Hindawi
Journal of Function Spaces
Volume 2019, Article ID 2759398, 5 pages
https://doi.org/10.1155/2019/2759398

Research Article

The Solution of the Optimal Insurance Problem with Background Risk

Yulian Fan

Department of Mathematics, North China University of Technology, Beijing 100144, China

Correspondence should be addressed to Yulian Fan; fanyl@ncut.edu.cn

Received 11 January 2019; Revised 5 March 2019; Accepted 28 March 2019; Published 4 April 2019

Academic Editor: Aurelian Gheondea

This paper proves the existence and uniqueness of the solution of the optimal insurance problem with background risk and presents the explicit form of the optimal solution.

1. Introduction

There are multiple kinds of risks which can adversely affect people's wealth or wellbeing, and therefore people usually buy insurance to reduce their losses. Unfortunately, in practice, the losses caused by some kind of risks are so great that the insurer is unable to insure it, and therefore exclusion clauses were frequently written in the insurance contract because of the limited capacity. For example, in car insurance, the insurer's liabilities are often exempted if the losses result from wars or earthquakes.

Hence, it is necessary to study insurance problems in the presence of background risk.

The authors in [1] firstly studied the optimal insurance demand in the presence of background risk. Then insurance contracts with background risks have been studied in various cases. But most of the studies have focused on the demand of the insurance when there is background risk; for example, see the survey paper of [2] which discussed the proportional coinsurance policy under independent and dependent background risk.

The authors in [3] studied the efficient insurance contract when there is background risk. The optimal insurance contract is proved to contain a deductible if the background risk is independent with the insurable risk. Under given conditions of the utility function, the existence of the background risk will reduce the deductible. If the background risk increases with the insurable risk, and the insured is prudent, the optimal insurance contract entails a disappearing deductible. Efficient insurance contracts with independent background risk were also studied in [4–8]. The cases of positively correlated background risk are discussed in [9, 10]. The authors in [11] examined the qualitative properties of efficient insurance policy under the assumption of "stochastic increasingness" between the background risk and the insurable risk.

The authors in [8] proposed a cumulative distribution function (CDF) based method to derive the explicit form of the optimal reinsurance contract for an insurer when there is background risk. But the objective of [8] is maximizing the insured's survival probability. In this paper, we will derive the explicit form of the optimal insurance policy which maximizes the insured's expected utility. We first prove that when the background risk is discrete, the optimal solution should be contingent upon the realized values of the background risk without any extra assumptions on the dependence between the insurable risk and the background risk, or the distributions of them, or the utility function. And then we give the explicit form of the optimal solution and prove the uniqueness of the solution.

The paper is organized as follows: Section 2 introduces the model. Section 3 proves that when the background risk is discrete, the optimal insurance contract should be contingent upon the realized values of the

background risk, and gives the explicit form of the optimal solution.

2. The Model

Let $(\Omega, \mathscr{F}, P)$ be a probability space and X, Y be two random variables defined on this space. We assume X is a continuous random variable and Y is a discrete random variable, $X : \Omega \longrightarrow \mathscr{C}$, and $Y : \Omega \longrightarrow \mathscr{D}$, where $\mathscr{C}$ and $\mathscr{D}$ are the sets of consequences of X and Y, respectively. We assume $\mathscr{C} = [0, \overline{x}]$, $\overline{x} > 0$, and $\mathscr{D}$ is a countable subset of $\{0\} \cup R^+$.

Let us assume an agent faces two kinds of random losses X and Y. The insurance market provides insurance only for loss X. In the absence of background risk, the insurance policy or indemnity function is a deterministic function $I(x)$ defined on $\mathscr{C}$ denoting the indemnity that is paid by the insurer if the observed loss is x. The indemnity function is usually assumed to satisfy $0 \le I(x) \le x$, which ensures the indemnity is nonnegative and does not exceed the occurred insurable loss. When there is background risk Y, the optimal insurance policy may be affected by Y, so we denote the insurance policy by $I_Y(X)$. We denote the premium paid by the insured by π. The reasonable premium should satisfy $0 \le \pi \le \overline{x}$.

Assume the insured's initial wealth is W_0. Facing a potential loss X and background loss Y, if the insured buys the contract, he is endowed with the random wealth $W = W_0 - \pi - Y - X + I_Y(X)$. In this paper, we assume the insured has von Neumann-Morgenstern utility function u, and his preference over random wealth is represented by $E[u(W)]$. We assume the utility function $u : R \longrightarrow R$ is increasing, strictly concave, and twice differentiable. The insured is supposed to choose a premium π and an insurance indemnity function $I_Y(\cdot)$ so that they maximize his/her expected utility on final wealth, i.e.,

$$\max_{\pi, I} E\left[u\left(W_0 - \pi - Y - X + I_Y(X)\right)\right]. \tag{1}$$

We also assume the insurer is risk-neutral. From [12, 13], if the cost of offering the insurance is proportional to the expected value of the indemnity $E[I_Y(X)]$, then, for the insurer, he/she requires

$$(1+\rho) E\left[I_Y(X)\right] \le \pi. \tag{2}$$

The expectation $E[I_Y(X)]$ is the actuarial value of the insurance policy. For the risk-neutral insurer, $(1+\rho)E[I_Y(X)]$ is the minimum price of the indemnity $I_Y(X)$ to participate in the business.

Then the optimal design of the insurance contract can be modelled by the optimization problem (3)

$$\begin{gathered} \max_{\pi, I} E\left[u\left(W_0 - \pi - Y - X + I_Y(X)\right)\right], \\ (1+\rho) E\left[I_Y(X)\right] \le \pi. \end{gathered} \tag{3}$$

The model of the optimal index insurance contract in [14] can be modelled by the following optimization problem (4):

$$\begin{gathered} \max_{I} E\left[u\left(W_0 - (1-\theta)\pi - Y + I(X)\right)\right], \\ \gamma E\left[I(X)\right] = \pi. \end{gathered} \tag{4}$$

The two problems (3) and (4) have similar structure, but in model (4), X is not a loss term of the insured, so the solving methods and results are completely different.

3. The Solution of the Optimization Problem with Background Risk

Assume the probability distribution of Y to be

$$P(Y = y_i) = p_i, \quad \forall y_i \in \mathscr{D}, \tag{5}$$

with $\sum_{y_i \in \mathscr{D}} p_i = 1$.

The conditional distribution of X given Y is $F(x \mid Y = y)$. Then the objective function in (3) can be written as

$$\begin{aligned} E\left[u\left(W_0 - \pi - Y - X + I_Y(X)\right)\right] &= E\left[E\left[u\left(W_0 - \pi\right.\right.\right. \\ &\left.\left.\left. - Y - X + I_Y(X)\right) \mid Y\right]\right] = \sum_{y_i \in \mathscr{D}} p_i \int_0^{\overline{x}} u\left(W_0 - \pi - y_i\right. \\ &\left. - x + I_{y_i}(x)\right) dF\left(x \mid Y = y_i\right). \end{aligned} \tag{6}$$

To be able to apply the Lagrange technique in the following, we write the constraint condition as

$$\begin{aligned} (1+\rho) E\left[I_Y(X)\right] &= (1+\rho) E\left[E\left[I_Y(X) \mid Y\right]\right] \\ &= (1+\rho) \sum_{y_i \in \mathscr{D}} p_i \int_0^{\overline{x}} I_{y_i}(x)\, dF\left(x \mid Y = y_i\right) \le \pi. \end{aligned} \tag{7}$$

Lemma 1. *For given π, and each fixed $y_i \in \mathscr{D}$, the solution of optimization problem*

$$\begin{gathered} \max_{0 \le I_{y_i}(x) \le x} \int_0^{\overline{x}} u\left(W_0 - \pi - y_i - x\right. \\ \left. + I_{y_i}(x)\right) dF\left(x \mid Y = y_i\right), \\ (1+\rho) \int_0^{\overline{x}} I_{y_i}(x)\, dF\left(x \mid Y = y_i\right) \le \pi. \end{gathered} \tag{8}$$

is

$$I_{y_i}(x) = x, \quad \text{with } (1+\rho) E\left[I_{y_i}(X) \mid Y = y_i\right] = \pi, \tag{9}$$

or

$$I_{y_i}(x) = \left(x - d_{y_i}\right)^+, \quad \text{where } d_{y_i} = W_0 - \pi - y_i - \left(u'\right)^{-1}\left(\lambda_{y_i}(1+\rho)\right) \text{ and } (1+\rho) E\left[I_{y_i}(X) \mid Y = y_i\right] = \pi. \tag{10}$$

Proof. For given π and fixed $y_i \in \mathscr{D}$, we solve the optimization problem (8). The Lagrangian of (8) is

$$\begin{aligned} L\left(I_{y_i}, \lambda_{y_i}\right) &= \int_0^{\overline{x}} u\left(W_0 - \pi - y_i - x + I_{y_i}(x)\right) dF\left(x \mid Y = y_i\right) \\ &\quad - \lambda_{y_i}\left((1+\rho)\int_0^{\overline{x}} I_{y_i}(x)\, dF\left(x \mid Y = y_i\right) - \pi\right). \end{aligned} \tag{11}$$

Let

$$\begin{aligned} H\left(I_{y_i}, \lambda_{y_i}\right) &= u\left(W_0 - \pi - y_i - x + I_{y_i}(x)\right) \\ &\quad - \lambda_{y_i}(1+\rho) I_{y_i}(x). \end{aligned} \tag{12}$$

Since u is concave, H is concave about $I_{y_i}(\cdot)$.

For x such that $(\partial H/\partial I_{y_i})(I_{y_i}, \lambda_{y_i})|_{I_{y_i}=x} = u'(W_0 - \pi - y_i) - \lambda_{y_i}(1+\rho) > 0$, we should let

$$I_{y_i}(x) = x. \tag{13}$$

For x such that $(\partial H/\partial I_{y_i})(I_{y_i}, \lambda_{y_i})|_{I_{y_i}=0} = u'(W_0 - \pi - y_i - x) - \lambda_{y_i}(1+\rho) < 0$, i.e., $x < W_0 - \pi - y_i - (u')^{-1}(\lambda_{y_i}(1+\rho))$ since u' is nonincreasing, we should let

$$I_{y_i}(x) = 0. \tag{14}$$

Otherwise,

$$\begin{aligned} \frac{\partial H}{\partial I_{y_i}}\left(I_{y_i}, \lambda_{y_i}\right) &= u'\left(W_0 - \pi - y_i - x + I_{y_i}(x)\right) \\ &\quad - \lambda_{y_i}(1+\rho) = 0, \end{aligned} \tag{15}$$

i.e.,

$$I_{y_i}(x) = x + \pi + y_i - W_0 + \left(u'\right)^{-1}\left(\lambda_{y_i}(1+\rho)\right). \tag{16}$$

Since u is concave, u' is nonincreasing. Therefore $u'(W_0 - \pi - y_i) - \lambda_{y_i}(1+\rho) > 0$, and $u'(W_0 - \pi - y_i - x) - \lambda_{y_i}(1+\rho) < 0$ cannot be satisfied simultaneously. Hence with the constraint condition, the optimal solution for (8) is

$$I_{y_i}(x) = x, \quad \text{if } (1+\rho) E\left[I_{y_i}(X) \mid Y = y_i\right] = \pi, \tag{17}$$

or

$$I_{y_i}(x) = \left(x - d_{y_i}\right)^+, \quad \text{where } d_{y_i} = W_0 - \pi - y_i - \left(u'\right)^{-1}\left(\lambda_{y_i}(1+\rho)\right) \text{ and } (1+\rho) E\left[I_{y_i}(X) \mid Y = y_i\right] = \pi. \tag{18}$$

We get the result.

For the case of (10), d_{y_i} is determined by π uniquely; we define a function $d_{y_i}(\pi) : \pi \longrightarrow d_{y_i}$. For the case of (9), we let $d_{y_i}(\pi) \equiv 0$. □

Lemma 2. *For the case of (10), the first- and the second-order derivatives of $d_{y_i}(\pi)$ for each $y_i \in \mathscr{D}$ are*

$$\begin{aligned} d'_{y_i}(\pi) &= \frac{1}{1+\rho}\frac{1}{F\left(d_{y_i}(\pi) \mid Y = y_i\right) - 1} < 0, \\ d''_{y_i}(\pi) &= -\frac{1}{1+\rho}\frac{F'\left(d_{y_i}(\pi) \mid Y = y_i\right) d'_{y_i}(\pi)}{\left(F\left(d_{y_i}(\pi) \mid Y = y_i\right) - 1\right)^2} \geq 0. \end{aligned} \tag{19}$$

Proof. By the constraint conditions in (10), we have

$$(1+\rho)\int_{d_{y_i}(\pi)}^{\overline{x}} \left(x - d_{y_i}(\pi)\right) dF\left(x \mid Y = y_i\right) = \pi, \tag{20}$$

and differentiating about π, we get (19). □

Remark 3. For the case of (9), the first- and the second-order derivatives of $d_{y_i}(\pi)$ for each $y_i \in \mathscr{D}$ are

$$\begin{aligned} d'_{y_i}(\pi) &= 0, \\ d''_{y_i}(\pi) &= 0. \end{aligned} \tag{21}$$

Lemma 4. *There exists a unique solution of the following optimal problem without constraint:*

$$\begin{aligned} \max_{0 \leq \pi \leq \overline{x}} \sum_{y_i \in \mathscr{D}} p_i \int_0^{\overline{x}} u\Big(W_0 - \pi - y_i - x \\ + \left(x - d_{y_i}(\pi)\right)^+\Big) dF\left(x \mid Y = y_i\right). \end{aligned} \tag{22}$$

Proof. We denote the objective function in (22) by $V(\pi)$, i.e.,

$$\begin{aligned} V(\pi) = \sum_{y_i \in \mathscr{D}} p_i \int_0^{\overline{x}} u\Big(W_0 - \pi - y_i - x \\ + \left(x - d_{y_i}(\pi)\right)^+\Big) dF\left(x \mid Y = y_i\right). \end{aligned} \tag{23}$$

The second derivative of $V(\cdot)$,

$$\begin{aligned} V''(\pi) &= \sum_{y_i\in\mathcal{D}} p_i \int_0^{\overline{x}} \Big[u''\Big(W_0 - \pi - y_i - x + \big(x - d_{y_i}(\pi)\big)^+\Big)\Big(-1 - \mathbf{I}_{\{x>d_{y_i}(\pi)\}} d'_{y_i}(\pi)\Big)^2 + u'\Big(W_0 - \pi - y_i - x + \big(x - d_{y_i}(\pi)\big)^+\Big) \\ &\quad \cdot \Big(-\mathbf{I}_{\{x>d_{y_i}(\pi)\}} d''_{y_i}(\pi)\Big)\Big] dF(x \mid Y = y_i). \end{aligned} \tag{24}$$

Since $u'' < 0$, $u' > 0$, and $d''_{y_i} \geq 0$, $(\partial^2 V/\partial\pi^2)(\pi) < 0$, that is, $V(\cdot)$ is strictly concave. The concave maximization problem (22) can be solved by minimizing the convex objective function $-V(\pi)$. By convex analysis ([15], Corollaire 3.20), and strict convexity of $-V(\pi)$, there exists a unique π that attains the minimum of $-V(\pi)$. □

Theorem 5. *The optimal solution π for the optimization problem (3) is the unique solution of the optimal problem without constraint (22), and the optimal solution $I_Y(X)$ for the optimization problem (3) is*

$$I_Y(X) = \sum_{y_i\in\mathcal{D}} \mathbf{I}_{\{Y=y_i\}} I_{y_i}(X), \tag{25}$$

where $y_i \in \mathcal{D}$ are the consequences of Y, and for each fixed y_i, $I_{y_i}(X)$ is the optimal solution for the optimization problem (8).

Proof. Since u is strictly concave, if the solution for problem (3) as well as (8) exists, it is unique.

From (6) and (7), the insured's optimization problem (3) can be written as

$$\begin{aligned} &\max_{\pi,I} \sum_{y_i\in\mathcal{D}} p_i \cdot \int_0^{\overline{x}} u\big(W_0 - \pi - y_i - x + I_{y_i}(x)\big) dF(x \mid Y = y_i), \\ &(1+\rho) \sum_{y_i\in\mathcal{D}} p_i \int_0^{\overline{x}} I_{y_i}(x)\, dF(x \mid Y = y_i) \leq \pi. \end{aligned} \tag{26}$$

Let $\widetilde{\pi}, \widetilde{I}_Y(X)$ be the solution of (3).

For each fixed $y_i \in \mathcal{D}$, solve the optimization problem

$$\begin{aligned} &\max_{0\leq \widetilde{I}_{y_i}(x)\leq x} \int_0^{\overline{x}} u\big(W_0 - \widetilde{\pi} - y_i - x + \widetilde{I}_{y_i}(x)\big) dF(x \mid Y = y_i), \\ &(1+\rho) \int_0^{\overline{x}} \widetilde{I}_{y_i}(x)\, dF(x \mid Y = y_i) \leq \widetilde{\pi}. \end{aligned} \tag{27}$$

Then

$$\begin{aligned} E\Big[u\big(W_0 - \widetilde{\pi} - Y - X + \widetilde{I}_Y(X)\big)\Big] &\leq \sum_{y_i\in\mathcal{D}} p_i \cdot \max_{0\leq \widetilde{I}_{y_i}(x)\leq x} \int_0^{\overline{x}} u\big(W_0 - \widetilde{\pi} - y_i - x + \widetilde{I}_{y_i}(x)\big) dF(x \mid Y = y_i). \end{aligned} \tag{28}$$

Since $\widetilde{\pi}, \widetilde{I}_Y(X)$ are the solution of (3),

$$\begin{aligned} E\Big[u\big(W_0 - \widetilde{\pi} - Y - X + \widetilde{I}_Y(X)\big)\Big] &\geq \sum_{y_i\in\mathcal{D}} p_i \cdot \max_{0\leq \widetilde{I}_{y_i}(x)\leq x} \int_0^{\overline{x}} u\big(W_0 - \widetilde{\pi} - y_i - x + \widetilde{I}_{y_i}(x)\big) dF(x \mid Y = y_i). \end{aligned} \tag{29}$$

Hence

$$\widetilde{I}_Y(X) = \mathbf{I}_{\{Y=y_i\}} \widetilde{I}_{y_i}(X), \tag{30}$$

From Lemma 1, the solutions of the optimization problems (27) for $y_i \in \mathcal{D}$ are (9) (with $d_{y_i}(\widetilde{\pi}) \equiv 0$) or (10) (with $d_{y_i}(\widetilde{\pi}) > 0$) satisfying

$$(1+\rho) \int_{d_{y_i}(\widetilde{\pi})}^{\overline{x}} \big(x - d_{y_i}(\widetilde{\pi})\big) dF(x \mid Y = y_i) = \widetilde{\pi}. \tag{31}$$

Therefore

$$\begin{aligned} &\sum_{y_i\in\mathcal{D}} p_i \max_{0\leq \widetilde{I}_{y_i}(x)\leq x} \int_0^{\overline{x}} u\big(W_0 - \widetilde{\pi} - y_i - x + \widetilde{I}_{y_i}(x)\big) dF(x \mid Y = y_i) \\ &= \sum_{y_i\in\mathcal{D}} p_i \int_0^{\overline{x}} u\Big(W_0 - \widetilde{\pi} - y_i - x + \big(x - d_{y_i}(\widetilde{\pi})\big)^+\Big) dF(x \mid Y = y_i) \\ &\leq \sum_{y_i\in\mathcal{D}} p_i \int_0^{\overline{x}} u\Big(W_0 - \pi - y_i - x + \big(x - d_{y_i}(\pi)\big)^+\Big) dF(x \mid Y = y_i) \end{aligned} \tag{32}$$

where the inequality is because of (22), and the equal sign "=" holds if and only if $\pi = \widetilde{\pi}$; otherwise "<" holds and it will contradict the assumption that $\widetilde{\pi}, \widetilde{I}_Y(X)$ is the solution of (3). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work is supported by NSF of China (No. 11871163).

References

[1] N. A. Doherty and H. Schlesinger, "Optimal insurance in incomplete markets," *Journal of Political Economy*, vol. 91, no. 6, pp. 1045–1054, 1983.

[2] H. Schlesinger, "The theory of insurance demand," in *Handbook of Insurance*, G. Dionne, Ed., pp. 131–151, Kluwer, Boston, Mass, USA, 2000.

[3] C. Gollier, "Optimum insurance of approximate losses," *Journal of Risk and Insurance*, vol. 63, no. 3, pp. 369–380, 1996.

[4] D. J. Meyer and J. Meyer, "Changes in background risk and the demand for insurance," *GENEVA Papers on Risk and Insurance Theory*, vol. 23, no. 1, pp. 29–40, 1998.

[5] O. Mahul, "Optimum crop insurance under joint yield and price risk," *Journal of Risk and Insurance*, vol. 67, no. 1, pp. 109–122, 2000.

[6] J. Vercammen, "Optimal insurance with nonseparable background risk," *Journal of Risk and Insurance*, vol. 68, no. 3, pp. 437–448, 2001.

[7] H.-H. Huang, Y.-M. Shiu, and C.-P. Wang, "Optimal insurance contract with stochastic background wealth," *Scandinavian Actuarial Journal*, vol. 2013, no. 2, pp. 119–139, 2013.

[8] C. Weng and S. C. Zhuang, "CDF formulation for solving an optimal reinsurance problem," *Scandinavian Actuarial Journal*, vol. 2017, no. 5, pp. 395–418, 2017.

[9] L. Guiso and T. Jappelli, "Background uncertainty and the demand for insurance against insurable risks," *GENEVA Papers on Risk and Insurance Theory*, vol. 23, no. 1, pp. 7–27, 1998.

[10] Z. Lu, L. Liu, J. Zhang, and L. Meng, "Optimal insurance under multiple sources of risk with positive dependence," *Insurance: Mathematics & Economics*, vol. 51, no. 2, pp. 462–471, 2012.

[11] R.-A. Dana and M. Scarsini, "Optimal risk sharing with background risk," *Journal of Economic Theory*, vol. 133, no. 1, pp. 152–176, 2007.

[12] K. J. Arrow, *Essays in the Theory of Risk-Bearing*, Chicago, Ill, USA, 1971.

[13] A. Raviv, "The design of an optimal insurance policy," *The American Economic Review*, vol. 69, no. 1, pp. 84–96, 1979.

[14] J. Zhang, K. Tan, and C. Weng, "Index insurance design," 2018, https://papers.ssrn.com/sol3/papers.cfm?abstract_id=3241297.

[15] H. Brezis, *Analyse Fonctionnelle*, Masson, Paris, France, 1983.