoted by $r(L_{\mu_1})$, satisfies

$$\int_0^1 K_1(1, s) s^{\alpha-\beta_1-1} ds \leq r(L_{\mu_1}) \leq \int_0^1 K_1(1, s) ds. \quad (74)$$

Note the definition of K_2 , and from Lemma 5(ii), we have

$$\begin{aligned} & \frac{1}{\Gamma(\beta_{n-1} - \beta_{n-2})} \\ & \cdot \int_0^\tau t^{\alpha-\beta_{n-1}-1} K_1(1, \tau) (\tau - s)^{\beta_{n-1}-\beta_{n-2}-1} d\tau \\ & \leq K_2(t, s) = \frac{1}{\Gamma(\beta_{n-1} - \beta_{n-2})} \\ & \cdot \int_0^\tau K_1(t, \tau) (\tau - s)^{\beta_{n-1}-\beta_{n-2}-1} d\tau \\ & \leq \frac{1}{\Gamma(\beta_{n-1} - \beta_{n-2})} \int_0^\tau K_1(1, \tau) (\tau - s)^{\beta_{n-1}-\beta_{n-2}-1} d\tau. \end{aligned} \quad (75)$$

Thus $t^{\alpha-\beta_{n-1}-1} K_2(1, s) \leq K_2(t, s) \leq K_2(1, s)$, for $t, s \in [0, 1]$. Using a similar method, we have that the spectral radius of L_{μ_2} , denoted by $r(L_{\mu_2})$, satisfies

$$\int_0^1 K_2(1, s) s^{\alpha-\beta_1-1} ds \leq r(L_{\mu_2}) \leq \int_0^1 K_2(1, s) ds. \quad (76)$$

Consequently, the spectral radius of the operator L_{μ_1, μ_2} satisfies

$$\begin{aligned} & \int_0^1 K_1(1, s) s^{\alpha-\beta_1-1} ds + \int_0^1 K_2(1, s) s^{\alpha-\beta_1-1} ds \\ & \leq r(L_{\mu_1, \mu_2}) \leq \int_0^1 K_1(1, s) ds + \int_0^1 K_2(1, s) ds. \end{aligned} \quad (77)$$

Thus

$$\begin{aligned} & \frac{1}{\int_0^1 K_1(1, s) s^{\alpha-\beta_1-1} ds + \int_0^1 K_2(1, s) s^{\alpha-\beta_1-1} ds} \geq \lambda_{\mu_1, \mu_2} \\ & \geq \frac{1}{\int_0^1 K_1(1, s) ds + \int_0^1 K_2(1, s) ds}. \end{aligned} \quad (78)$$

Now take $\rho_1 \in (0, 1/(\int_0^1 K_1(1, s) ds + \int_0^1 K_2(1, s) ds))$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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