Hindawi
Journal of Function Spaces
Volume 2019, Article ID 3695869, 4 pages
https://doi.org/10.1155/2019/3695869

Research Article

The Quasi-Sure Limit of Convex Combinations of Nonnegative Measurable Functions

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Received 24 September 2018; Revised 14 November 2018; Accepted 21 November 2018; Published 1 January 2019

Academic Editor: Pasquale Vetro

This study shows that, for a sequence of nonnegative valued measurable functions, a sequence of convex combinations converges to a nonnegative function in the quasi-sure sense. This can be used to prove some existence results in multiprobabilities models, and an example application in finance is discussed herein.

1. Introduction

Stochastic models are widely used in practical applications and most are built on a probability space. However, in practice, situations with many possible probabilities occur. A few examples include the uncertain drift or uncertain volatility cases in economic models. These multiprobabilities are called ambiguity, model uncertainty, or Knight uncertainty. Reference [1] emphasized the significant distinction between risk and ambiguity and [2] showed that the distinction between risk and ambiguity is behaviorally significant. To study economic problems while considering ambiguity, mathematical models must be built on a multiprobability space.

Reference [3] first developed the theory of nonlinear g-expectation which nontrivially generalizes the classical linear expectation from a probability space to a space with a set of uniformly absolute continuous probabilities. Considering this theory, [4] studied the stochastic differential recursive utility with drift ambiguity. However, many economic and financial problems involve significant volatility uncertainty, which is characterized by a family of nondominated probability measures. Motivated by volatility uncertainty in statistics, risk measures, and super-hedging in finance, [5] introduced a nonlinear expectation, called the G-expectation, which can be regarded as the upper expectation of a specific family of nondominated probability measures. Subsequently, [6, 7] introduced a new type of "G-Brownian motion" and presented the related calculus of Itô's type. Reference [8] developed a representation of the G-expectation and G-Brownian motion. Reference [9] studied the martingale representation theorem for the G-expectation. Reference [10] determined the properties of hitting times for the G-martingales. References [11, 12] studied the G-BSDEs in G-expectation space.

This paper provides a convergence result in the multiprobability space. In such spaces, under the upper expectation of $\mathbb{E}[\cdot]$ defined in (4), the corresponding Fatou's lemma and dominated convergence theorem no longer hold. Therefore, the convergence result reported in this paper will be useful for future studies.

This paper is organized as follows. In Section 2, we prove that, for a sequence of nonnegative measurable functions, there is a sequence of convex combinations which converges to a nonnegative function in the quasi-sure sense. In Section 3, we use the results of Section 2 to prove an existence result in a multiprobabilities model.

2. Main Results

Let Ω be a complete separable metric space, $\mathscr{B}(\Omega)$ the Borel σ-algebra of Ω, and $\mathscr{M}$ the collection of all probability measures on $(\Omega, \mathscr{B}(\Omega))$. Let $L^0(\Omega)$ be the space of all $\mathscr{B}(\Omega)$-measurable real functions.

Consider a given subset $\mathscr{P} \subseteq \mathscr{M}$.

Denote

$$c(A) := \sup_{P \in \mathscr{P}} P(A), \quad A \in \mathscr{B}(\Omega). \tag{1}$$

Then, $c(\cdot)$ is a Choquet capacity; see [13–15]. A set A is called polar if $c(A)=0$, and we say a property holds "quasi-surely"(q.s.) if it holds outside a polar set. Let $X_n, X \in L^0(\Omega)$, $n\in\mathbb{N}$. The sequence X_n is said to converge in capacity c to X, denoted by $X_n \xrightarrow{c} X$, $n\longrightarrow\infty$, if

$$\lim_{n\longrightarrow\infty} c\left(\{|X_n - X| \geq \epsilon\}\right) = 0, \quad \forall\epsilon > 0. \tag{2}$$

The sequence X_n is said to mutually converge in capacity c, if

$$\lim_{m,n\longrightarrow\infty} c\left(\{|X_m - X_n| \geq \epsilon\}\right) = 0, \quad \forall\epsilon > 0. \tag{3}$$

The upper expectation $\mathbb{E}[\cdot]$ of $\mathscr{P}$ is defined as follows (see [16]): for each $X\in L^0(\Omega)$ such that $E^P[X]$ exists for each $P\in\mathscr{P}$,

$$\mathbb{E}[X] \coloneqq \sup_{P\in\mathscr{P}} E^P[X]. \tag{4}$$

Lemma 1. *Let $X_n, X\in L^0(\Omega)$, $n\in\mathbb{N}$. Then,*

(i) $X_n\longrightarrow X, n\longrightarrow\infty$, q.s., if

$$c\left(\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X| \geq \epsilon\}\right) = 0, \quad \forall\epsilon > 0. \tag{5}$$

(ii) X_n is Cauchy q.s., if

$$c\left(\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X_n| \geq \epsilon\}\right) = 0, \quad \forall\epsilon > 0. \tag{6}$$

Proof. (i) We choose $\{\epsilon_k\}$, such that $\epsilon_k > 0$ and $\epsilon_k\longrightarrow 0$, $k\longrightarrow\infty$.

$$\begin{aligned} c\left(\{X_n\longrightarrow X\}^c\right) &= c\left(\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X| \geq \epsilon_k\}\right) \\ &\leq \sum_{k=1}^{\infty} c\left(\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X| \geq \epsilon_k\}\right) \\ &= 0. \end{aligned} \tag{7}$$

Thus, we obtain $X_n\longrightarrow X$ q.s.

(ii) Similar to proof (i), we choose $\{\epsilon_k\}$, such that $\epsilon_k > 0$ and $\epsilon_k\longrightarrow 0$, $k\longrightarrow\infty$.

$$\begin{aligned} &c\left(\{|X_n - X_m|\longrightarrow 0\}^c\right) \\ &\quad= c\left(\bigcup_{k=1}^{\infty}\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X_n| \geq \epsilon_k\}\right) \\ &\quad\leq \sum_{k=1}^{\infty} c\left(\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X_n| \geq \epsilon_k\}\right) = 0. \end{aligned} \tag{8}$$

Therefore, X_n is Cauchy q.s. □

Lemma 2. *Let $X_n, X\in L^0(\Omega)$, $n\in\mathbb{N}$, $\{\epsilon_n\}$ be a positive number sequence, and $\epsilon_n\longrightarrow 0$.*

(i) If

$$\sum_{n=1}^{\infty} c\left(\{|X_n - X| \geq \epsilon_n\}\right) < \infty, \tag{9}$$

then $X_n\longrightarrow X$ q.s.

(ii) If

$$\sum_{n=1}^{\infty} c\left(\{|X_{n+\nu} - X_n| \geq \epsilon_n\}\right) < \infty, \tag{10}$$

then X_n is Cauchy q.s.

Proof. (i) For any $\epsilon > 0$, there exists n_0 such that, for all $n\geq n_0$, $\epsilon_n < \epsilon$. Then, we have

$$\begin{aligned} c\left(\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X| \geq \epsilon\}\right) &\leq \sum_{n=k}^{\infty} c\left(\{|X_n - X| \geq \epsilon\}\right) \\ &\leq \sum_{n=k}^{\infty} c\left(\{|X_n - X| \geq \epsilon_n\}\right). \end{aligned} \tag{11}$$

Letting $k\longrightarrow\infty$ on the right side of the above inequality yields

$$c\left(\bigcap_{n=1}^{\infty}\bigcup_{\nu=1}^{\infty}\{|X_{n+\nu} - X| \geq \epsilon\}\right) = 0, \tag{12}$$

and by Lemma 1, we have $X_n\longrightarrow X$ q.s.

(ii) Similar to proof (i). □

Lemma 3. *Let $X_n\in L^0(\Omega)$, $n\in\mathbb{N}$.*

(i) If $X_n \xrightarrow{c} X$, then there exists a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ such that $X_{n_k}\longrightarrow X$ q.s.

(ii) If $X_n, n\in\mathbb{N}$, mutually converges in capacity c, then there exist a subsequence $\{X_{n_k}\}$ of $\{X_n\}$ and $X\in L^0(\Omega)$ such that $X_{n_k}\longrightarrow X$ q.s.

Proof. (i) Since $X_n\xrightarrow{c} X$, for each $\epsilon_k = 1/2^k$, $k\in\mathbb{N}$, there exists $n_k\in\mathbb{N}$, such that

$$c\left(\left\{|X_n - X| \geq \frac{1}{2^k}\right\}\right) < \frac{1}{2^k}, \quad n\geq n_k. \tag{13}$$

We can choose $n_k\uparrow\infty, k\longrightarrow\infty$. Then, we have

$$\sum_{k=1}^{\infty} c\left(\left\{|X_{n_k} - X| \geq \frac{1}{2^k}\right\}\right) < \infty. \tag{14}$$

By Lemma 2, $X_{n_k}\longrightarrow X, k\longrightarrow\infty$, q.s.

(ii) Similar to proof (i), there exists a subsequence $\{X_{n_k}\}$, which is Cauchy q.s. Then, there exists $X\in L^0(\Omega)$ such that $X_{n_k}\longrightarrow X$ q.s.

If $L^0(\Omega)$ is the space of all $\mathscr{B}(\Omega)$-measurable functions in $R\cup\{+\infty\}$, the sequence X_n mutually converges in capacity c, if

$$\begin{aligned} &\lim_{m,n\longrightarrow\infty} c\left(\left\{|X_m - X_n| \geq \epsilon \text{ and } \min(X_n, X_m) \leq \epsilon^{-1}\right\}\right) \\ &\quad= 0, \quad \forall\epsilon > 0. \end{aligned} \tag{15}$$

The results of the above lemmas still hold. □

Let $conv(X_n, X_{n+1}, \ldots)$ denote the convex combination of $X_n, X_{n+1}, \ldots$. Using a similar argument as in Lemma A1.1 of [17], we obtain the following.

Theorem 4. *Let $(X_n)_{n\geq 1}$ be a sequence of $[0,\infty[$ valued measurable functions. There exists a sequence $g_n \in conv(X_n, X_{n+1}, \ldots)$ such that $(g_n)_{n\geq 1}$ converges to a $[0,\infty]$ valued function g q.s.*

Proof. Let $u : R_+ \cup \{0\} \cup \{+\infty\} \longrightarrow [0,1]$ be defined as $u(x) = e^{-x}$. Define s_n as

$$s_n = \inf \{u(g) \mid g \in conv(X_n, X_{n+1}, \ldots)\} \quad (16)$$

and choose $g_n \in conv(X_n, X_{n+1}, \ldots)$ so that

$$u(g_n) \leq s_n + \epsilon_n, \quad (17)$$

where $0 \leq \epsilon_n \longrightarrow 0$.

It is clear that s_n is a bounded increasing sequence, so there exists s_0 such that $\lim_{n\longrightarrow\infty} u(g_n) = s_0$.

On the compact (metrisable) space $[0,\infty]$, $(x_n)_{n\geq 1}$ is Cauchy if and only if for each $\alpha > 0$ there is n_0 so that for all $n, m \geq n_0$ we have $|x_n - x_m| \leq \alpha$ or $\min(x_n, x_m) \geq \alpha^{-1}$. From the properties of $u(x)$, we have that for $\alpha > 0$ there is $\beta > 0$ so that $|x - y| > \alpha$ and $\min(x, y) \leq \alpha^{-1}$ imply $(1/2)u(x) + (1/2)u(y) > u((x + y)/2) + \beta$.

For a given $\alpha > 0$, we can take β as above and, with the convexity of u, we obtain

$$\begin{aligned} &\beta 1_{\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\}} + u\left(\frac{g_n + g_m}{2}\right) \\ &\quad \leq \frac{1}{2} u(g_n) + \frac{1}{2} u(g_m). \end{aligned} \quad (18)$$

Then,

$$\begin{aligned} &-\beta 1_{\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\}} \\ &\quad \geq u\left(\frac{g_n + g_m}{2}\right) - \frac{1}{2} u(g_n) - \frac{1}{2} u(g_m). \end{aligned} \quad (19)$$

Without loss of generality, we can set $m \geq n$. By (16), $u((g_n + g_m)/2) \geq s_n$. Taking the expectation about each $P \in \mathscr{P}$ and letting $n \longrightarrow \infty$, we obtain

$$\begin{aligned} &-\beta \lim_{n\longrightarrow\infty} P\left(\left\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \right.\right. \\ &\quad \left.\left. \leq \alpha^{-1}\right\}\right) \geq \lim_{n\longrightarrow\infty} E^P\left[u\left(\frac{g_n + g_m}{2}\right) - \frac{1}{2} u(g_n) - \frac{1}{2} \right. \\ &\quad \left. \cdot u(g_m)\right] = E^P\left[\lim_{n\longrightarrow\infty}\left(u\left(\frac{g_n + g_m}{2}\right) - \frac{1}{2} u(g_n) \right.\right. \\ &\quad \left.\left. - \frac{1}{2} u(g_m)\right)\right] \geq E^P\left[\lim_{n\longrightarrow\infty}\left(s_n - \frac{1}{2} s_n - \frac{1}{2} s_m - \epsilon_n \right.\right. \\ &\quad \left.\left. - \epsilon_m\right)\right] = 0. \end{aligned} \quad (20)$$

Thus, we have

$$\begin{aligned} &\lim_{n\longrightarrow\infty} P\left(\left\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\right\}\right) \\ &\quad = 0. \end{aligned} \quad (21)$$

Therefore,

$$\begin{aligned} &\lim_{n\longrightarrow\infty} c\left(\left\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\right\}\right) \\ &\geq \sup_{P\in\mathscr{P}} \lim_{n\longrightarrow\infty} P\left(\left\{|g_n - g_m| > \alpha \text{ and } \min(g_n, g_m) \leq \alpha^{-1}\right\}\right) \\ &= 0. \end{aligned} \quad (22)$$

That is, g_n mutually converges in capacity c, and, by Lemma 3, there exist a subsequence $(g_{n_k})_{k\geq 1}$ and a $[0,\infty]$ valued function g such that $g_{n_k} \longrightarrow g$, q.s. □

3. Application of Theorem 4

We set, for $p > 0$,

$$\mathscr{L}^p := \left\{X \in L^0(\Omega) : \mathbb{E}\left[|X|^p\right] < \infty\right\}, \quad (23)$$

$$\mathscr{N} := \left\{X \in L^0(\Omega) : X = 0, q.s.\right\}, \quad (24)$$

and denote $\mathbb{L}^p = \mathscr{L}^p / \mathscr{N}$.

Consider the following optimization problem in finance:

$$\begin{gathered} \inf_{X_T \in L_\xi} \mathbb{E}\left[l(\xi - X_T)\right], \\ \text{under the constraint } \mathbb{E}\left[X_T\right] \leq x, \end{gathered} \quad (25)$$

where X_T is the terminal wealth of the hedging portfolio at terminal time T, $L_\xi = \{X : 0 \leq X \leq \xi\}$, ξ is the nonnegative contingent claim which the investor attempts to hedge, $\mathbb{E}[|\xi|^2] < \infty$, l is the loss function which is an increasing convex function defined on $[0,\infty)$, and x is the constraint regarding the initial wealth. The corresponding hedging problem in the single probability model was introduced in [18] and is referred to as efficient hedging.

We use the result of the Theorem 4 to prove the existence of the solution of problem (25).

Theorem 5. *There is a solution $\widetilde{X}_T \in L_\xi$ to problem (25).*

Proof. Let L_ξ^x consist of elements of L_ξ that satisfy $\mathbb{E}[X_T] \leq x$ and let (X_T^n) be a minimizing sequence for (25) in L_ξ^x. By Theorem 4, there exists a sequence $\widetilde{X}_T^n$ belonging to $conv\{X_T^n, X_T^{n+1}, \ldots\}$ such that $\widetilde{X}_T^n \longrightarrow \widetilde{X}_T$, q.s. Since $X_T^n \in L_\xi^x$, i.e., $0 \leq X_T^n \leq \xi$ and $\mathbb{E}[X_T^n] \leq x$, we obtain $0 \leq \widetilde{X}_T^n, \widetilde{X}_T \leq \xi$, $\mathbb{E}[\widetilde{X}_T^n] \leq x$, and

$$\begin{aligned} \mathbb{E}\left[\widetilde{X}_T\right] &= \sup_{P\in\mathscr{P}} E^P\left[\lim_{n\longrightarrow\infty} \widetilde{X}_T^n\right] = \sup_{P\in\mathscr{P}} \lim_{n\longrightarrow\infty} E^P\left[\widetilde{X}_T^n\right] \\ &\leq \sup_{P\in\mathscr{P}} \lim_{n\longrightarrow\infty} \mathbb{E}\left[\widetilde{X}_T^n\right] = \lim_{n\longrightarrow\infty} \mathbb{E}\left[\widetilde{X}_T^n\right] \leq x, \end{aligned} \quad (26)$$

where the second equality sign is a result of the dominated convergence theorem under probability P. Therefore, $\widetilde{X}_T \in L_\xi^x$.

Similarly, since $\widetilde{X}_T^n \longrightarrow \widetilde{X}_T$, q.s., we have

$$\begin{aligned}\mathbb{E}\left[l\left(\xi-\widetilde{X}_T\right)\right] &= \sup_{P\in\mathscr{P}} E^P\left[l\left(\xi-\widetilde{X}_T\right)\right] \\ &= \sup_{P\in\mathscr{P}} E^P\left[\lim_{n\longrightarrow\infty} l\left(\xi-\widetilde{X}_T^n\right)\right] \\ &= \sup_{P\in\mathscr{P}} \lim_{n\longrightarrow\infty} E^P\left[l\left(\xi-\widetilde{X}_T^n\right)\right] \\ &\le \sup_{P\in\mathscr{P}} \lim_{n\longrightarrow\infty} \mathbb{E}\left[l\left(\xi-\widetilde{X}_T^n\right)\right] \\ &= \lim_{n\longrightarrow\infty} \mathbb{E}\left[l\left(\xi-\widetilde{X}_T^n\right)\right]\end{aligned} \tag{27}$$

By the convexity of $\mathbb{E}[\cdot]$ and function l and $\widetilde{X}_T^n$ belonging to $conv\{X_T^n, X_T^{n+1}, \ldots\}$, we can conclude that $\mathbb{E}[l(\xi-\widetilde{X}_T^n)]$ is not larger than the corresponding convex combination of $\mathbb{E}[l(\xi-X_T^m)]$, $m \ge n$. And because (X_T^n) is a minimizing sequence for (25) in L_ξ^x,

$$\lim_{n\longrightarrow\infty} \mathbb{E}\left[l\left(\xi-X_T^n\right)\right] = \inf_{X_T\in L_\xi^x} \mathbb{E}\left[l\left(\xi-X_T\right)\right]. \tag{28}$$

Then we have

$$\mathbb{E}\left[l\left(\xi-\widetilde{X}_T\right)\right] \le \inf_{X_T\in L_\xi^x} \mathbb{E}\left[l\left(\xi-X_T\right)\right]. \tag{29}$$

So

$$\mathbb{E}\left[l\left(\xi-\widetilde{X}_T\right)\right] = \inf_{X_T\in L_\xi^x} \mathbb{E}\left[l\left(\xi-X_T\right)\right]. \tag{30}$$

□

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

Acknowledgments

This work was supported by the National Natural Science Foundation of China under Grant no. 11301011 and the Beijing Natural Science Foundation under Grant no. 1112009.

References

[1] F. H. Knight, *Risk, Uncertainty and Profit*, Dover Publications, Boston, MA, USA, 1921.

[2] D. Ellsberg, "Risk, ambiguity, and the savage axioms," *The Quarterly Journal of Economics*, vol. 75, no. 4, pp. 643–669, 1961.

[3] S. Peng, "Backward SDE and related g-expectation," in *Backward stochastic differential equations (Paris, 1995-1996)*, vol. 364 of *Pitman Res. Notes Math. Ser.*, pp. 141–159, Longman, Harlow, UK, 1997.

[4] Z. Chen and L. Epstein, "Ambiguity, risk, and asset returns in continuous time," *Econometrica*, vol. 70, no. 4, pp. 1403–1443, 2002.

[5] S. Peng, "G-Expectation, G-Brownian Motion and related stochastic calculus of Itô's type," in *Proceedings of the The Abel Symposium 2005, Abel Symposia 2*, Benth et al., Ed., pp. 541–567, Springer.

[6] S. Peng, "Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation," *Stochastic Processes and Their Applications*, vol. 118, no. 12, pp. 2223–2253, 2008.

[7] S. G. Peng, "Survey on normal distributions, central limit theorem, Brownian motion and the related stochastic calculus under sublinear expectations," *Science China Mathematics*, vol. 52, no. 7, pp. 1391–1411, 2009.

[8] M. Hu and S. Peng, "On the representation theorem of G-expectation and paths of G-Brownian motion," *Acta Mathematicae Applicatae Sinica*, vol. 25, no. 3, pp. 539–546, 2009.

[9] H. M. Soner, N. Touzi, and J. Zhang, "Martingale representation theorem for the G-expectation," *Stochastic Processes and Their Applications*, vol. 121, no. 2, pp. 265–287, 2011.

[10] Y. Song, "Properties of hitting times for G-martingales and their applications," *Stochastic Processes and Their Applications*, vol. 121, no. 8, pp. 1770–1784, 2011.

[11] M. Hu, S. Ji, S. Peng, and Y. Song, "Backward stochastic differential equations driven by G-Brownian motion," *Stochastic Processes and Their Applications*, vol. 124, no. 1, pp. 759–784, 2014.

[12] M. Hu, S. Ji, S. Peng, and Y. Song, "Comparison theorem, Feynman–Kac formula and Girsanov transformation for BSDEs driven by G-Brownian motion," *Stochastic Processes and Their Applications*, vol. 124, no. 2, pp. 1170–1195, 2014.

[13] C. Dellacherie, *Capacitis et Processus Stochastiques*, Springer, Berlin, Germany, 1972.

[14] G. Choquet, "Theory of capacities," *Annales de l'Institut Fourier*, vol. 5, pp. 131–295, 1955.

[15] L. Denis, M. Hu, and S. Peng, " Functional spaces and capacity related to a sulinear expectations: application to G-Brownian motion paths," *Potential Analysis*, vol. 34, no. 2, pp. 139–161, 2011.

[16] P. J. Huber and V. Strassen, "Minimax tests and the Neyman-Pearson lemma for capacities," *The Annals of Statistics*, vol. 1, pp. 251–263, 1973.

[17] F. Delbaen and W. Schachermayer, "A general version of the fundamental theorem of asset pricing," *Mathematische Annalen*, vol. 300, no. 3, pp. 463–520, 1994.

[18] H. Föllmer and P. Leukert, "Efficient hedging: cost versus shortfall risk," *Finance and Stochastics*, vol. 4, no. 2, pp. 117–146, 2000.