Research Article

Continuous $\star$-$K$-$G$-Frame in Hilbert $C^*$-Modules

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Frame theory is exciting and dynamic with applications to a wide variety of areas in mathematics and engineering. In this paper, we introduce the concept of Continuous $\star$-$K$-$G$-frame in Hilbert $C^*$-Modules and we give some properties.

1. Introduction and Preliminaries

The concept of frames in Hilbert spaces has been introduced by Duffin and Schaeffer [1] in 1952 to study some deep problems in nonharmonic Fourier series, after the fundamental paper [2] by Daubechies, Grossman and Meyer, frame theory began to be widely used, particularly in the more specialized context of wavelet frames and Gabor frames [3].

Traditionally, frames have been used in signal processing, image processing, data compression, and sampling theory. Discrete frames are referred to as coherent states [5].

As frames are referred to as coherent states [5]. Askari-Hemmat, Dehghan, and Radjabalipour in [7] called them generalized frames and in mathematical physics they are referred to as coherent states [5].

In this paper, we introduce the notion of Continuous $\star$-$K$-$g$-frame which are generalization of $\star$-$K$-$g$-Frame in Hilbert $C^*$-Modules introduced by M. Rossafi and S. Kabbaj [8] and we establish some new results.

The paper is organized as follows: we continue this introductory section we briefly recall the definitions and basic properties of $C^*$-algebra, Hilbert $C^*$-modules. In Section 2, we introduce the Continuous $\star$-$K$-$g$-Frame, the Continuous pre-$\star$-$K$-$g$-frame operator, and the Continuous $\star$-$K$-$g$-frame operator; also we establish here properties.

In the following we briefly recall the definitions and basic properties of $C^*$-algebra, Hilbert $\mathcal{A}$-modules. Our reference for $C^*$-algebras is [9, 10]. For a $C^*$-algebra $\mathcal{A}$ if $a \in \mathcal{A}$ is positive we write $a \geq 0$ and $\mathcal{A}^+$ denotes the set of positive elements of $\mathcal{A}$.

Definition 1 (see [11]). Let $\mathcal{A}$ be a unital $C^*$-algebra and $\mathcal{H}$ a left $\mathcal{A}$-module, such that the linear structures of $\mathcal{A}$ and $\mathcal{H}$ are compatible. $\mathcal{H}$ is a pre-Hilbert $\mathcal{A}$-module if $\mathcal{H}$ is equipped with an $\mathcal{A}$-valued inner product $\langle \ldots , \ldots \rangle_{\mathcal{A}} : \mathcal{H} \times \mathcal{H} \longrightarrow \mathcal{A}$, such that is sesquilinear, positive definite, and respects the module action. In the other words,

(i) $\langle x, x \rangle_{\mathcal{A}} \geq 0$ for all $x \in \mathcal{H}$ and $\langle x, x \rangle_{\mathcal{A}} = 0$ if and only if $x = 0$.

(ii) $\langle ax + y, z \rangle_{\mathcal{A}} = a\langle x, y \rangle_{\mathcal{A}} + \langle y, z \rangle_{\mathcal{A}}$ for all $a \in \mathcal{A}$ and $x, y, z \in \mathcal{H}$.

(iii) $\langle x, y \rangle_{\mathcal{A}} = \langle y, x \rangle_{\mathcal{A}}^*$ for all $x, y \in \mathcal{H}$.

For $x \in \mathcal{H}$, we define $\|x\| = \|\langle x, x \rangle_{\mathcal{A}}\|^{1/2}$. If $\mathcal{H}$ is complete with $\|\|$, it is called a Hilbert $\mathcal{A}$-module and a Hilbert $C^*$-module over $\mathcal{A}$. For every $a \in C^*$-algebra $\mathcal{A}$, we have $|a| = (a^*a)^{1/2}$ and the $\mathcal{A}$-valued norm on $\mathcal{H}$ is defined by $\|x\| = \langle x, x \rangle_{\mathcal{A}}^{1/2}$ for $x \in \mathcal{H}$.

Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-modules. A map $T : \mathcal{H} \longrightarrow \mathcal{K}$ is said to be adjointable if there exists a map $T^* : \mathcal{K} \longrightarrow \mathcal{H}$ such that $\langle Tx, y \rangle_{\mathcal{A}} = \langle x, T^* y \rangle_{\mathcal{A}}$ for all $x \in \mathcal{H}$ and $y \in \mathcal{K}$. 


We reserve the notation $\text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ for the set of all
adjointable operators from $\mathcal{H}$ to $\mathcal{K}$ and $\text{End}_{\mathcal{A}}^*(\mathcal{H}, \mathcal{K})$ is abbreviated to $\text{End}_{\mathcal{A}}^*(\mathcal{H})$.

The following lemmas will be used to prove our main results

Lemma 2 (see [11]). Let $\mathcal{H}$ be Hilbert $\mathcal{A}$-module. If $T \in \text{End}_{\mathcal{A}}^*(\mathcal{H})$, then
\[
(Tx, Tx) \leq \|T\|^2 (x, x), \quad \forall x \in \mathcal{H}. \tag{1}
\]

Lemma 3 (see [12]). Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-Modules and $T \in \text{End}^*(\mathcal{H}, \mathcal{K})$. Then the following statements are equivalent:

(i) $T$ is surjective.

(ii) $T^*$ is bounded below with respect to norm; i.e., there is $m > 0$ such that $\|T^*x\| \geq m \|x\|$ for all $x \in \mathcal{H}$.

(iii) $T^*$ is bounded below with respect to the inner product; i.e., there is $m' > 0$ such that $(T^*x, T^*x) \geq m' (x, x)$ for all $x \in \mathcal{H}$.

Lemma 4 (see [13]). Let $\mathcal{H}$ and $\mathcal{K}$ be two Hilbert $\mathcal{A}$-Modules and $T \in \text{End}^*(\mathcal{H}, \mathcal{K})$. Then,

(i) if $T$ is injective and $T$ has closed range, then the adjointable map $T^* T$ is invertible and
\[
\left\|(T^* T)^{-1}\right\|_\mathcal{H} \leq T^* T \leq \|T\|^2 I_\mathcal{H}. \tag{2}
\]

(ii) If $T$ is surjective, then the adjointable map $T^* T$ is invertible and
\[
\left\|(T^* T)^{-1}\right\|_\mathcal{H} \leq T^* T \leq \|T\|^2 I_\mathcal{H}. \tag{3}
\]

2. Continuous $*$-K-frame in Hilbert $C^*$-Modules

Let $X$ be a Banach space, $(\Omega, \mu)$ a measure space, and function $f : \Omega \to X$ a measurable function. Integral of the Banach-valued function $f$ has been defined by Bochner and others. Most properties of this integral are similar to those of the integral of real-valued functions. Because every $C^*$-algebra and Hilbert $C^*$-module is a Banach space thus we can use this integral and its properties.

Let $(\Omega, \mu)$ be a measure space, let $U$ and $V$ be two Hilbert $C^*$-modules, $\{V_w : w \in \Omega\}$ is a sequence of subspaces of $V$, and $\text{End}_{\mathcal{A}}^*(U, V_w)$ is the collection of all adjointable $\mathcal{A}$-linear maps from $U$ into $V_w$. We define
\[
\bigoplus_{w \in \Omega} V_w = \left\{ x = \{x_w : x_w \in V_w\}, \int_\Omega |x_w|^2 d\mu(w) < \infty \right\}. \tag{4}
\]

For any $x = \{x_w : w \in \Omega\}$ and $y = \{y_w : w \in \Omega\}$, if the $\mathcal{A}$-valued inner product is defined by $(x, y) = \int_\Omega (x_w, y_w) d\mu(w)$, the norm is defined by $\|x\| = \|\{x, y\}\|^{1/2}$, the $\bigoplus_{w \in \Omega} V_w$ is a Hilbert $C^*$-module.

Definition 5. Let $K \in \text{End}_{\mathcal{A}}^*(U)$; we call $\{A_w \in \text{End}_{\mathcal{A}}^*(U, V_w) : w \in \Omega\}$ a Continuous $*$-K-frame for Hilbert $C^*$-module $U$ with respect to $\{V_w : w \in \Omega\}$ if

(a) for any $x \in U$, the function $\bar{x} : \Omega \to V_w$ defined by $\bar{x}(w) = A_w x$ is measurable;

(b) there exist two strictly nonzero elements $A$ and $B$ in $\mathcal{A}$ such that
\[
A \langle K^* x, K^* x \rangle A^* \leq \int_\Omega \langle A_w x, A_w x \rangle d\mu(w) \leq B \langle x, x \rangle B^*, \quad \forall x \in U. \tag{5}
\]

The elements $A$ and $B$ are called Continuous $*$-K-frame bounds.

If $A = B$ we call this Continuous $*$-K-frame a continuous tight $*$-K-frame, and if $A = B = 1$ it is called a continuous Parseval $*$-K-frame. If only the right-hand inequality of (5) is satisfied, we call $\{A_w : w \in \Omega\}$ a continuous $*$-K-Bessel for $U$ with respect to $\{A_w : w \in \Omega\}$ with Bessel bound $B$.

Example 6. Let $f^{(1)}$ be the set of all bounded complex-valued sequences. For any $u = \{u_n\}_{n \in \mathbb{N}}, v = \{v_n\}_{n \in \mathbb{N}} \in f^{(1)}$, we define
\[
u v = \{u_n v_n\}_{n \in \mathbb{N}}, \quad u^* = \{\overline{u_n}\}_{n \in \mathbb{N}}, \tag{6}
\]

Then $\mathcal{A} = \{f^{(1)}, \|x\|\}$ is a $C^*$-algebra.

Let $\mathcal{H} = C_0$ be the set of all sequences converging to zero. For any $u, v \in \mathcal{H}$ we define
\[
\langle u, v \rangle = uv^* = \{u_n \overline{v_n}\}_{n \in \mathbb{N}}. \tag{7}
\]

Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module.

Define $f_j = \{f_j^n\}_{n \in \mathbb{N}}$ by $f_j^n = 1/2^n + 1/2^j$ if $i = j$ and $f_j^n = 0$ if $i \neq j \forall j \in \mathbb{N}^*$.

Now define the adjointable operator $A_j : \mathcal{H} \to \mathcal{A}, \Lambda_j x = \langle x, f_j \rangle$.

Then for every $x \in \mathcal{H}$ we have
\[
\sum_{j \in \mathbb{N}} \langle A_j x, A_j x \rangle = \left(\frac{1}{2} + \frac{1}{2^j}\right)_{j \in \mathbb{N}^*}, \tag{8}
\]

So $\{A_j\}_{j \in \mathbb{N}}$ is a $\{1/2 + 1/2^j\}_{j \in \mathbb{N}^*}$-tight $*$-frame.

Let $K : \mathcal{H} \to \mathcal{A}$ defined by $K x = \{x_j\}_{j \in \mathbb{N}^*}$.

Then for every $x \in \mathcal{H}$ we have
\[
\langle K^* x, K^* x \rangle_{\mathcal{A}} \leq \sum_{j \in \mathbb{N}} \langle A_j x, A_j x \rangle = \left(\frac{1}{2} + \frac{1}{2^j}\right)_{j \in \mathbb{N}^*}. \tag{9}
\]

Now, let $(\Omega, \mu)$ be a $\sigma$-finite measure space with infinite measure and $\{H_w\}_{w \in \Omega}$ be a family of Hilbert $\mathcal{A}$-module $(H_w = C_0, \forall w \in \Omega)$. 

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Since \( \Omega \) is a \( \sigma \)-finite, it can be written as a disjoint union
\( \Omega = \bigcup \Omega_w \) of countably many subsets \( \Omega_w \subseteq \Omega \), such that
\( \mu(\Omega_w) < \infty \), \( \forall k \in \mathbb{N} \). Without less of generality, assume that
\( \mu(\Omega_w) > 0 \) \( \forall k \in \mathbb{N} \).

For each \( \omega \in \Omega \), define the operator: \( \Lambda_{\omega} : H \to H_w \) by
\[
\Lambda_{\omega}(x) = \frac{1}{\mu(\Omega_w)} \langle x, f_k \rangle h_{\omega}, \quad \forall x \in H
\]  
(10)
where \( k \) is such that \( w \in \Omega_w \) and \( h_{\omega} \) is an arbitrary element of
\( H_w \), such that \( \|h_{\omega}\| = 1 \).

For each \( x \in H \), \( \{\Lambda_{\omega}x\}_{\omega \in \Omega} \) is strongly measurable (since
\( h_{\omega} \) are fixed) and
\[
\int_{\Omega} \langle \Lambda_{\omega}x, \Lambda_{\omega}x \rangle \, d\mu(\omega) = \sum_{j \in \mathbb{N}} \langle x, f_j \rangle \langle f_j, x \rangle
\]  
(11)
so, therefore
\[
\langle K^*x, K^*x \rangle \leq \int_{\Omega} \langle \Lambda_{\omega}x, \Lambda_{\omega}x \rangle \, d\mu(\omega)
\]
\[
= \sum_{j \in \mathbb{N}} \langle x, f_j \rangle \langle f_j, x \rangle
\]  
(12)
Indeed,
if \( K \) is surjective there exists \( m > 0 \) such that
\[
m \langle x, x \rangle \leq \langle K^*x, K^*x \rangle
\]  
(16)
then
\[
(A \sqrt{m}) \langle x, x \rangle (A \sqrt{m})^* \leq A \langle K^*x, K^*x \rangle A^*
\]  
(17)
or if \( \{\Lambda_{\omega} \in \text{End}_{\omega}^*(U, V_w) : w \in \Omega \} \) is a continuous
\( * \)-K-g-frame, we have
\[
(A \sqrt{m}) \langle x, x \rangle (A \sqrt{m})^* \leq \int_{\Omega} \langle \Lambda_{\omega}x, \Lambda_{\omega}x \rangle \, d\mu(w)
\]  
(18)
\[
\leq B \langle x, x \rangle B^*
\]
hence \( \{\Lambda_{\omega} \in \text{End}_{\omega}^*(U, V_w) : w \in \Omega \} \) is a continuous
\( * \)-g-frame for \( U \) with lower and upper bounds \( A \sqrt{m} \) and \( B \), respectively.

Let \( K \in \text{End}_{\omega}^*(U) \), and \( \{\Lambda_{\omega} \in \text{End}_{\omega}^*(U, V_w) : w \in \Omega \} \) be a continuous
\( * \)-K-g-frame for Hilbert \( C^* \)-module \( U \) with respect to \( \{V_w : w \in \Omega \} \),
then \( T \) is called the continuous \( * \)-K-g-frame transform.
So its adjoint operator is \( T^* : \bigoplus_{\omega \in \Omega} V_w \to U \) given by
\[
T^*(\{x_\omega \}_{\omega \in \Omega}) = \int_{\Omega} \Lambda^*_\omega x_\omega \, d\mu(\omega)
\]  
(20)
By composing \( T \) and \( T^* \), the frame operator \( S = T^*T \) given by
\[
Sx = \int_{\Omega} \Lambda^*_\omega \Lambda_\omega x \, d\mu(\omega), \quad S \text{ is called continuous } * \text{-K-g frame operator}
\]

**Theorem 8.** The continuous \( * \)-K-g frame operator \( S \) is bounded, positive, self-adjoint, and \( \|A^{-1}\| \|K\| \leq \|S\| \leq \|B\| \).

**Proof.** First we show, \( S \) is a self-adjoint operator. By definition we have \( \forall x, y \in U \)
\[
\langle Sx, y \rangle = \int_{\Omega} \Lambda^*_\omega \Lambda_\omega x \, d\mu(\omega), \quad y
\]
\[
= \int_{\Omega} \langle \Lambda^*_\omega \Lambda_\omega x, y \rangle \, d\mu(\omega)
\]
\[
= \int_{\Omega} \langle x, \Lambda^*_\omega \Lambda_\omega y \rangle \, d\mu(\omega)
\]
\[
= \langle x, \int_{\Omega} \Lambda^*_\omega \Lambda_\omega y \, d\mu(\omega) \rangle = \langle x, Sy \rangle.
\]  
(21)
Then \( S \) is a self-adjoint. Clearly \( S \) is positive.

By definition of a continuous \( * \)-K-g-frame we have
\[
A \langle K^*x, K^*x \rangle A^* \leq \int_{\Omega} \langle \Lambda_{\omega}x, \Lambda_{\omega}x \rangle \, d\mu(\omega)
\]
\[
\leq B \langle x, x \rangle B^*.
\]  
(22)
So
\[ A(K^*x, K^*x) A^* \leq \langle Sx, x \rangle \leq B \langle x, x \rangle B^*. \] (23) This gives
\[ \|A^{-1}\|^2 \|KK^*, x, x\| \leq \|Sx, x\| \leq \|B\|^2 \|x, x\|. \] (24)
If we take supremum on all \( x \in U \), where \( \|x\| \leq 1 \), we have
\[ \|A^{-1}\|^2 \|K\|^2 \leq \|S\| \leq \|B\|^2. \] (25)
\[ \square \]
\[ \text{Theorem 9.} \text{ Let } K \in \text{End}_d^d(H) \text{ be surjective and } \{\Lambda_w \in \text{End}_d^d(U, V_w) : w \in \Omega\} \text{ a continuous } \ast\text{-K-g-frame for } U, \text{ with lower and upper bounds } A \text{ and } B, \text{ respectively, and with the continuous } \ast\text{-K-g-frame operator } S.
\] Let \( T \in \text{End}_d^d(U) \) be invertible; then \( \{\Lambda_w T : w \in \Omega\} \) is a continuous \( \ast\text{-K-g-frame for } U \) with continuous \( \ast\text{-K-g-frame operator } T^*ST. \)
\[ \text{Proof.} \text{ We have}
\[ A(K^*Tx, K^*Tx) A^* \leq \int_{\Omega} \langle \Lambda_w Tx, \Lambda_w Tx \rangle d\mu(w) \] (26)
\[ \leq B \langle Tx, Tx \rangle B^*, \hspace{1em} \forall x \in U. \]
Using Lemma 3, we have \( \|T^*T^{-1}\|^2 \langle x, x \rangle \leq \langle Tx, Tx \rangle, \forall x \in U. \)
K is surjective, then there exists \( m \) such that
\[ m \langle Tx, Tx \rangle \leq \langle K^*Tx, K^*Tx \rangle \] (27)
and then
\[ m \|T^*T^{-1}\|^2 \langle x, x \rangle \leq \langle K^*Tx, K^*Tx \rangle \] (28)
so
\[ m \|T^*T^{-1}\|^2 A \langle x, x \rangle A^* \leq A \langle K^*Tx, K^*Tx \rangle A^* \] (29)
Or \( \|T^{-1}\|^2 \leq \|T^*T^{-1}\|^2 \), this implies
\[ \left( \|T^{-1}\|^2 \sqrt{m} \right) \langle x, x \rangle \left( \|T^{-1}\|^2 \sqrt{m} \right)^* \]
\[ \leq A \langle K^*Tx, K^*Tx \rangle A^*, \hspace{1em} \forall x \in U. \] (30)
And we know that \( \langle Tx, Tx \rangle \leq \|T\|^2 \langle x, x \rangle, \forall x \in U. \) This implies that
\[ B \langle Tx, Tx \rangle B^* \leq \langle \|T\| B, \langle x, x \rangle \langle \|T\| B \rangle \rangle^*, \hspace{1em} \forall x \in U. \] (31)
Using (26), (30), (31) we have
\[ \left( \|T^{-1}\|^2 \sqrt{m} \right) \langle x, x \rangle \left( \|T^{-1}\|^2 \sqrt{m} \right)^* \]
\[ \leq \int_{\Omega} \langle \Lambda_w Tx, \Lambda_w Tx \rangle d\mu(w) \]
\[ \leq \langle \|T\| B, \langle x, x \rangle \rangle \langle \|T\| B \rangle^* \]
So \( \{\Lambda_w T : w \in \Omega\} \) is a continuous \( \ast\text{-K-g-frame for } U. \)
Moreover for every \( x \in U \), we have
\[ T^*STx = T^* \int_{\Omega} \Lambda_w^* \Lambda_w Tx d\mu(w) \]
\[ = \int_{\Omega} T^* \Lambda_w^* \Lambda_w Tx d\mu(w) \]
\[ = \int_{\Omega} (\Lambda_w T)^* (\Lambda_w T) x d\mu(w). \] (33)
This completes the proof. \[ \square \]
\[ \text{Corollary 10.} \text{ Let } \{\Lambda_w \in \text{End}_d^d(U, V_w) : w \in \Omega\} \text{ be a continuous } \ast\text{-K-g-frame for } U \text{ and let } K \in \text{End}_d^d(U) \text{ be surjective, with continuous } \ast\text{-K-g-frame operator } S. \text{ Then } \{\Lambda_w S^{-1} : w \in \Omega\} \text{ is a continuous } \ast\text{-K-g-frame for } U. \)
\[ \text{Proof.} \text{ Result from the last theorem by taking } T = S^{-1}. \][3]
\[ \square \]
The following theorem characterizes a continuous \( \ast\text{-K-g-frame} \) by its frame operator.
\[ \text{Theorem 11.} \text{ Let } \{\Lambda_w \in \Omega\} \text{ be a continuous } \ast\text{-K-g-Bessel for } H \text{ with bounded } A \text{ and } B \text{ if and only if}
\[ A \langle K^*x, K^*x \rangle A^* \leq \int_{\Omega} \langle \Lambda_w x, \Lambda_w x \rangle d\mu(w) \]
\[ \leq B \langle x, x \rangle B^* \] (34)
If and only if
\[ A \langle KK^*x, x \rangle A^* \leq \int_{\Omega} \langle \Lambda_w^* \Lambda_w x, x \rangle d\mu(w) \]
\[ \leq B \langle x, x \rangle B^* \] (35)
If and only if
\[ A \langle KK^*x, x \rangle A^* \leq \langle Sx, x \rangle \leq B \langle x, x \rangle B^* \] (36)
where \( S \) is the continuous \( \ast\text{-K-g-frame operator for } \{\Lambda_w \in \Omega\}. \)
Therefore, the conclusion holds. \[ \square \]
\[ \text{Data Availability} \]
No data were used to support this study.
\[ \text{Conflicts of Interest} \]
The authors declare that they have no conflicts of interest.
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