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Research Article

Positive Solutions for a System of Neumann Boundary Value Problems of Second-Order Difference Equations Involving Sign-Changing Nonlinearities

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In this paper, we study the existence of positive solutions for the system of second-order difference equations involving Neumann boundary conditions: $-\Delta^2 u_1(t-1) = f_1(t, u_1(t), u_2(t))$, $t \in [1, T]_{\mathbb{Z}}$, $-\Delta^2 u_2(t-1) = f_2(t, u_1(t), u_2(t))$, $t \in [1, T]_{\mathbb{Z}}$, $\Delta u_i(0) = \Delta u_i(T) = 0$, $i = 1, 2$, where $T > 1$ is a given positive integer, $\Delta u(t) = u(t+1) - u(t)$, and $\Delta^2 u(t) = \Delta(\Delta u(t))$. Under some appropriate conditions for our sign-changing nonlinearities, we use the fixed point index to establish our main results.

1. Introduction

For $a, b \in \mathbb{Z}$ with $a < b$, let $[a, b]_{\mathbb{Z}} = \{a, a+1, a+2, \ldots, b-1, b\}$. Consider the system of second-order difference equations involving Neumann boundary conditions:

$$\begin{aligned} -\Delta^2 u_1(t-1) &= f_1(t, u_1(t), u_2(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ -\Delta^2 u_2(t-1) &= f_2(t, u_1(t), u_2(t)), \quad t \in [1, T]_{\mathbb{Z}}, \\ \Delta u_i(0) &= \Delta u_i(T) = 0, \quad i = 1, 2, \end{aligned} \tag{1}$$

where $f_i : [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+ \longrightarrow \mathbb{R}$ $(i = 1, 2)$ are two continuous functions and there exist $h_i : [1, T]_{\mathbb{Z}} \longrightarrow \mathbb{R}^+$ with $h_i(t) \not\equiv 0$ on $[1, T]_{\mathbb{Z}} (i = 1, 2)$ and a positive number $L > 0$ such that

(H0) $f_i(t, u_1, u_2) + L u_i + h_i(t) \geq 0$, $(t, u_1, u_2) \in [1, T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+$, i=1,2.

As known to all, semipositone problems arise in bulking of mechanical systems, design of suspension bridges, chemical reactions, astrophysics, combustion, and management of natural resources; for example, see [1–4]. We note that studying positive solutions for semipositone problems is more difficult than that for positive problems. There are many methods to deal with semipositone (positive) problems, with the usual approaches being variational methods, fixed point theory, subsuper solutions methods, and degree theory; for example, see [3–32] and references therein.

In [5], the author used the Guo-Krasnosel'skii fixed point theorem to study the existence of at least one positive solution for the discrete fractional equation:

$$\begin{aligned} -\Delta^\nu y(t) &= \lambda f(t + \nu - 1, y(t + \nu - 1)), \\ &\qquad t \in [1, b+1]_{\mathbb{N}}, \\ y(\nu - 2) &= \sum_{i=1}^{N} F_i^1\left(y\left(t_i^1\right)\right), \\ y(\nu + b + 1) &= \sum_{i=1}^{M} F_i^2\left(y\left(t_i^2\right)\right), \end{aligned} \tag{2}$$

where $\lambda > 0$ is a parameter; the semipositone nonlinear term f satisfies the condition

$$\lim_{y\longrightarrow+\infty}\frac{f(t,y)}{y}=+\infty,$$
$$\lim_{y\longrightarrow0^+}\frac{f(t,y)}{y}=0, \tag{3}$$
$$\text{uniformly for } t\in[\nu,\nu+b]_{\mathbb{N}}.$$

For semipositone systems, the authors [9, 17, 18] used the similar conditions of (3) to obtain some results for boundary value problems of differential (difference) equations.

However, we note that systems of boundary value problems for difference problems have seldom been considered in the literature; we refer to only [8, 9, 33–37] and references therein.

In [33], the authors used the Krasnosel'skii-Zabreiko fixed point theorem to investigate the existence and multiplicity of positive solutions for the system of second-order discrete boundary value problems:

$$\begin{gathered}\Delta^2u(k-1)+f(k,u(k),v(k))=0,\\ k\in\{1,2,\dots,T\},\\ \Delta^2v(k-1)+g(k,u(k),v(k))=0,\\ k\in\{1,2,\dots,T\},\\ u(0)=u(T+1)=v(0)=v(T+1)=0,\end{gathered} \tag{4}$$

where f and g are nonnegative continuous functions on $\{1,2,\dots,T\}\times\mathbb{R}^+\times\mathbb{R}^+$.

Inspired by the works aforementioned, in this paper we use the fixed point index to consider the existence of positive solutions for (1). The novelty is threefold: (1) The nonlinearities may be either bounded or unbounded below; ultimately nonpositive or nonnegative or oscillating, see [6, Page 2]; this improves some conditions for the nonlinearities in [36, 37]. (2) Some appropriate nonnegative concave and convex functions are employed to depict the coupling behaviors of nonlinearities. (3) Our conditions are better than (3). (4) Our a priori estimates for positive solutions are derived by unknown functions $u_i-w_i, i=1,2$; see Section 3. This is different from [36, 37].

2. Preliminary

For convenience, let

$$\begin{aligned}A&=\frac12\left(L+2+\sqrt{L^2+4L}\right),\\ \rho&=\left(A^T-A^{-T}\right)\left(A^2-1\right).\end{aligned} \tag{5}$$

Then

$$G(t,s)=\frac1\rho\cdot\begin{cases}\left(A^s+A^{-s+1}\right)\left(A^{t-T}+A^{T-t+1}\right), & 1\le s\le t\le T+1,\\ \left(A^t+A^{-t+1}\right)\left(A^{s-T}+A^{T-s+1}\right), & 0\le t\le s\le T,\end{cases} \tag{6}$$

is the Green's function associated with the linear Neumann boundary value problems

$$\begin{aligned}-\Delta^2u_i(t-1)+Lu_i(t)=h_i(t),\quad t\in[1,T]_{\mathbb{Z}},\\ \Delta u_i(0)=\Delta u_i(T)=0,\quad i=1,2,\end{aligned} \tag{7}$$

which is equivalent to

$$w_i(t)=\sum_{s=1}^TG(t,s)h_i(s),\quad i=1,2. \tag{8}$$

Let $q^*(t)=(1/2A^T)\min\{A^{t-T}+A^{T-t+1},A^t+A^{-t+1}\}$, $t\in[0,T+1]_{\mathbb{Z}}$. Then $0<q^*(t)<1$, for $t\in[1,T]_{\mathbb{Z}}$. Consequently, from [6] we obtain that the Green's function G has the following properties.

Lemma 1. *(i) $G(t,s)>0$ for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.*

(ii) $q^*(t)G(s,s)\le G(t,s)\le G(s,s)$, for all $(t,s)\in[0,T+1]_{\mathbb{Z}}\times[1,T]_{\mathbb{Z}}$.

(iii) If $\varphi(s)=G(s,s)$, for $s\in[1,T]_{\mathbb{Z}}$, then we have

$$\sum_{t=1}^Tq^*(t)\varphi(t)\cdot\varphi(s)\le\sum_{t=1}^TG(t,s)\varphi(t)\le\sum_{t=1}^T\varphi(t)\cdot\varphi(s). \tag{9}$$

These involve direct computations, and so we omit their proofs. For convenience, we denote

$$\begin{aligned}\kappa_1&=\sum_{t=1}^Tq^*(t)\varphi(t),\\ \kappa_2&=\sum_{t=1}^T\varphi(t).\end{aligned} \tag{10}$$

Define

$$\tilde f_i(t,u_1,u_2)=\begin{cases}f_i(t,u_1,u_2)+Lu_i+h_i(t), & (t,u_1,u_2)\in[1,T]_{\mathbb{Z}}\times\mathbb{R}^+\times\mathbb{R}^+,\\ f_i(t,0,0)+h_i(t), & (t,u_1,u_2)\in[1,T]_{\mathbb{Z}}\times\mathbb{R}^-\times\mathbb{R}^-,\ i=1,2,\end{cases} \tag{11}$$

and consider the following modified discrete Neumann boundary value problems

$$\begin{aligned} &-\Delta^2 u_1(t-1) + Lu_1(t) \\ &\quad = \tilde{f}_1(t, u_1(t) - w_1(t), u_2(t) - w_2(t)), \\ &\qquad\qquad t \in [1,T]_{\mathbb{Z}}, \\ &-\Delta^2 u_2(t-1) + Lu_2(t) \\ &\quad = \tilde{f}_2(t, u_1(t) - w_1(t), u_2(t) - w_2(t)), \\ &\qquad\qquad t \in [1,T]_{\mathbb{Z}}, \\ &\Delta u_i(0) = \Delta u_i(T) = 0, \quad i = 1, 2. \end{aligned} \tag{12}$$

Lemma 2 (see [6]). *(u_1, u_2) is a positive solution of (1) if and only if $(v_1, v_2) = (u_1 + w_1, u_2 + w_2)$ is a solution of (12) with $v_i(t) \ge w_i(t)$ for $t \in [1,T]_{\mathbb{Z}}$.*

Let E be the collection of all maps from $[0, T+1]_{\mathbb{Z}}$ to $\mathbb{R}$ equipped with the max norm, $\|\cdot\|$. Then E is a Banach space. Define a set $P \subset E$ by $P = \{y \in E : y(t) \ge 0,\ t \in [1,T]_{\mathbb{Z}}\}$. Then P is a cone on E. Moreover, $E \times E$ is a Banach space with the norm $\|(x, y)\| := \max\{\|x\|, \|y\|\}$, and $P \times P$ is a cone on $E \times E$.

Note that (12) can be expressed in the form

$$\begin{aligned} &u_1(t) \\ &\quad = \sum_{s=1}^{T} G(t,s) \tilde{f}_1(s, u_1(s) - w_1(s), u_2(s) - w_2(s)), \\ &u_2(t) \\ &\quad = \sum_{s=1}^{T} G(t,s) \tilde{f}_2(s, u_1(s) - w_1(s), u_2(s) - w_2(s)). \end{aligned} \tag{13}$$

As a result, for $u_i \in P(i = 1, 2)$, and $t \in [1,T]_{\mathbb{Z}}$, we define the operators

$$\begin{aligned} &B_i(u_1, u_2)(t) \\ &\quad = \sum_{s=1}^{T} G(t,s) \tilde{f}_i(s, u_1(s) - w_1(s), u_2(s) - w_2(s)), \end{aligned} \tag{14}$$

and

$$B(u_1, u_2)(t) = (B_1, B_2)(u_1, u_2)(t). \tag{15}$$

Then we use the Arzelà-Ascoli theorem in a standard way to establish that $B : P \times P \longrightarrow P \times P$ is a completely continuous operator. It is clear that $(u_1, u_2) \in (P \times P) \setminus \{\mathbf{0}\}$ is a positive solution for (12) if and only if $(u_1, u_2) \in (P \times P) \setminus \{\mathbf{0}\}$ is a fixed point of B.

On the other hand, let $P_0 = \{y \in P : y(t) \ge q^*(t)\|u_2\|,\ \forall t \in [1,T]_{\mathbb{Z}}\}$. Then from Lemma 1(ii) we have

$$B_i(P \times P) \subset P_0, \quad i = 1, 2. \tag{16}$$

Therefore, if we seek a fixed point (v_1, v_2) of B with $(v_1, v_2)(t) \ge (w_1, w_2)(t)$ (i.e., $(v_1 - w_1, v_2 - w_2)(t)$ is a positive solution for (1)). Then if $v_i \in P_0$ $(i = 1, 2)$ we have

$$\begin{aligned} v_1(t) - w_1(t) &\ge q^*(t)\|v_1\| - \sum_{s=1}^{T} G(s,s) h_1(s) \\ &\ge q_0\|v_1\| - \sum_{s=1}^{T} G(s,s) h_1(s), \\ &\qquad \text{for } t \in [1,T]_{\mathbb{Z}}, \end{aligned} \tag{17}$$

where $q_0 = \min_{t \in [1,T]_{\mathbb{Z}}} q^*(t) > 0$. As a result, $\|v_i\| \ge q_0^{-1} \sum_{s=1}^{T} G(s,s) h_i(s) := \mathscr{M}_i$ $(i = 1, 2)$ implies that $(v_1, v_2)(t) \ge (w_1, w_2)(t)$ for $t \in [1,T]_{\mathbb{Z}}$.

Lemma 3 (see [38]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists $\omega_0 \in P \setminus \{0\}$ such that*

$$\omega - A\omega \ne \lambda\omega_0, \quad \forall \lambda \ge 0,\ \omega \in \partial\Omega \cap P, \tag{18}$$

then $i(A, \Omega \cap P, P) = 0$, where i denotes the fixed point index on P.

Lemma 4 (see [38]). *Let E be a real Banach space and P a cone on E. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \ne 0, \quad \forall \lambda \in [0, 1],\ \omega \in \partial\Omega \cap P, \tag{19}$$

then $i(A, \Omega \cap P, P) = 1$.

3. Main Results

For convenience, we use $c_1, c_2, \ldots$ to denote distinct positive constants. Let $B_\varrho := \{x \in E : \|x\| < \varrho\}$ for $\varrho > 0$. Now, we list our assumptions on $\tilde{f}_i (i = 1, 2)$.

(H1) There exist $p, q \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that

(i) p is concave and strictly increasing on $\mathbb{R}^+$,

(ii) there exist $c_1 > 0, d_1 \in (0, \kappa_2^{-1})$ such that

$$\begin{aligned} &\tilde{f}_1(t, x, y) \ge d_1 x + p(y) - c_1, \\ &\tilde{f}_2(t, x, y) \ge q(x) - c_1, \\ &\qquad \forall (t, x, y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{20}$$

(iii) there is a $\gamma_1 > \kappa_1^{-2}(1 - d_1\kappa_1)$ such that

$$p(\kappa_2 q(x(t))) \ge \kappa_2\gamma_1(x(t)) - c_1 \tag{21}$$

for $x \in \mathbb{R}^+$ and $t \in [1,T]_{\mathbb{Z}}$.

(H2) Let $\mathscr{M}_3 = \max\{\mathscr{M}_1, \mathscr{M}_3\}$. Then for any $(t, x, y) \in [1,T]_{\mathbb{Z}} \times [0, \mathscr{M}_3] \times [0, \mathscr{M}_3]$, we suppose that

$$\tilde{f}_i(t, x, y) < \kappa_2^{-1}\mathscr{M}_3. \tag{22}$$

(H3) There exist $\xi, \eta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that
(i) ξ is convex and strictly increasing on $\mathbb{R}^+$,
(ii) there exists $d_2 \in (0, \kappa_2^{-1})$ such that

$$\begin{aligned} \widetilde{f}_1(t,x,y) &\le d_2 x + \xi(y), \\ \widetilde{f}_2(t,x,y) &\le \eta(x), \\ &\forall (t,x,y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+, \end{aligned} \tag{23}$$

(iii) there are $c_2 > 0$ and $\gamma_2 < (0, (1 - d_2\kappa_2)\kappa_2^{-2})$ such that

$$\xi(\kappa_2 \eta(x(t))) \le \kappa_2 \gamma_2 (x(t)) + c_2 \tag{24}$$

for $x \in \mathbb{R}^+$ and $t \in [1,T]_{\mathbb{Z}}$.

(H4) For any $(t,x,y) \in [1,T]_{\mathbb{Z}} \times [0, \mathscr{M}_3] \times [0, \mathscr{M}_3]$, we suppose that

$$\widetilde{f}_i(t,x,y) > q_0^{-1}\kappa_2^{-1}\mathscr{M}_3. \tag{25}$$

We now present a list of remarks and examples in which we discuss how our hypotheses and assumptions are better (weaker) hypotheses and assumptions in some of the closely related papers cited in the reference.

Remark 5. We first provide the growth conditions for the nonlinearities of [36, (H3)(ii) on page 4] given by

$$\begin{aligned} &\limsup_{e_1x+e_2y\longrightarrow+\infty} \frac{\widetilde{f}(t,x,y)}{e_1x+e_2y} \le 1, \\ &\limsup_{e_3x+e_4y\longrightarrow+\infty} \frac{\widetilde{g}(t,x,y)}{e_3x+e_4y} \le 1, \\ &\text{uniformly on } t \in [\nu-1, b+\nu+1]_{\mathbb{N}_{\nu-1}}, \end{aligned} \tag{26}$$

where $e_i (i = 1,2,3,4)$ are nonnegative real numbers. However, note that our condition (H3)(i), (ii) of this paper is given by

$$\begin{aligned} &\limsup_{d_2x+\xi(y)\longrightarrow+\infty} \frac{\widetilde{f}_1(t,x,y)}{d_2x+\xi(y)} \le 1, \\ &\text{uniformly on } [1,T]_{\mathbb{Z}}, \end{aligned} \tag{27}$$

and obviously, this includes (26) as a special case.

On the other hand, we note that our growth conditions for nonlinearity $\widetilde{f}_1$ depends on two variables x, y; however, in [37, (H2)(ii) and (H4)(i)], the corresponding conditions only involve one variable. Finally, our nonlinearities here are allowed to be unbounded from below, which are better than the nonlinearities in [36, 37], which are bounded below due to being semipositone.

Remark 6. Note that (3) is the superlinear condition; i.e., the degree is 1; however, for our conditions, the corresponding degree can be any arbitrary positive number. For example, if we take $\widetilde{f}_2(t,x,y) = q(x) = x^\gamma$ with $\gamma > 0$ for $(t,x,y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+$, we see that this function does not satisfy the condition (3) if $\gamma \in (0,1)$.

Remark 7. In this paper, we use the functions p, q, ξ, η (see (H1) and (H3)) to act on $u_i - w_i$ and then estimate the norms of u_i; however, in [36, 37] the corresponding parts only involve $u_i (i = 1,2)$. Moreover, when the nonlinearities in [36] grow sublinearly at $+\infty$, nonnegative matrices are used to depict the coupling behaviors, yet this is not used in our paper.

Example 8. Let $p(y) = y^{4/5}$, $q(x) = x^2$, $x, y \in \mathbb{R}^+$. Then $\lim_{x\longrightarrow+\infty}(p(\kappa_2 q(x))/x) = \lim_{x\longrightarrow+\infty}(\kappa_2^{4/5}x^{8/5}/x) = +\infty$, and p, q satisfy (H1). Moreover, we take $\widetilde{f}_1, \widetilde{f}_2$ as follows:

$$\begin{aligned} \widetilde{f}_1(t,x,y) &= d_1\kappa_2^{-1}x + \frac{1}{\beta_1 + e^{|\sin t|}}\left(\kappa_2^{-1} - d_1\right)\kappa_2^{-1}y, \\ \widetilde{f}_2(t,x,y) &= \frac{1}{\beta_2 + e^{|\cos t|}}\kappa_2^{-1}\mathscr{M}_3^{1-\beta_3}x^{\beta_3}, \end{aligned} \tag{28}$$

where $\beta_1, \beta_2 > 0, \beta_3 > 2$, $(t,x,y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+$. We next show that $\widetilde{f}_1, \widetilde{f}_2$ satisfy (H1)(ii) and (H2). Suppose that $(t,x,y) \in [1,T]_{\mathbb{Z}} \times [0,\mathscr{M}_3] \times [0,\mathscr{M}_3]$; we have

$$\begin{aligned} \widetilde{f}_1(t,x,y) &\le d_1\kappa_2^{-1}\mathscr{M}_3 \\ &\quad + \frac{1}{\beta_1 + e^{|\sin t|}}\left(\kappa_2^{-1} - d_1\right)\kappa_2^{-1}\mathscr{M}_3 \\ &< \kappa_2^{-1}\mathscr{M}_3, \\ \widetilde{f}_2(t,x,y) &\le \frac{1}{\beta_2 + e^{|\cos t|}}\kappa_2^{-1}\mathscr{M}_3^{1-\beta_3}\mathscr{M}_3^{\beta_3} < \kappa_2^{-1}\mathscr{M}_3. \end{aligned} \tag{29}$$

On the other hand, we also have

$$\begin{aligned} &\liminf_{y\longrightarrow+\infty} \frac{\widetilde{f}_1(t,x,y)}{d_1x + p(y)} \\ &= \liminf_{y\longrightarrow+\infty} \frac{d_1\kappa_2^{-1}x + \left(1/\left(\beta_1 + e^{|\sin t|}\right)\right)\left(\kappa_2^{-1} - d_1\right)\kappa_2^{-1}y}{d_1x + y^{4/5}} \\ &= +\infty, \quad \text{uniformly on } (t,x) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+, \end{aligned} \tag{30}$$

and

$$\begin{aligned} &\liminf_{x\longrightarrow+\infty} \frac{\widetilde{f}_2(t,x,y)}{q(x)} \\ &= \liminf_{x\longrightarrow+\infty} \frac{\left(1/\left(\beta_2 + e^{|\cos t|}\right)\right)\kappa_2^{-1}\mathscr{M}_3^{1-\beta_3}x^{\beta_3}}{x^2} = +\infty, \\ &\text{uniformly on } (t,y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+. \end{aligned} \tag{31}$$

And so, (H1)(ii) and (H2) hold.

It follows, from (11), that

$$\begin{aligned} f_1(t,x,y) &= \widetilde{f}_1(t,x,y) - Lx - h_1(t) \\ &= d_1\kappa_2^{-1}x + \frac{1}{\beta_1 + e^{|\sin t|}}\left(\kappa_2^{-1} - d_1\right)\kappa_2^{-1}y \\ &\quad - Lx - h_1(t), \end{aligned} \tag{32}$$

and this function may be unbounded from below if L and h_1 are large enough. So, this function is not applicable in [36, 37]. Moreover, we also note that $\widetilde{f}_1$ is a linear function about x, y, and it does not satisfy the condition (3).

Example 9. Let $\xi(y) = y^2$, $\eta(x) = \ln(x+1)$, and $x, y \in \mathbb{R}^+$. Then $\lim_{x\longrightarrow+\infty}(\xi(\kappa_2\eta(x))/x) = \lim_{x\longrightarrow+\infty}(\kappa_2^2\ln^2(x+1)/x) = 0$, and ξ, η satisfy (H3). Moreover, we chose $\widetilde{f}_1, \widetilde{f}_2$ as follows:

$$\begin{aligned}\widetilde{f}_1(t,x,y) &= d_2x \\ &\quad + \left(q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3} + \beta_4 + |\sin t|\right)e^{-y}, \\ \widetilde{f}_2(t,x,y) &= \left(q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3} + \beta_5 + |\cos ty|\right)e^{-x},\end{aligned} \tag{33}$$

where $\beta_4, \beta_5 > 0$, $(t,x,y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+ \times \mathbb{R}^+$. In what follows, we prove that $\widetilde{f}_1, \widetilde{f}_2$ satisfy (H3)(ii) and (H4). Indeed, if $(t,x,y) \in [1,T]_{\mathbb{Z}} \times [0,\mathscr{M}_3] \times [0,\mathscr{M}_3]$, we have

$$\begin{aligned}\widetilde{f}_1(t,x,y) &\geq \left(q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3} + \beta_4 + |\sin t|\right)e^{-y} \\ &> q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3}e^{-\mathscr{M}_3} = q_0^{-1}\kappa_2^{-1}\mathscr{M}_3,\end{aligned} \tag{34}$$

and

$$\begin{aligned}\widetilde{f}_2(t,x,y) &> q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3}e^{-x} \geq q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3}e^{-\mathscr{M}_3} \\ &= q_0^{-1}\kappa_2^{-1}\mathscr{M}_3.\end{aligned} \tag{35}$$

On the other hand, we obtain

$$\begin{aligned}&\limsup_{y\longrightarrow+\infty}\frac{\widetilde{f}_1(t,x,y)}{d_2x+\xi(y)} \\ &\quad= \limsup_{y\longrightarrow+\infty}\frac{d_2x + \left(q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3} + \beta_4 + |\sin t|\right)e^{-y}}{d_2x + y^2} \\ &\quad= 0, \quad \text{uniformly on } (t,x) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+,\end{aligned} \tag{36}$$

and

$$\begin{aligned}&\limsup_{x\longrightarrow+\infty}\frac{\widetilde{f}_2(t,x,y)}{\eta(x)} \\ &\quad= \limsup_{x\longrightarrow+\infty}\frac{\left(q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3} + \beta_5 + |\cos ty|\right)e^{-x}}{\ln(x+1)} \\ &\quad= 0, \quad \text{uniformly on } (t,y) \in [1,T]_{\mathbb{Z}} \times \mathbb{R}^+.\end{aligned} \tag{37}$$

Consequently, (H3)(ii) and (H4) hold.

Next, from (11) we see that

$$\begin{aligned}f_2(t,x,y) &= \widetilde{f}_2(t,x,y) - Ly - h_2(t) \\ &= \left(q_0^{-1}\kappa_2^{-1}\mathscr{M}_3e^{\mathscr{M}_3} + \beta_5 + |\cos ty|\right)e^{-x} \\ &\quad - Ly - h_2(t),\end{aligned} \tag{38}$$

and this function may be unbounded from below if L and h_2 are large enough. So, this function is not applicable in [36, 37]. Moreover, we also note that $\widetilde{f}_2$ is a sublinear function about x, and it does not satisfy the condition (3).

Theorem 10. *Suppose that (H0)-(H2) hold. Then (1) has at least one positive solution.*

Proof. There exists a sufficiently large $R > \max\{\mathscr{M}_1, \mathscr{M}_2\} = \mathscr{M}_3$, for which we will prove that

$$\begin{aligned}&(u_1,u_2) \neq B(u_1,u_2) + \lambda(x_0,y_0), \\ &\qquad \forall (u_1,u_2) \in \partial(B_R \times B_R) \cap (P \times P), \ \lambda \geq 0,\end{aligned} \tag{39}$$

where $x_0, y_0 \in P_0$ are two given functions. Indeed, if not, there exist $(u_1,u_2) \in \partial(B_R \times B_R) \cap (P \times P)$ and $\lambda \geq 0$ such that $(u_1,u_2) = B(u_1,u_2) + \lambda(x_0,y_0)$, and then

$$\begin{aligned}u_1(t) &= B_1(u_1,u_2)(t) + \lambda x_0(t), \\ u_2(t) &= B_2(u_1,u_2)(t) + \lambda y_0(t), \\ &\qquad \text{for } t \in [1,T]_{\mathbb{Z}}.\end{aligned} \tag{40}$$

This implies $u_1(t) \geq B_1(u_1,u_2)(t)$, and $u_2(t) \geq B_2(u_1,u_2)(t)$ for $t \in [1,T]_{\mathbb{Z}}$. Note that $(u_1,u_2) \in \partial(B_R \times B_R) \cap (P \times P)$ with $R > \max\{\mathscr{M}_1, \mathscr{M}_2\}$; this implies $u_i(t) \geq w_i(t)$ for $t \in [1,T]_{\mathbb{Z}}$, $i = 1, 2$. From (H1) we have

$$\begin{aligned}&u_1(t) - w_1(t) \geq B_1(u_1,u_2)(t) - w_1(t) \geq \sum_{s=1}^{T}G(t,s) \\ &\quad \cdot \left[d_1(u_1(s) - w_1(s)) + p(u_2(s) - w_2(s)) - c_1\right] \\ &\quad - w_1(t) \geq \sum_{s=1}^{T}G(t,s) \\ &\quad \cdot \left[d_1(u_1(s) - w_1(s)) + p(u_2(s) - w_2(s))\right] - c_3, \\ &\qquad \text{for } t \in [1,T]_{\mathbb{Z}},\end{aligned} \tag{41}$$

and

$$\begin{aligned}u_2(t) - w_2(t) &\geq B_2(u_1,u_2)(t) - w_2(t) \\ &\geq \sum_{s=1}^{T}G(t,s)\left[q(u_1(s) - w_1(s)) - c_1\right] \\ &\quad - w_2(t) \\ &\geq \sum_{s=1}^{T}G(t,s)q(u_1(s) - w_1(s)) - c_3, \\ &\qquad \text{for } t \in [1,T]_{\mathbb{Z}}.\end{aligned} \tag{42}$$

As a result, for $t \in [1,T]_{\mathbb{Z}}$, we have

$$\begin{aligned}&p(u_2(t) - w_2(t)) + p(c_3) \geq p(u_2(t) - w_2(t) + c_3) \\ &\quad \geq p\left[\sum_{s=1}^{T}G(t,s)q(u_1(s) - w_1(s))\right] \\ &\quad = p\left[\sum_{s=1}^{T}G(t,s)\kappa_2^{-1}\kappa_2q(u_1(s) - w_1(s))\right]\end{aligned}$$

$$\begin{aligned}
&\geq \sum_{s=1}^{T} G(t,s)\,\kappa_2^{-1} p\left(\kappa_2 q\left(u_1(s)-w_1(s)\right)\right) \\
&\geq \sum_{s=1}^{T} G(t,s)\,\kappa_2^{-1}\left[\kappa_2\gamma_1\left(u_1(s)-w_1(s)\right)-c_1\right] \\
&\geq \gamma_1 \sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right)-c_4.
\end{aligned} \tag{43}$$

This means

$$p\left(u_2(t)-w_2(t)\right) \geq \gamma_1 \sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right) - c_5. \tag{44}$$

Therefore,

$$\begin{aligned}
u_1(t)-w_1(t) &\geq \sum_{s=1}^{T} G(t,s) \\
&\cdot\left[d_1\left(u_1(s)-w_1(s)\right)+p\left(u_2(s)-w_2(s)\right)\right]-c_3 \\
&\geq d_1 \sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right)+\sum_{s=1}^{T} G(t,s) \\
&\cdot\left[\gamma_1 \sum_{\tau=1}^{T} G(s,\tau)\left(u_1(\tau)-w_1(\tau)\right)-c_5\right]-c_3 \\
&\geq d_1 \sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right)+\gamma_1 \sum_{s=1}^{T} G(t,s) \\
&\cdot \sum_{\tau=1}^{T} G(s,\tau)\left(u_1(\tau)-w_1(\tau)\right)-c_6, \\
&\qquad \text{for } t \in [1,T]_{\mathbb{Z}}.
\end{aligned} \tag{45}$$

Multiply both sides of the above inequality by $\varphi(t)$ and sum from 1 to T. Then together with Lemma 1(iii) we obtain

$$\begin{aligned}
&\sum_{t=1}^{T}\left(u_1(t)-w_1(t)\right)\varphi(t) \geq \sum_{t=1}^{T}\varphi(t) \\
&\quad\cdot\left[d_1 \sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right)\right. \\
&\quad\left.+\gamma_1 \sum_{s=1}^{T} G(t,s) \sum_{\tau=1}^{T} G(s,\tau)\left(u_1(\tau)-w_1(\tau)\right)-c_6\right] \\
&\quad\geq \kappa_1 d_1 \sum_{t=1}^{T}\left(u_1(t)-w_1(t)\right)\varphi(t)+\gamma_1\kappa_1^2 \sum_{t=1}^{T}\left(u_1(t)\right. \\
&\quad\left.-w_1(t)\right)\varphi(t)-c_7.
\end{aligned} \tag{46}$$

Hence, we have

$$\begin{aligned}
&\sum_{t=1}^{T}\left(u_1(t)-w_1(t)\right)\varphi(t) \leq \frac{c_7}{\gamma_1\kappa_1^2+d_1\kappa_1-1}, \\
&\sum_{t=1}^{T} u_1(t)\varphi(t) \leq \frac{c_7}{\gamma_1\kappa_1^2+d_1\kappa_1-1}+\sum_{t=1}^{T} w_1(t)\varphi(t) \\
&\quad := \mathcal{N}_1.
\end{aligned} \tag{47}$$

From (16), (40), and $x_0 \in P_0$ we have $u_1 \in P_0$. This implies

$$\begin{aligned}
\kappa_1\left\|u_1\right\| = \left\|u_1\right\| \sum_{t=1}^{T} q^*(t)\varphi(t) \leq \sum_{t=1}^{T} u_1(t)\varphi(t) \leq \mathcal{N}_1, \\
\left\|u_1\right\| \leq \kappa_1^{-1}\mathcal{N}_1.
\end{aligned} \tag{48}$$

Note that from (16), (40), and $y_0 \in P_0$, we find $u_2 \in P_0$. From the definition of w_2, we know $w_2 \in P_0$, and this implies $u_2 - w_2 \in P_0$. Moreover, we may assume $u_2(t) - w_2(t) \not\equiv 0$, for $t \in [1,T]_{\mathbb{Z}}$. Then $\|u_2 - w_2\| > 0$ and $p(\|u_2 - w_2\|) > 0$. Thus, from the concavity of p, we have

$$\begin{aligned}
&\kappa_1\left\|u_2-w_2\right\| = \left\|u_2-w_2\right\| \sum_{t=1}^{T} q^*(t)\varphi(t) \\
&\leq \sum_{t=1}^{T}\left(u_2(t)-w_2(t)\right)\varphi(t) \\
&= \frac{\left\|u_2-w_2\right\|}{p\left(\left\|u_2-w_2\right\|\right)} \sum_{t=1}^{T} \frac{u_2(t)-w_2(t)}{\left\|u_2-w_2\right\|} p\left(\left\|u_2-w_2\right\|\right)\varphi(t) \\
&\leq \frac{\left\|u_2-w_2\right\|}{p\left(\left\|u_2-w_2\right\|\right)} \sum_{t=1}^{T} p\left(u_2(t)-w_2(t)\right)\varphi(t).
\end{aligned} \tag{49}$$

This implies that

$$p\left(\left\|u_2-w_2\right\|\right) \leq \kappa_1^{-1} \sum_{t=1}^{T} p\left(u_2(t)-w_2(t)\right)\varphi(t). \tag{50}$$

On the other hand, from (41) and Lemma 1(ii) we obtain

$$\begin{aligned}
u_1(t)+c_3 &\geq \sum_{s=1}^{T} G(t,s)\, p\left(u_2(s)-w_2(s)\right) \\
&\geq \sum_{s=1}^{T} q^*(t)\varphi(s)\, p\left(u_2(s)-w_2(s)\right) \\
&\geq q_0 \sum_{s=1}^{T} p\left(u_2(t)-w_2(t)\right)\varphi(t).
\end{aligned} \tag{51}$$

Combining the above two inequalities, we get

$$\begin{aligned}
p\left(\left\|u_2-w_2\right\|\right) &\leq \left(\kappa_1 q_0\right)^{-1}\left(u_1(t)+c_3\right) \\
&\leq \left(\kappa_1 q_0\right)^{-1}\left(\kappa_1^{-1}\mathcal{N}_1+c_3\right).
\end{aligned} \tag{52}$$

Note that triangular inequality $p(\|u_2\|) \le p(\|u_2 - w_2\|) + p(\|w_2\|)$ and from (H1), $\lim_{z\to+\infty} p(z) = +\infty$, and thus there exists $\mathcal{N}_2 > 0$ such that $\|u_2\| \le \mathcal{N}_2$.

Consequently, we obtain that $\|u_1\| \le \kappa_1^{-1}\mathcal{N}_1$ and $\|u_2\| \le \mathcal{N}_2$. As a result, we can choose $R > \max\{\mathcal{M}_3, \kappa_1^{-1}\mathcal{N}_1, \mathcal{N}_2\}$ and thus (39) holds true. Consequently, Lemma 3 indicates that

$$i\left(B, (B_R \times B_R) \cap (P \times P), P \times P\right) = 0. \tag{53}$$

Then we show that

$$(u_1, u_2) \neq \lambda B(u_1, u_2), \quad \forall (u_1, u_2) \in \partial\left(B_{\mathcal{M}_3} \times B_{\mathcal{M}_3}\right) \cap (P \times P), \ \lambda \in [0, 1]. \tag{54}$$

Indeed, if not, there exist $(u_1, u_2) \in \partial(B_{\mathcal{M}_3} \times B_{\mathcal{M}_3}) \cap (P \times P)$ and $\lambda_0 \in [0, 1]$ such that $(u_1, u_2) = \lambda_0 B(u_1, u_2)$. This implies that

$$\begin{aligned} &u_1(t) \le B_1(u_1, u_2)(t), \\ &u_2(t) \le B_2(u_1, u_2)(t), \\ &\qquad \text{for } t \in [1, T]_{\mathbb{Z}}. \end{aligned} \tag{55}$$

Hence, $\|u_1\| \le \|B_1(u_1, u_2)\|$ and $\|u_2\| \le \|B_2(u_1, u_2)\|$. However, from (H2) we have

$$\begin{aligned} B_1(u_1, u_2)(t) &= \sum_{s=1}^{T} G(t, s)\, \tilde{f}\left(s, u_1(s) - w_1(s), u_2(s) - w_2(s)\right) \\ &< \sum_{s=1}^{T} \varphi(s)\, \kappa_2^{-1}\mathcal{M}_3 = \mathcal{M}_3, \end{aligned} \tag{56}$$

for all $t \in [1, T]_{\mathbb{Z}}$. Similarly, $\|B_2(u_1, u_2)\| < \mathcal{M}_3$. Note that

$$\begin{aligned} \|(u_1, u_2)\| &= \max\{\|u_1\|, \|u_2\|\} \\ &\le \max\{B_1(u_1, u_2), B_2(u_1, u_2)\} < \mathcal{M}_3 \\ &= \|(u_1, u_2)\| \end{aligned} \tag{57}$$

with $(u_1, u_2) \in \partial(B_{\mathcal{M}_3} \times B_{\mathcal{M}_3}) \cap (P \times P)$. This is a contradiction. So (54) is true. It follows from Lemma 4 that

$$i\left(B, \left(B_{\mathcal{M}_3} \times B_{\mathcal{M}_3}\right) \cap (P \times P), P \times P\right) = 1. \tag{58}$$

From (53) and (58) we have

$$\begin{aligned} &i\left(B, \left((B_R \times B_R) \setminus \left(\overline{B}_{\mathcal{M}_3} \times \overline{B}_{\mathcal{M}_3}\right)\right) \cap (P \times P), P \times P\right) \\ &\quad = 0 - 1 = -1. \end{aligned} \tag{59}$$

Therefore B has at least one fixed point (u_1, u_2) in $((B_R \times B_R) \setminus (\overline{B}_{\mathcal{M}_3} \times \overline{B}_{\mathcal{M}_3})) \cap (P \times P)$ with $\|u_1\| \ge \mathcal{M}_1, \|u_2\| \ge \mathcal{M}_2$ (note that $\|u_i\| = \mathcal{M}_3 \ge \mathcal{M}_i, i = 1, 2$), and then $(u_1 - w_1, u_2 - w_2)$ is a positive solution for (1). This completes the proof. □

Theorem 11. *Suppose that (H0), (H3), and (H4) hold. Then (1) has at least one positive solution.*

Proof. There exists a sufficiently large $R > \max\{\mathcal{M}_1, \mathcal{M}_2\}$, for which we shall prove that

$$(u_1, u_2) \neq \lambda B(u_1, u_2), \quad \forall (u_1, u_2) \in \partial(B_R \times B_R) \cap (P \times P), \ \lambda \in [0, 1]. \tag{60}$$

Indeed, if not, there exist $(u_1, u_2) \in \partial(B_R \times B_R) \cap (P \times P)$ and $\lambda_0 \in [0, 1]$ such that $(u_1, u_2) = \lambda_0 B(u_1, u_2)$. This indicates that

$$\begin{aligned} &u_1(t) \le B_1(u_1, u_2)(t), \\ &u_2(t) \le B_2(u_1, u_2)(t), \\ &\qquad \text{for } t \in [1, T]_{\mathbb{Z}}. \end{aligned} \tag{61}$$

Note that $(u_1, u_2) \in \partial(B_R \times B_R) \cap (P \times P)$ with $R > \max\{\mathcal{M}_1, \mathcal{M}_2\}$; this implies $u_i(t) \ge w_i(t)$ for $t \in [1, T]_{\mathbb{Z}}, i = 1, 2$. From (H3) we have

$$\begin{aligned} u_1(t) - w_1(t) &\le \sum_{s=1}^{T} G(t, s) \\ &\quad \cdot \left[d_2\left(u_1(s) - w_1(s)\right) + \xi\left(u_2(s) - w_2(s)\right)\right] \\ &\quad - w_1(t) \le \sum_{s=1}^{T} G(t, s) \\ &\quad \cdot \left[d_2\left(u_1(s) - w_1(s)\right) + \xi\left(u_2(s) - w_2(s)\right)\right], \end{aligned} \tag{62}$$

and

$$\begin{aligned} u_2(t) - w_2(t) &\le \sum_{s=1}^{T} G(t, s)\, \eta\left(u_1(s) - w_1(s)\right) - w_2(t) \\ &\le \sum_{s=1}^{T} G(t, s)\, \eta\left(u_1(s) - w_1(s)\right), \end{aligned} \tag{63}$$

for $t \in [1, T]_{\mathbb{Z}}$. Consequently, for all $t \in [1, T]_{\mathbb{Z}}$, we have

$$\begin{aligned} \xi\left(u_2(t) - w_2(t)\right) &\le \xi\left[\sum_{s=1}^{T} G(t, s)\, \eta\left(u_1(s) - w_1(s)\right)\right] \\ &= \xi\left[\sum_{s=1}^{T} G(t, s)\, \kappa_2^{-1}\kappa_2\eta\left(u_1(s) - w_1(s)\right)\right] \\ &\le \sum_{s=1}^{T} G(t, s)\, \kappa_2^{-1}\xi\left(\kappa_2\eta\left(u_1(s) - w_1(s)\right)\right) \\ &\le \sum_{s=1}^{T} G(t, s)\, \kappa_2^{-1}\left[\kappa_2\gamma_2\left(u_1(s) - w_1(s)\right) + c_2\right] \\ &\le \gamma_2 \sum_{s=1}^{T} G(t, s)\left(u_1(s) - w_1(s)\right) + c_8. \end{aligned} \tag{64}$$

This, together with (62), implies that

$$\begin{aligned} u_1(t)-w_1(t) &\le \sum_{s=1}^{T} G(t,s) \\ &\quad \cdot \left[d_2\left(u_1(s)-w_1(s)\right)+\xi\left(u_2(s)-w_2(s)\right)\right] \\ &\le d_2\sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right)+\gamma_2\sum_{s=1}^{T} G(t,s) \\ &\quad \cdot \sum_{\tau=1}^{T} G(s,\tau)\left(u_1(\tau)-w_1(\tau)\right)+c_9. \end{aligned} \tag{65}$$

Multiply both sides of the above inequality by $\varphi(t)$ and sum from 1 to T. Then together with Lemma 1(iii) we obtain

$$\begin{aligned} \sum_{t=1}^{T}\left(u_1(t)-w_1(t)\right)\varphi(t) &\le \sum_{t=1}^{T}\varphi(t) \\ &\quad \cdot \left[d_2\sum_{s=1}^{T} G(t,s)\left(u_1(s)-w_1(s)\right)\right. \\ &\quad \left. +\gamma_2\sum_{s=1}^{T} G(t,s)\sum_{\tau=1}^{T} G(s,\tau)\left(u_1(\tau)-w_1(\tau)\right)+c_9\right] \\ &\le d_2\kappa_2\sum_{t=1}^{T}\left(u_1(t)-w_1(t)\right)\varphi(t)+\gamma_2\kappa_2^2\sum_{t=1}^{T}\left(u_1(t)\right. \\ &\quad \left. -w_1(t)\right)\varphi(t)+c_9\sum_{t=1}^{T}\varphi(t). \end{aligned} \tag{66}$$

Hence, we have

$$\begin{aligned} \sum_{t=1}^{T}\left(u_1(t)-w_1(t)\right)\varphi(t) &\le \frac{\kappa_2 c_9}{1-d_2\kappa_2-\gamma_2\kappa_2^2}, \\ \sum_{t=1}^{T} u_1(t)\varphi(t) &\le \frac{\kappa_2 c_9}{1-d_2\kappa_2-\gamma_2\kappa_2^2}+\sum_{t=1}^{T} w_1(t)\varphi(t) \\ &:= \mathscr{N}_3. \end{aligned} \tag{67}$$

Note that $u_1 \in P_0$ from the fact that $B_1(P\times P)\subset P_0$. This implies

$$\begin{aligned} \left\|u_1\right\|\sum_{t=1}^{T} q^*(t)\varphi(t) &\le \sum_{t=1}^{T} u_1(t)\varphi(t) \le \mathscr{N}_3, \\ \left\|u_1\right\| &\le \kappa_1^{-1}\mathscr{N}_3. \end{aligned} \tag{68}$$

On the other hand, for $t \in [1,T]_{\mathbb{Z}}$, by (64) and Lemma 1(ii) we obtain

$$\begin{aligned} \xi\left(u_2(t)-w_2(t)\right) &\le \gamma_2\sum_{s=1}^{T}\varphi(s)u_1(s)+c_8 \\ &\le \gamma_2\mathscr{N}_3+c_8. \end{aligned} \tag{69}$$

Note that $u_2 \in P_0$ for $B_2(P\times P)\subset P_0$, and from the definition of w_2, we also have $w_2 \in P_0$. This, for $t\in[1,T]_{\mathbb{Z}}$, implies that

$$\xi\left(q_0\left\|u_2-w_2\right\|\right) \le \xi\left(u_2(t)-w_2(t)\right) \le \gamma_2\mathscr{N}_3+c_8. \tag{70}$$

As a result, combining with triangular inequality of norm, there exists $\mathscr{N}_4>0$ such that $\|u_2\|\le\mathscr{N}_4$.

Consequently, we can conclude that $\|u_1\| \le \kappa_1^{-1}\mathscr{N}_3$ and $\|u_2\|\le\mathscr{N}_4$. Therefore, we choose $R>\max\{\mathscr{M}_3,\kappa_1^{-1}\mathscr{N}_3,\mathscr{N}_4\}$ such that (60) holds true. Lemma 4 implies that

$$i\left(\mathrm{B},\left(B_R\times B_R\right)\cap(P\times P),P\times P\right)=1. \tag{71}$$

Note that the definition of $\mathscr{M}_3$. Secondly, we prove that

$$\begin{aligned} &\left(u_1,u_2\right)\neq B\left(u_1,u_2\right)+\lambda\left(x_0,y_0\right), \\ &\quad \forall\left(u_1,u_2\right)\in\partial\left(B_{\mathscr{M}_3}\times B_{\mathscr{M}_3}\right)\cap(P\times P),\ \lambda\ge 0, \end{aligned} \tag{72}$$

where $x_0, y_0 \in P$ are two fixed functions. Indeed, if not, there exist $(u_1,u_2)\in\partial(B_{\mathscr{M}_3}\times B_{\mathscr{M}_3})\cap(P\times P)$ and $\lambda_0\ge 0$ such that $(u_1,u_2)=B(u_1,u_2)+\lambda_0(x_0,y_0)$. This implies that

$$\begin{aligned} &u_1(t)\ge B_1\left(u_1,u_2\right)(t), \\ &u_2(t)\ge B_2\left(u_1,u_2\right)(t), \\ &\qquad \text{for } t\in[1,T]_{\mathbb{Z}}. \end{aligned} \tag{73}$$

Hence, $\|u_1\| \ge \|B_1(u_1,u_2)\|$ and $\|u_2\| \ge \|B_2(u_1,u_2)\|$. However, from (H4) we have

$$\begin{aligned} &B_1\left(u_1,u_2\right)(t) \\ &\quad =\sum_{s=1}^{T} G(t,s)\tilde{f}\left(s,u_1(s)-w_1(s),u_2(s)-w_2(s)\right) \\ &\quad >\sum_{s=1}^{T} q^*(t)\varphi(s)q_0^{-1}\kappa_2^{-1}\mathscr{M}_3\ge\mathscr{M}_3, \end{aligned} \tag{74}$$

for all $t\in[1,T]_{\mathbb{Z}}$. This implies $\|B_1(u_1,u_2)\|>\mathscr{M}_3$. Similarly, $\|B_2(u_1,u_2)\|>\mathscr{M}_3$. This implies

$$\begin{aligned} \left\|\left(u_1,u_2\right)\right\| &= \max\left\{\left\|u_1\right\|,\left\|u_2\right\|\right\} \\ &\ge \max\left\{\left\|B_1\left(u_1,u_2\right)\right\|,\left\|B_2\left(u_1,u_2\right)\right\|\right\}>\mathscr{M}_3 \\ &= \left\|\left(u_1,u_2\right)\right\| \end{aligned} \tag{75}$$

with $(u_1,u_2)\in\partial(B_{\mathscr{M}_3}\times B_{\mathscr{M}_3})\cap(P\times P)$. This is a contradiction. Hence, (72) is true. Lemma 3 yields that

$$i\left(B,\left(B_{\mathscr{M}_3}\times B_{\mathscr{M}_3}\right)\cap(P\times P),P\times P\right)=0. \tag{76}$$

From (71) and (76) we have

$$\begin{aligned} &i(B,\left(\left(B_R\times B_R\right)\setminus\left(\overline{B}_{\mathscr{M}_3}\times\overline{B}_{\mathscr{M}_3}\right)\cap(P\times P),P\times P\right) \\ &\quad =1-0=1. \end{aligned} \tag{77}$$

Hence B has at least one fixed point $((B_R\times B_R)\setminus(\overline{B}_{\mathscr{M}_3}\times\overline{B}_{\mathscr{M}_3})\cap(P\times P)$ with $\|u_1\|\ge\mathscr{M}_1$, $\|u_2\|\ge\mathscr{M}_2$(note that $\|u_i\|=\mathscr{M}_3\ge\mathscr{M}_i, i=1,2$), and thus (u_1-w_1,u_2-w_2) is a positive solution for (1). This completes the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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