Hindawi
Journal of Function Spaces
Volume 2019, Article ID 3486410, 12 pages
https://doi.org/10.1155/2019/3486410

*Research Article*

# Existence of Nontrivial Solutions for Fractional Differential Equations with p-Laplacian

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Received 13 September 2018; Accepted 9 December 2018; Published 1 January 2019

Academic Editor: Xinguang Zhang



Combining the properties of the Green function with some point theorems, we consider the existence of nontrivial solutions for fractional equations with $p$-Laplacian operator $D_{0^+}^{\beta}\phi_p[D_{0^+}^{\alpha}(p(t)u'(t))] + f(t,u(t)) = 0,\ 0 < t < 1$, $au(0) - bp(0)u'(0) = 0$, and $cu(1) + dp(1)u'(1) = 0$, $D_{0^+}^{\alpha}(p(t)u'(t))|_{t=0} = 0$, where $a, b, c, d$ are constants and $p(\cdot) : [0,1] \longrightarrow (0,+\infty)$ is continuous.

## 1. Introduction

Fractional-order models are better than integer order models to describe the real word, which appear frequently in various fields, such as electrical circuits, biology, material, control theory, and physics (see [1–5]). With the rapid development of the theory of fractional differential equations, during the last two decades, the existence of nontrivial solutions of fractional differential equations has been studied by many researchers in nonsingular case as well as singular case. See [6–19]. Usually, the proof is based on either the method of upper and lower solutions, fixed point theorems, alternative principle of Leray-Schauder, topological degree theory, or critical point theory. To our attention, based on a fixed point theorem in cones, K. Lan and W. Lin [20] obtain some new results on existence of multiple positive solutions of systems of nonlinear Caputo fractional differential equations with some of general separated boundary conditions

$$\begin{aligned} -{}^cD^q z_i(t) &= f_i(t, z(t)), \quad t \in (0,1), \\ \alpha z_i(0) - \beta z_i'(0) &= 0, \\ \gamma z_i(1) + \delta z_i'(1) &= 0, \end{aligned} \tag{1}$$

where $z(t) = (z_1(t), \ldots, z_n(t))$, $f_i : [0,1] \times \mathbb{R}_+^n \longrightarrow \mathbb{R}_+$ is continuous on $[0,1] \times \mathbb{R}_+^n$, and ${}^cD^q$ is the Caputo differential operator of order $q \in (1,2)$. $\alpha, \beta, \gamma, \delta$ are positive real numbers. The relations between the linear Caputo fractional differential equations and the corresponding linear Hammerstein integral equations are studied, which show that suitable Lipschitz type conditions are needed when one studies the nonlinear Caputo fractional differential equations.

Recently, fractional differential equations with $p$-Laplacian operator have gained its popularity and importance due to its distinguished applications in numerous diverse fields of science and engineering, such as viscoelasticity mechanics, non-Newtonian mechanics, electrochemistry, fluid mechanics, combustion theory, and material science. There have appeared some results for the existence of solutions or positive solutions of boundary value problems for fractional differential equations with $p$-Laplacian operator; see [15, 21–26] and the references therein. For example, under different conditions K. Hasib, W. Chen and H. Sun [21] apply some classical fixed-point theorems to study the existence of positive solution for a class of singular fractional differential equations with nonlinear $p$-Laplacian operator in Caputo sense

$$\begin{aligned} D^{\beta}\phi_p(D^{\epsilon}u(t)) + \Theta(t)\varphi_1(t, u(t)) &= 0, \\ \phi_p(D^{\epsilon}u(t))^{(i)}\Big|_{t=0} &= 0, \\ i &= 1, \ldots, n-1 \end{aligned}$$

$$u^{(j)}(0) = 0 = u''(1), \quad j = 1, \ldots, n, \tag{2}$$

where $D^{\beta}, D^{\epsilon}$ is Caputo fractional derivative, $n-1 < \beta, \epsilon \leq n, \phi_p(r) = |r|^{p-2}r$ is $p$-Laplacian operator, and $\Theta(\cdot)$ is continuous functions. In addition, Hyers-Ulam stability of the proposed problem is also considered.

Inspired by the references, based on some fixed point theorems in cones, under different combinations of local superlinearity and local sublinearity of the function $f$, we will deal with the existence of nontrivial solutions for a certain $p$-Laplacian fractional differential equation

$$\begin{aligned} &D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\right) + f(t, u(t)) = 0, \\ &\qquad 0 < \alpha, \beta < 1, \\ &au(0) - bp(0)u'(0) = 0, \\ &cu(1) + dp(1)u'(1) = 0, \\ &\left(D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\right)\Big|_{t=0} = 0, \end{aligned} \tag{3}$$

where $a, b, c, d$ are constants with satisfying $0 < ad + bc + ac\int_0^1 (1/p(s))ds < +\infty$, $p(\cdot) : [0,1] \longrightarrow \mathbb{R}_+$ is continuous, and $\phi_p(r) = |r|^{p-2}r$ is $p$-Laplacian operator, where $1/p + 1/q = 1$ and $\phi_q$ denotes inverse of $p$-Laplacian operator. Now we give some notations as follows.

*Definition 1* (see [2]). The fractional integral of order $\alpha > 0$ of a function $u : (0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$I_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}u(s)\,ds, \tag{4}$$

provided that the right-hand side integral is pointwise defined on $(0, +\infty)$, where $\Gamma(\alpha) = \int_0^{+\infty} e^{-s}s^{\alpha-1}ds$.

*Definition 2* (see [2]). The Caputo fractional derivative of order $\alpha > 0$ of a continuous function $u : (0, +\infty) \longrightarrow \mathbb{R}$ is given by

$$D_{0^+}^{\alpha}u(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}u^{(n)}(s)\,ds, \tag{5}$$

for $n = [\alpha] + 1$, where $[\alpha]$ is used for the integer part of $\alpha$, provided that the right-hand side integral is pointwise defined on $(0, +\infty)$.

**Lemma 3** (see [2, Theorem 2.22]). *Let $\alpha \in (n-1, n]$, $u \in AC^{n-1}$. Then*

$$I_{0^+}^{\alpha}D_{0^+}^{\alpha}u(t) = u(t) + c_0 + c_1 t + c_2 t^2 + \cdots + c_{n-1}t^{n-1}, \tag{6}$$

*where $c_i = -u^{(i)}(0)/i!$, $i = 1, \ldots, n-1$.*

The paper is organized as follows. In Section 2, we give some notations and the Green function is examined whether it is increasing or decreasing and positive or negative function. In Section 3, we will give the main results, which are illustrated by some examples.

## 2. Preliminaries

**Lemma 4.** *Let $h(t) \in AC([0,1])$ and $0 < \alpha, \beta < 1$. Then the solution of the fractional differential equation with $p$-Laplacian operator*

$$\begin{aligned} &D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\right) + h(t) = 0, \\ &au(0) - bp(0)u'(0) = 0, \\ &cu(1) + dp(1)u'(1) = 0, \\ &D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\Big|_{t=0} = 0 \end{aligned} \tag{7}$$

*can be expressed by*

$$\begin{aligned} &u(t) \\ &\quad = \int_0^1 G(t,\tau)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^{\tau}(\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau, \end{aligned} \tag{8}$$

*where*

$$\begin{aligned} G(t,\tau) &= \frac{1}{\rho\Gamma(\alpha)}\begin{cases} \left(b + a\int_0^t \frac{1}{p(s)}ds\right)\left(d(1-\tau)^{\alpha-1} + c\int_t^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right) - H(t,\tau), & 0 \leq \tau \leq t \leq 1; \\ \left(b + a\int_0^t \frac{1}{p(s)}ds\right)\left(d(1-\tau)^{\alpha-1} + c\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right), & 0 \leq t \leq \tau \leq 1, \end{cases} \\ \rho &= ad + bc + ac\int_0^1 \frac{1}{p(s)}ds, \\ H(t,\tau) &= a\left(d + c\int_t^1 \frac{1}{p(s)}ds\right)\int_\tau^t \frac{(s-\tau)^{\alpha-1}}{p(s)}ds. \end{aligned} \tag{9}$$

*Proof.* Via some computations, from Lemma 3 it follows that

$$p(t)u'(t)+c_1 = -I_{0^+}^{\alpha}\left(\phi_q\left(c_0+\frac{1}{\Gamma(\beta)}\int_0^t (t-s)^{\beta-1}h(s)\,ds\right)\right). \tag{10}$$

Since $p(t)>0$, we have

$$u'(t) = -\frac{c_1}{p(t)} - \frac{1}{\Gamma(\alpha)p(t)}\int_0^t (t-s)^{\alpha-1} \cdot \phi_q\left(c_0+\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds. \tag{11}$$

Integrating both sides from 0 to $t$, we can obtain

$$u(t) = u(0) - c_1\int_0^t \frac{1}{p(s)}ds - \int_0^t \frac{1}{\Gamma(\alpha)p(s)}\int_0^s (s-\tau)^{\alpha-1} \cdot \phi_q\left(c_0+\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds. \tag{12}$$

Due to $D_{0^+}^{\alpha}(p(t)u'(t))|_{t=0} = 0$, we get $c_0 = 0$ and $p(0)u'(0) = -c_1$. According to the boundary conditions $au(0)-bp(0)u'(0)=0$, we have $au(0)+bc_1=0$ and $u(0) = -(b/a)c_1$. Since

$$cu(1)+dp(1)u'(1)=0,$$

$$p(1)u'(1) = -c_1 - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds,$$

$$u(1) = u(0) - c_1\int_0^1 \frac{1}{p(s)}ds - \int_0^1 \frac{1}{\Gamma(\alpha)p(s)}\int_0^s (s-\tau)^{\alpha-1} \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds, \tag{13}$$

it is clear that

$$0 = c\left\{u(0) - c_1\int_0^1 \frac{1}{p(s)}ds - \int_0^1 \frac{1}{\Gamma(\alpha)p(s)}\int_0^s (s-\tau)^{\alpha-1} \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\right\} + d\left\{-c_1 - \frac{1}{\Gamma(\alpha)}\int_0^1 (1-s)^{\alpha-1}\phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds\right\}. \tag{14}$$

Since $u(0) = -(b/a)c_1$, we have

$$c_1\left(-\frac{bc+ad}{a} - c\int_0^1 \frac{1}{p(s)}ds\right) = \int_0^1 \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1} \cdot h(\tau)\,d\tau\right)ds + \int_0^1\int_0^s \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds, \tag{15}$$

which follows that

$$c_1 = -\frac{a}{\rho}\left\{\int_0^1 \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds + \int_0^1\int_0^s \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\right\}. \tag{16}$$

Then substituting $c_1$ and $u(0)$, we obtain

$$u(t) = \frac{b}{\rho}\left\{\int_0^1 \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds + \int_0^1\int_0^s \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\right\} + \frac{a}{\rho}\int_0^t \frac{1}{p(s)}ds$$

$$
\begin{aligned}
&\cdot\left\{\int_0^1 \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds\right.\\
&\left.+\int_0^1\int_0^s \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\right\}\\
&-\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\\
&=\frac{b+a\int_0^t(1/p(s))\,ds}{\rho}\left\{\int_0^1 \frac{d(1-s)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^s (s-\tau)^{\beta-1}h(\tau)\,d\tau\right)ds\right.\\
&\left.+\int_0^1\int_0^s \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\right\}\\
&-\int_0^t\int_0^s \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\,ds\\
&=\frac{b+a\int_0^t(1/p(s))\,ds}{\rho}\left\{\int_0^1 \frac{d(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\right.\\
&\left.+\int_0^1\int_\tau^1 \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\right\}-\int_0^t\int_\tau^t \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds\\
&\cdot\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\\
&=\frac{b+a\int_0^t(1/p(s))\,ds}{\rho}\left\{\int_0^1\left(\frac{d(1-\tau)^{\alpha-1}}{\Gamma(\alpha)}+\int_\tau^1 \frac{c(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds\right)\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau\right\}\\
&-\int_0^t\int_\tau^t \frac{(s-\tau)^{\alpha-1}}{\Gamma(\alpha)p(s)}ds\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}h(\omega)\,d\omega\right)d\tau,
\end{aligned}
\tag{17}
$$

which implies the expression of the Green function $G(t,s)$. □

**Lemma 5.** *Assume that $a,b,c,d>0$, and $p(t):[0,1]\longrightarrow(0,+\infty)$. The Green function $G(t,\tau)$ has the following properties:*

(i) $G(t,\tau)>0$, *for* $0\le t,\tau\le 1$;

(ii) $G(t,\tau)$ *is an increasing function and* $\max_{t\in[0,1]}G(t,\tau)=G(1,\tau)$;

(iii) *For* $0\le t,\tau\le 1$, *there exists a* $C(t)=(b+a\int_0^t(1/p(s))ds)/(b+a\int_0^1(1/p(s))ds)\in(0,1)$ *such that*

$$G(t,\tau)\ge C(t)\,G(1,\tau). \tag{18}$$

*Proof.* (i) For $0\le t\le\tau\le 1$,

$$
\begin{aligned}
G(t,\tau)=&\frac{1}{\rho\Gamma(\alpha)}\left(b+a\int_0^t\frac{1}{p(s)}ds\right)\\
&\cdot\left(d(1-\tau)^{\alpha-1}+c\int_\tau^1\frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right).
\end{aligned}
\tag{19}
$$

Since $a,b,c,d>0$ and $p(t)>0$, we have $\rho=ad+bc+ac\int_0^1(1/p(s))ds>0$ and $G(t,\tau)>0$.

For $0\le\tau\le t\le 1$,

$$
\begin{aligned}
G(t,\tau)=&\frac{1}{\rho\Gamma(\alpha)}\left\{\left(b+a\int_0^t\frac{1}{p(s)}ds\right)\right.\\
&\cdot\left(d(1-\tau)^{\alpha-1}+\int_t^1\frac{c(s-\tau)^{\alpha-1}}{p(s)}ds\right)\\
&\left.-a\left(d+c\int_t^1\frac{1}{p(s)}ds\right)\int_\tau^t\frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right\}.
\end{aligned}
\tag{20}
$$

Let

$$
\begin{aligned}
h(t)=&\left(b+a\int_0^t\frac{1}{p(s)}ds\right)\\
&\cdot\left(d(1-\tau)^{\alpha-1}+\int_t^1\frac{c(s-\tau)^{\alpha-1}}{p(s)}ds\right)\\
&-a\left(d+c\int_t^1\frac{1}{p(s)}ds\right)\int_\tau^t\frac{(s-\tau)^{\alpha-1}}{p(s)}ds.
\end{aligned}
\tag{21}
$$

Via some computations, we get

$$\begin{aligned} h'(t) &= -\left(b + a\int_0^t \frac{1}{p(s)}ds\right)\frac{c(t-\tau)^{\alpha-1}}{p(t)} \\ &\quad + \frac{a}{p(t)}\left(d(1-\tau)^{\alpha-1} + \int_t^1 \frac{c(s-\tau)^{\alpha-1}}{p(s)}ds\right) \\ &\quad - \frac{a\left(d + c\int_t^1 (1/p(s))\,ds\right)}{p(t)}(t-\tau)^{\alpha-1} \\ &\quad + \frac{ac}{p(t)}\int_\tau^t \frac{(s-\tau)^{\alpha-1}}{p(s)}ds \\ &= -\frac{\rho(t-\tau)^{\alpha-1}}{p(t)} + \frac{ad(1-\tau)^{\alpha-1}}{p(t)} \\ &\quad + \frac{ac}{p(t)}\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds. \end{aligned} \tag{22}$$

Let $F(t) = -\rho(t-\tau)^{\alpha-1} + ad(1-\tau)^{\alpha-1} + ac\int_\tau^1((s-\tau)^{\alpha-1}/p(s))ds$. Since $F'(t) = -(\alpha-1)(t-\tau)^{\alpha-2}\rho > 0$ and $F(\tau) = ad(1-\tau)^{\alpha-1} + ac\int_\tau^1((s-\tau)^{\alpha-1}/p(s))ds > 0$, we obtain $F(t) \geq F(\tau) > 0$, which implies that $h'(t) = F(t)/p(t) > 0$ and $h(t)$ is increasing on $[\tau, t)$. Furthermore, for $0 \leq \tau \leq t \leq 1$, we have

$$\begin{aligned} G(t,\tau) &= \frac{h(t)}{\rho\Gamma(\alpha)} \geq \frac{h(\tau)}{\rho\Gamma(\alpha)} = \left(b + a\int_0^\tau \frac{1}{p(s)}ds\right) \\ &\quad \cdot \left(d(1-\tau)^{\alpha-1} + \int_\tau^1 \frac{c(s-\tau)^{\alpha-1}}{p(s)}ds\right) > 0. \end{aligned} \tag{23}$$

(ii) For $0 \leq t \leq \tau \leq 1$, since

$$\begin{aligned} &\frac{\partial G(t,\tau)}{\partial t} \\ &\quad = \frac{a}{\rho\Gamma(\alpha)}\left(d(1-\tau)^{\alpha-1} + c\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right)\frac{1}{p(t)} \\ &\quad > 0, \end{aligned} \tag{24}$$

we obtain that $G(t,\tau)$ is increasing on $t$ in $(0, 1]$, which implies that $G(t,\tau) \leq G(1,\tau)$.

For $0 \leq \tau \leq t \leq 1$, from the proof of (i) it follows that $G(t,\tau)$ is increasing on $t$ in $(0,1)$ and $G(t,\tau) \leq G(1,\tau)$.

Then from the above discussion, we can obtain the conclusion

$$\max_{t\in[0,1]} G(t,\tau) = G(1,\tau). \tag{25}$$

(iii) For $0 \leq t \leq \tau \leq 1$,

$$\frac{G(t,\tau)}{G(1,\tau)} = \frac{\rho\Gamma(\alpha)\left(b + a\int_0^t (1/p(s))\,ds\right)\left(d(1-\tau)^{\alpha-1} + \int_\tau^1 \left(c(s-\tau)^{\alpha-1}/p(s)\right)ds\right)}{\rho\Gamma(\alpha)\left(b + a\int_0^1 (1/p(s))\,ds\right)\left(d(1-\tau)^{\alpha-1} + \int_\tau^1 \left(c(s-\tau)^{\alpha-1}/p(s)\right)ds\right)} = \frac{b + a\int_0^t (1/p(s))\,ds}{b + a\int_0^1 (1/p(s))\,ds} = C(t). \tag{26}$$

For $0 \leq \tau \leq t \leq 1$,

$$\begin{aligned} \frac{G(t,\tau)}{G(1,\tau)} &= \frac{\rho\Gamma(\alpha)\left\{\left(b + a\int_0^t (1/p(s))\,ds\right)\left(d(1-\tau)^{\alpha-1} + c\int_t^1 \left((s-\tau)^{\alpha-1}/p(s)\right)ds\right)\right.}{\rho\Gamma(\alpha)\left\{d(1-\tau)^{\alpha-1}\left(b + a\int_0^1 (1/p(s))\,ds\right) - ad\int_\tau^1 \left((s-\tau)^{\alpha-1}/p(s)\right)ds\right\}} \\ &\quad - \frac{a\left(d + c\int_t^1 (1/p(s))\,ds\right)\int_\tau^t \left((s-\tau)^{\alpha-1}/p(s)\right)ds\}}{\rho\Gamma(\alpha)\left\{d(1-\tau)^{\alpha-1}\left(b + a\int_0^1 (1/p(s))\,ds\right) - ad\int_\tau^1 \left((s-\tau)^{\alpha-1}/p(s)\right)ds\right\}} \\ &\geq \frac{d(1-\tau)^{\alpha-1}\left(b + a\int_0^t (1/p(s))\,ds\right) + c\left(b + a\int_0^t (1/p(s))\,ds\right)\int_t^1 \left((s-\tau)^{\alpha-1}/p(s)\right)ds}{d(1-\tau)^{\alpha-1}\left(b + a\int_0^1 (1/p(s))\,ds\right)} \\ &\quad - \frac{a\left(d + c\int_t^1 (1/p(s))\,ds\right)\int_\tau^t \left((s-\tau)^{\alpha-1}/p(s)\right)ds}{d(1-\tau)^{\alpha-1}\left(b + a\int_0^1 (1/p(s))\,ds\right)}. \end{aligned} \tag{27}$$

Let

$$\begin{aligned} K(t) &= c\left(b + a\int_0^t \frac{1}{p(s)}ds\right)\int_t^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds \\ &\quad - a\left(d + c\int_t^1 \frac{1}{p(s)}ds\right)\int_\tau^t \frac{(s-\tau)^{\alpha-1}}{p(s)}ds, \\ K_1(t) &= ac\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds - (t-\tau)^{\alpha-1}\rho. \end{aligned} \tag{28}$$

On one hand, since

$$\begin{aligned}K'(t) &= \frac{ac}{p(t)}\int_t^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds - c\left(b + a\int_0^t \frac{1}{p(s)}ds\right)\\ &\cdot \frac{(t-\tau)^{\alpha-1}}{p(t)} + \frac{ac}{p(t)}\int_\tau^t \frac{(s-\tau)^{\alpha-1}}{p(s)}ds - a\left(d\right.\\ &\left.+ c\int_t^1 \frac{1}{p(s)}ds\right)\frac{(t-\tau)^{\alpha-1}}{p(t)}\\ &= \frac{1}{p(t)}\left\{ac\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right.\\ &\left.- (t-\tau)^{\alpha-1}\left(bc + ad + ac\int_0^1 \frac{1}{p(s)}ds\right)\right\}\\ &= \frac{1}{p(t)}\left\{ac\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds - (t-\tau)^{\alpha-1}\rho\right\},\end{aligned} \tag{29}$$

we have $K'(t) = K_1(t)/p(t)$. On the other hand, it is not hard to obtain that $K_1'(t) = -(\alpha-1)(t-\tau)^{\alpha-2}\rho > 0$. Then $K_1(t)$ is increasing on $t$, which implies that

$$K_1(t) \geq K_1(\tau) = ac\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds > 0. \tag{30}$$

Hence, we get $K'(t) > 0$. Furthermore, $K(t)$ is increasing on $t$ and

$$\begin{aligned}K(t) \geq K(\tau) &= c\left(b + a\int_0^\tau \frac{1}{p(s)}ds\right)\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\\ &> 0.\end{aligned} \tag{31}$$

Then, we have

$$\begin{aligned}&\frac{G(t,\tau)}{G(1,\tau)}\\ &\geq \frac{d\left(b + a\int_0^t (1/p(s))\,ds\right)(1-\tau)^{\alpha-1} + K(t)}{d\left(b + a\int_0^1 (1/p(s))\,ds\right)(1-\tau)^{\alpha-1}}\\ &\geq \frac{d\left(b + a\int_0^t (1/p(s))\,ds\right)(1-\tau)^{\alpha-1}}{d\left(b + a\int_0^1 (1/p(s))\,ds\right)(1-\tau)^{\alpha-1}}\\ &= \frac{b + a\int_0^t (1/p(s))\,ds}{b + a\int_0^1 (1/p(s))\,ds} = C(t).\end{aligned} \tag{32}$$

Therefore, from the above discussion, we get the conclusion $G(t,\tau) \geq C(t)G(1,\tau), t \in (0,1)$ and it is clear to see that $0 < C(t) < 1$. □

At the end of this section, we give some notations and crucial lemmas.

Let $E = C[0,1]$ be the Banach space and it endowed with the norm $\|u\| = \max_{t\in[0,1]}\{|u(t)| : u \in E\}$. Define a subcone $K$ as

$$K = \{u \in E : u(t) \geq C(t)\|u\|,\ t \in [0,1]\}. \tag{33}$$

For any given $r > 0$, let

$$\begin{aligned}\Omega(r) &= \{u \in K : \|u\| < r\},\\ \partial\Omega(r) &= \{u \in K : \|u\| = r\}.\end{aligned} \tag{34}$$

**Lemma 6** (see [27]). *Let $E$ be a Banach space, $E_1$ a closed, convex subset of $E$, $\Omega$ an open subset of $E_1$, and $0 \in \Omega$. Suppose that $T : \overline{\Omega} \longrightarrow E_1$ is completely continuous. Then either*

(i) *$T$ has a fixed point in $\overline{\Omega}$, or*

(ii) *there are an $u \in \partial\Omega$ (the boundary of $\Omega$ in $E_1$) and $\lambda \in (0,1)$ with $u = \lambda Tu$.*

**Lemma 7** (see [27]). *Let $E$ be a Banach space and $K \subset E$ be a cone in $E$. Assume $\Omega_1, \Omega_2$ are open subsets of $E$ with $0 \in \Omega_1$, $\overline{\Omega_1} \subset \Omega_2$, and let $T : K \cap (\overline{\Omega_2} \setminus \Omega_1) \longrightarrow K$ be a completely continuous operator such that either*

(i) *$\|Tu\| \leq \|u\|, u \in K\cap\partial\Omega_1$ and $\|Tu\| \geq \|u\|, u \in K\cap\partial\Omega_2$; or*

(ii) *$\|Tu\| \geq \|u\|, u \in \partial\Omega_1$ and $\|Tu\| \leq \|u\|, u \in K \cap \partial\Omega_2$.*

Then $T$ has a fixed point in $K \cap (\overline{\Omega_2} \setminus \Omega_1)$.

**Lemma 8** (see [21]). *Let $\phi_p$ be a $p$–Laplacian operator. Then*

(i) *If $1 < p \leq 2, x_1x_2 > 0$ and $|x_1|, |x_2| \geq \lambda > 0$, then*

$$\left|\phi_p(x_1) - \phi_p(x_2)\right| \leq (p-1)\lambda^{p-2}|x_1 - x_2|. \tag{35}$$

(ii) *If $p > 2$, and $|x_1|, |x_2| \leq \lambda^*$, then*

$$\left|\phi_p(x_1) - \phi_p(x_2)\right| \leq (p-1)\lambda^{*(p-2)}|x_1 - x_2|. \tag{36}$$

## 3. Existence Results

For convenience, the following assumptions hold throughout this paper:

$(A1)$ $f : (0,1)\times(0,+\infty) \longrightarrow \mathbb{R}$ is continuous;

$(A1')$ $f : (0,1)\times(0,+\infty) \longrightarrow [0,+\infty)$ is continuous;

$(A2)$ there exists positive constants $\mu_1, \mu_2$ and $k \in [0,1]$ such that $f$ satisfies

$$f(t,u(t)) \leq \phi_p\left(\mu_1|u(t)|^k + \mu_2\right); \tag{37}$$

$(A3)$ there exists a positive constant $L$ such that for all $u, v \in V$,

$$\left|f(t,u) - f(t,v)\right| \leq L|u(t) - v(t)|. \tag{38}$$

In addition, let

$$\begin{aligned}\varpi_0 &= \int_0^1 \tau^\beta G(1,\tau)\,d\tau,\\ \overline{\varpi} &= \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}\,d\omega\right)d\tau,\\ &\max_{0\leq t\leq 1} f(t,u(t)) = \overline{\mu}.\end{aligned} \tag{39}$$

**Theorem 9.** *Suppose that* $(A1)$, $(A2)$, *and* $(A3)$ *hold and* $p > 2$. *Then problem (3) has a unique solution if*

$$\frac{L(q-1)(\lambda^*)^{q-2}}{\Gamma(\beta+1)}\varpi_0 < 1 \quad \text{and} \quad (2-q)(q-1)^{(q-1)/(2-q)} L^{1/(2-q)}\varpi^{1/(2-q)} + \overline{\mu} \le 0. \tag{40}$$

*Proof.* By Lemma 4, (3) is equivalent to the following integral equation:

$$u(t) = \int_0^1 G(t,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega, u(\omega))\, d\omega\right) d\tau. \tag{41}$$

Define an operator $T : C[0,1] \longrightarrow C[0,1]$ by

$$Tu(t) = \int_0^1 G(t,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega, u(\omega))\, d\omega\right) d\tau. \tag{42}$$

From $(A1)$ and $(A2)$, the operator $T$ is well defined.

Let $B_{r_0} = \{u \in C[0,1] : \|u\| \le r_0\}$ with $r_0 = (q-1)^{(q-1)/(2-q)} L^{(q-1)/(2-q)}\varpi^{1/(2-q)}$. Then we have

$$\begin{aligned}
\|Tu\| &= \left\| \int_0^1 G(t,\tau)\, \phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega, u(\omega))\, d\omega\right) d\tau \right\| \\
&\le \int_0^1 G(1,\tau)\, \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} \cdot \left(f(\omega,u(\omega)) - f(\omega,0) + f(\omega,0)\right) d\omega\right) d\tau \\
&\le \int_0^1 G(1,\tau)\, \phi_q\left(\left(Lr_0+\overline{\mu}\right)\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} d\omega\right) d\tau = \phi_q\left(Lr_0+\overline{\mu}\right) \\
&\quad \cdot \int_0^1 G(1,\tau)\, \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} d\omega\right) d\tau \\
&= \varpi\left(Lr_0+\overline{\mu}\right)^{q-1}.
\end{aligned} \tag{43}$$

Let $g(r) = Lr - \varpi^{-1/(q-1)} r^{1/(q-1)} + \overline{\mu}$, and then we have

$$\begin{aligned}
g(r_0) &= L^{(q-1)^{(q-1)/(2-q)} L^{(q-1)/(2-q)}\varpi^{1/(2-q)}} - \varpi^{-1/(q-1)}\left[(q-1)^{(q-1)/(2-q)} \cdot L^{(q-1)/(2-q)}\varpi^{1/(2-q)}\right]^{1/(q-1)} + \overline{\mu} = (q-1)^{(q-1)/(2-q)} \\
&\quad \cdot L^{1/(2-q)}\varpi^{1/(2-q)} - (q-1)^{1/(2-q)} L^{1/(2-q)}\varpi^{1/(2-q)} + \overline{\mu} = (q-1)^{1/(2-q)} L^{1/(2-q)}\varpi^{1/(2-q)}\left[(q-1)^{-1} - 1\right] \\
&\quad + \overline{\mu} = (2-q)(q-1)^{(q-1)/(2-q)} L^{1/(2-q)}\varpi^{1/(2-q)} + \overline{\mu} \le 0,
\end{aligned} \tag{44}$$

which implies that $\|Tu\| = \varpi(Lr_0+\overline{\mu})^{q-1} \le r_0$. Therefore, we proved that $T : B_{r_0} \longrightarrow B_{r_0}$.

For any $u, v \in E$, by Lemma 8, we have

$$\begin{aligned}
\|Tu - Tv\| &= \left\| \int_0^1 G(t,\tau) \cdot \left\{\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\, d\omega\right) - \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,v(\omega))\, d\omega\right)\right\} d\tau \right\| \\
&\le \int_0^1 G(1,\tau)\left\{(q-1)(\lambda^*)^{q-2}\left|\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\, d\omega - \frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,v(\omega))\, d\omega\right|\right\} d\tau \\
&\le \frac{(q-1)(\lambda^*)^{q-2}}{\Gamma(\beta)}\int_0^1 G(1,\tau)\int_0^\tau (\tau-\omega)^{\beta-1} \cdot \left|f(\omega,u(\omega)) - f(\omega,v(\omega))\right| d\omega\, d\tau \\
&\le \frac{L(q-1)(\lambda^*)^{q-2}}{\Gamma(\beta+1)}\int_0^1 \tau^\beta G(1,\tau)\, d\tau\, \|u-v\| \\
&\le \frac{L(q-1)(\lambda^*)^{q-2}}{\Gamma(\beta+1)}\varpi_0 \|u-v\|.
\end{aligned} \tag{45}$$

Since $(L(q-1)(\lambda^*)^{q-2}/\Gamma(\beta+1))\varpi_0 < 1$, from Banach's contraction mapping principle it follows that there exists a unique fixed point for the operator $T$, which corresponds to the unique solution for problem (3). □

**Lemma 10.** *Assume that* $(A1')$ *and* $(A2)$ *hold. Then the operator* $T : K \longrightarrow K$ *is completely continuous.*

*Proof.* For any $u \in K$, according to Lemma 5, we can get

$$Tu(t) = \int_0^1 G(t,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\, d\omega\right) d\tau$$

$$\leq \int_0^1 G(1,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau, \tag{46}$$

and

$$\begin{aligned} Tu(t) &= \int_0^1 G(t,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau \\ &\geq C(t)\int_0^1 G(1,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau. \end{aligned} \tag{47}$$

So we have

$$Tu(t) \geq C(t)\,\|Tu\|, \quad t\in[0,1]. \tag{48}$$

This implies $T: K \longrightarrow K$.

Given $R > r > 0$, now we show that $T$ is completely continuous on $\overline{\Omega(R)} \setminus \Omega(r)$.

Firstly, we will show that $T$ is continuous, and we only need to prove that $\|T(u_n) - T(u)\| \longrightarrow 0$ for any $u_n \longrightarrow u$ as $n \longrightarrow \infty$. It is clear that

$$\begin{aligned} \|T(u_n) - T(u)\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u_n(\omega))\,d\omega\right)d\tau \right. \\ &\quad \left. - \int_0^1 G(t,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\right| \\ &\leq \int_0^1 G(1,\tau)\left|\phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u_n(\omega))\,d\omega\right) - \phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)\right|d\tau. \end{aligned} \tag{49}$$

From the continuity of $f$, $\phi_q(\cdot)$, we have

$$\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u_n(\omega))\,d\omega\right) \longrightarrow \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right) \tag{50}$$

as $n \longrightarrow \infty$. Therefore, $\|T(u_n) - T(u)\| \longrightarrow 0$, as $n \longrightarrow \infty$, and $T$ is continuous.

Next, we show that the operator $T$ is uniformly bounded. By $(A2)$, we get

$$\begin{aligned} \|T(u)\| &= \max_{0\leq t\leq 1}\left|\int_0^1 G(t,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\right| \\ &\leq \int_0^1 G(1,\tau)\left|\phi_q\left(\frac{1}{\Gamma(\beta)} \cdot \int_0^\tau (\tau-\omega)^{\beta-1}\phi_p\left(\mu_1|u(t)|^k + \mu_2\right)d\omega\right)\right|d\tau \\ &= \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}\,d\omega\right)d\tau \cdot \left(\mu_1|u(t)|^k + \mu_2\right) \leq \overline{\omega}\left(\mu_1\|u(t)\|^k + \mu_2\right), \end{aligned} \tag{51}$$

which implies that the operator $T$ is uniformly bounded on $\overline{\Omega(R)} \setminus \Omega(r)$.

Finally, we show that the operator $T$ is equicontinuous. Before that, we will proceed with $\partial G(t,\tau)/\partial t$ being bounded.

For $0 \leq t \leq \tau \leq 1$,

$$\frac{\partial G(t,\tau)}{\partial t} = \frac{a}{\rho\Gamma(\alpha)\,p(t)}\left(d(1-\tau)^{\alpha-1} + c\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right). \tag{52}$$

Since the function $p(\cdot)$, $d(1-\tau)^{\alpha-1}$ and $\int_\tau^1((s-\tau)^{\alpha-1}/p(s))ds$ are bounded, there exists a positive constant $M_1$ such that $|\partial G(t,\tau)/\partial t| \leq M_1$. In a similar way, for $0 \leq \tau \leq t \leq 1$, there exists a positive constant $M_2$ such that

$$\left|\frac{\partial G(t,\tau)}{\partial t}\right| = \left|\frac{1}{\rho\Gamma(\alpha)\,p(t)}\left(-\rho(t-\tau)^{\alpha-1} + ad(1-\tau)^{\alpha-1} + ac\int_\tau^1 \frac{(s-\tau)^{\alpha-1}}{p(s)}ds\right)\right| \leq M_2. \tag{53}$$

Choosing $\overline{M} = \max\{M_1, M_2\}$, we can obtain $|\partial G(t,\tau)/\partial t| \leq \overline{M}$.

Finally, for any $t_1, t_2 \in [0,1]$ with $t_1 < t_2$, there exists a $\xi \in (t_1, t_2)$ such that

$$|Tu(t_2) - Tu(t_1)| = \left|\int_0^1 G(t_2,\tau) \cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\right.$$

$$
\begin{aligned}
&- \int_0^1 G(t_1,\tau) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\Bigg| \\
&= \Bigg|\int_0^1 (G(t_2,\tau)-G(t_1,\tau)) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\Bigg| \\
&= \Bigg|\int_0^1 \left.\frac{\partial G(t,\tau)}{\partial t}\right|_{t=\xi}(t_2-t_1) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\Bigg| \\
&\le \Bigg|\int_0^1 \overline{M}(t_2-t_1)\left(\mu_1|u(t)|^k+\mu_2\right) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}\,d\omega\right)d\tau\Bigg| \\
&\le \overline{M}\left(\mu_1\|u(t)\|^k+\mu_2\right) \\
&\cdot \int_0^1 \left|\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}\,d\omega\right)\right| d\tau\,|t_2-t_1|.
\end{aligned}
\tag{54}
$$

Thus, the operator $T$ is equicontinuous. According to Arzela-Ascoli theorem, $T:\overline{\Omega(R)}\setminus\Omega(r)\longrightarrow E$ is compact. □

**Theorem 11.** *Suppose that* $(A1)$ *and* $(A2)$ *hold. In addition,*

(C1) *there exists a continuous function* $g:\mathbb{R}_+\longrightarrow\mathbb{R}_+$ *such that*

$$
f(t,u(t))\le\phi_p(g(\|u\|)),\quad \text{for any } t\in[0,1],\ u\in\mathbb{R};
\tag{55}
$$

(C2) *there exists a constant* $R$ *such that* $R/\varpi g(R)>1$.

*Then problem (3) has at least one solution.*

*Proof.* Now we show the (ii) of Lemma 6 does not hold. If $u$ is a solution of (3), then, for $\lambda\in(0,1)$, we have

$$
\begin{aligned}
\|u\| &= \lambda\|Tu\| = \lambda\Bigg\|\int_0^1 G(t,\tau) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f(\omega,u(\omega))\,d\omega\right)d\tau\Bigg\| \\
&\le \lambda\int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\right. \\
&\left.\cdot \int_0^\tau (\tau-\omega)^{\beta-1}\phi_p(g(\|u\|))\,d\omega\right)d\tau \\
&\le \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1}\,d\omega\right) \\
&\cdot \phi_p\left(\phi_q(g(\|u\|))\right)d\tau \le g(\|u\|)\,\varpi.
\end{aligned}
\tag{56}
$$

Let $B_R=\{u\in E:\|u\|<R\}$. From the above inequality and (C2), it yields a contradiction. Therefore, the operator $T$ has a fixed point in $B_R$. □

For $r>0$, define the following functions:

$$
\begin{aligned}
f_M(t,r) &= \max\{f(t,u(t))\mid C(t)r\le u\le r\}, \\
f_m(t,r) &= \min\{f(t,u(t))\mid C(t)r\le u\le r\}.
\end{aligned}
\tag{57}
$$

**Theorem 12.** *Suppose that* $(A1')$ *and* $(A2)$ *hold. In addition, there exist* $r_0,R_0\in\mathbb{R}^+$ *such that one of the following conditions satisfied:*

$(B1)$

$$
\begin{aligned}
r_0 &\le \int_0^1 G(1,\tau) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f_m(\omega,r_0)\,d\omega\right)d\tau \\
&< +\infty
\end{aligned}
\tag{58}
$$

and

$$
\begin{aligned}
&\int_0^1 G(1,\tau) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f_M(\omega,r_0)\,d\omega\right)d\tau \\
&\le R_0;
\end{aligned}
\tag{59}
$$

or

$(B2)$

$$
\begin{aligned}
&\int_0^1 G(1,\tau) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f_M(\omega,x_0)\,d\omega\right)d\tau \\
&< r_0
\end{aligned}
\tag{60}
$$

and

$$
\begin{aligned}
R_0 &\le \int_0^1 G(1,\tau) \\
&\cdot \phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau (\tau-\omega)^{\beta-1} f_M(\omega,r_0)\,d\omega\right)d\tau \\
&< +\infty.
\end{aligned}
\tag{61}
$$

Then problem (3) has a positive solution $u_0\in K$ such that $r_0\le\|u_0\|\le R_0$.

*Proof.* We only verify the case $(B1)$. On one hand, for any $u \in \partial\Omega(r_0)$, we have $C(t)r_0 \le u \le r_0$ and

$$\begin{aligned}
\|T(u)\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\right|\\
&\ge \max_{0\le t\le 1}\left|\int_0^1 C(t)\,G(1,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\right|\\
&= \max_{0\le t\le 1}C(t)\left|\int_0^1 G(1,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\right|\\
&\ge \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\right.\\
&\quad\left.\cdot\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\\
&\ge \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\right.\\
&\quad\left.\cdot\int_0^\tau(\tau-\omega)^{\beta-1}f_m(\omega,r_0)\,d\omega\right)d\tau \ge r_0 = \|u\|.
\end{aligned}\tag{62}$$

Thus, $\|T(u)\| \ge \|u\|$, for any $u \in \partial\Omega(r_0)$.

On the other hand, for any $u \in \partial\Omega(R_0)$, we have $C(t)R_0 \le u \le R_0$, $t \in [0,1]$ and

$$\begin{aligned}
\|T(u)\| &= \max_{0\le t\le 1}\left|\int_0^1 G(t,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\right|\\
&\le \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\right.\\
&\quad\left.\cdot\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\\
&\le \int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\right.\\
&\quad\left.\cdot\int_0^\tau(\tau-\omega)^{\beta-1}f_M(\omega,R_0)\,d\omega\right)d\tau \le R_0\\
&= \|u\|.
\end{aligned}\tag{63}$$

Then, $\|T(u)\| \ge \|u\|$, for any $u \in \partial\Omega(R_0)$.

Therefore, By Lemma 7, the operator $T$ has a fixed point $u_0 \in \overline{\Omega(R_0)} \setminus \Omega(r_0)$ with $r_0 \le \|u_0\| \le R_0$. □

**Theorem 13.** *Suppose that $(A1')$ holds. In addition*

$(D1)$ $\lim_{u\longrightarrow 0^+}(f(t,u)/u^{1/(q-1)}) = 0$;

$(D2)$ *there exists a constant $K > 0$ such that $f(t,u) \le K$;*

$(D3)$ *there exists $\overline{R} > 0$ and $\theta \in (0,1/2)$ such that*

$$\min_{\vartheta\overline{R}\le u\le\overline{R}} f(t,u) > \left(\sigma\overline{R}\right)^{1/(q-1)}, \tag{64}$$

*where*

$$\begin{aligned}
&0 < \vartheta = \min_{\theta\le t\le 1-\theta} C(t) < 1,\\
&\sigma = \left(\min_{\theta\le t\le 1-\theta} C(t)\int_\theta^{1-\theta} G(1,\tau)\right.\\
&\qquad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}\,d\omega\right)d\tau\right)^{-1}.
\end{aligned}\tag{65}$$

*Then problem (3) has at least two solutions.*

*Proof.* Since $\lim_{u\longrightarrow 0^+}(f(t,u)/u^{1/(q-1)}) = 0$, there exist $\varepsilon > 0$ and $r > 0$ such that $f(t,u) < \varepsilon u^{1/(q-1)}$, for $0 \le u \le r$, $t \in [0,1]$, where $\varepsilon$ satisfies $\varepsilon^{q-1}\varpi < 1$. For $u \in \partial\Omega(r) = \{u \in E : \|u\| < r\}$, we have

$$\begin{aligned}
\|Tu\| &= \left\|\int_0^1 G(t,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\right\|\\
&\le \left\|\int_0^1 G(1,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}\left(\varepsilon u^{1/(q-1)}\right)d\omega\right)d\tau\right\|\\
&\le \varepsilon^{q-1}\|u\|\int_0^1 G(1,\tau)\,\phi_q\left(\frac{1}{\Gamma(\beta)}\right.\\
&\quad\left.\cdot\int_0^\tau(\tau-\omega)^{\beta-1}\,d\omega\right)d\tau \le \varepsilon^{q-1}\varpi\|u\| < \|u\|.
\end{aligned}\tag{66}$$

Choosing $R > K^{q-1}\varpi$. For $u \in \partial\Omega(R)$, we have

$$\begin{aligned}
\|Tu\| &= \left\|\int_0^1 G(t,\tau)\right.\\
&\quad\left.\cdot\,\phi_q\left(\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}f(\omega,u(\omega))\,d\omega\right)d\tau\right\|\\
&\le \int_0^1 G(1,\tau)\,\phi_q\left(K\frac{1}{\Gamma(\beta)}\int_0^\tau(\tau-\omega)^{\beta-1}\,d\omega\right)d\tau\\
&= K^{q-1}\varpi < R = \|u\|.
\end{aligned}\tag{67}$$

For any $u \in \partial\overline{\Omega}(\overline{R})$, choosing $t_0 \in (\theta, 1-\theta)$, we have $u(t_0) \in [\vartheta\overline{R}, \overline{R}]$. Furthermore, we have

$$\begin{aligned}
\|Tu(t_0)\| &= \left\| \int_0^1 G(t_0,\tau) \right. \\
&\quad \left. \cdot \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega, u(\omega))\, d\omega\right) d\tau \right\| \\
&\geq \left\| \int_0^1 C(t_0) G(1,\tau) \right. \\
&\quad \left. \cdot \phi_q\left(\frac{1}{\Gamma(\beta)} \int_0^\tau (\tau-\omega)^{\beta-1} f(\omega, u(\omega))\, d\omega\right) d\tau \right\| \\
&\geq \int_\theta^{1-\theta} C(t_0) G(1,\tau) \phi_q\left(\frac{1}{\Gamma(\beta)}\right. \\
&\quad \left. \cdot \int_0^\tau (\tau-\omega)^{\beta-1} \min_{\vartheta\overline{R}\leq u\leq \overline{R}} f(\omega, u(\omega))\, d\omega\right) d\tau \\
&\geq \left[\min_{\theta\leq t\leq 1-\theta} C(t)\right] \int_\theta^{1-\theta} G(1,\tau) \phi_q\left(\frac{1}{\Gamma(\beta)}\right. \\
&\quad \left. \cdot \int_0^\tau (\tau-\omega)^{\beta-1} d\omega\right) \phi_q\left(\left(\sigma\overline{R}\right)^{1/(q-1)}\right) d\tau \\
&= \delta^{-1}\delta\overline{R} = \overline{R} = \|u\|.
\end{aligned} \tag{68}$$

By Lemma 7, problem (3) has at least two positive solution $r \leq \|u_1(t)\| \leq \overline{R}$ and $\overline{R} \leq \|u_2(t)\| \leq R$. □

At the end of this section, we give some examples to illustrate our main results.

*Example 1.* Let us consider the problem

$$\begin{aligned}
&D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\right) \\
&\quad + \phi_p\left(\sin t\left(\arctan u^{1/3} + \cos u^{1/2} + 3\right)\right) = 0, \\
&au(0) - bp(0)u'(0) = 0, \\
&cu(1) + dp(1)u'(1) = 0, \\
&D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\Big|_{t=0} = 0.
\end{aligned} \tag{69}$$

It is clear to see that $|f(t,u)| = |\phi_p(\sin t(\arctan u^{1/3} + \cos u^{1/2} + 3))| \leq \phi_p(\|u\|^{1/3} + \|u\|^{1/2} + 3) = \phi_p(g(\|u\|))$, for $t \in [0,1]$, $u \in \mathbb{R}$. In addition, there exists a sufficiently large $R > 0$, we have $R/\varpi g(R) > 1$. Therefore, problem (69) has at least one solution by Theorem 11.

*Example 2.* Let us consider the problem

$$\begin{aligned}
&D_{0^+}^{\beta}\phi_p\left(D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\right) \\
&\quad + \frac{(\sigma p)^{p-1}+1}{(p\vartheta)^p e^{-p}\vartheta}(\sin t + 2)e^{-u}u^p = 0, \\
&au(0) - bp(0)u'(0) = 0, \\
&cu(1) + dp(1)u'(1) = 0, \\
&D_{0^+}^{\alpha}\left(p(t)u'(t)\right)\Big|_{t=0} = 0.
\end{aligned} \tag{70}$$

Since $f(t,u) = (((\sigma p)^{p-1}+1)/(p\vartheta)^p e^{-p}\vartheta)(\sin t + 2)e^{-u}u^p$, we have $f(t,u)/u^{p-1} = (((\sigma p)^{p-1}+1)/(p\vartheta)^p e^{-p}\vartheta)(\sin t + 2)ue^{-u} \longrightarrow 0$ as $u \longrightarrow 0^+$, and $(D2)$ hold. By some calculations, we get $f'(t,u) = (((\sigma p)^{p-1}+1)/(p\vartheta)^p e^{-(p-1)}\vartheta)(\sin t + 2)e^{-u}u^{p-1}(1/p - u)$, and it is clear to see that $f'(t,u) > 0$, for $u \in (0, 1/p)$, and $f'(t,u) < 0$, for $u \in (1/p, +\infty)$. Let $\overline{R} = 1/p$, and then for any $u \in (\vartheta(1/p), 1/p)$, we have $\min_{u\in[\vartheta(1/p),1/p]} f(t,u) = (((\sigma p)^{p-1}+1)/(p\vartheta)^p e^{-p}\vartheta)(\sin t + 2)(p\vartheta)^p e^{-(1/p)\vartheta} \geq (p\sigma)^{p-1} + 1 \geq (\overline{R}\vartheta)^{p-1}$, and $(D3)$ holds. Therefore, problem (70) has at least two positive solutions $u(t)$ by Theorem 13.

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

## Acknowledgments

Fanglei WANG was supported by NNSF of China (no. 11501165) and the Fundamental Research Funds for the Central Universities (2018B58614).

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