Research Article

Toeplitz Operator and Carleson Measure on Weighted Bloch Spaces

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In this paper, we consider Toeplitz operator acting on weighted Bloch spaces. Meanwhile, the inclusion map from weighted Bloch spaces into tent space is also investigated.

1. Introduction

Denote the open unit disk of the complex plane \( \mathbb{C} \) by \( D \) and the boundary of \( D \) by \( \partial D \). Let \( H(D) \) denote the space of all functions analytic in \( D \). For any \( a \in D \),

\[ \varphi_a(z) = \frac{a-z}{1-ar{a}z}, \quad z \in D \]  

(1)

is the automorphism of \( D \) which exchanges \( 0 \) for \( a \). Recall that

\[ \beta(z,a) = \frac{1}{2} \log \frac{1+|\varphi_a(z)|}{1-|\varphi_a(z)|} \]  

(2)

is the Bergman metric. For any \( 0 < r < \infty \), \( a \in D \),

\[ D(a,r) = \{ z \in D : \beta(z,a) < r \} \]  

(3)

is the Bergman disk. Let \( |D(a,r)| \) denote the normalized area of \( D(a,r) \). From [1], we see that \( |D(a,r)| \approx (1-|a|^2)^2 \) when \( r \) is fixed.

For \( 0 < p < \infty \) and \( \alpha > -1 \), the weighted Bergman space \( B^p_\alpha \) is the space of all functions \( f \in H(D) \) such that

\[ \| f \|_{B^p_\alpha} = \sup_{z \in D} \left( 1 - |z|^2 \right)^{\alpha} |f(z)| < \infty. \]  

(4)

When \( \alpha = 0 \), \( B^p_\alpha \) is the classical Bergman space. We refer the readers to [1, 2] for more results on weighted Bergman spaces.

Let \( 0 < \alpha < \infty \). An \( f \in H(D) \) is said to belong to the weighted Bloch space, denoted by \( \mathcal{B}^\alpha \), if

\[ \| f \|_{\mathcal{B}^\alpha} = \sup_{z \in D} \left( 1 - |z|^2 \right)^{\alpha} \| f'(z) \| < \infty. \]  

(5)

The space \( \mathcal{B}^\alpha \) has been studied extensively in [3]. See [1, 4–8] for the study of some operators on weighted Bloch spaces.

Let \( \varphi \in L^\infty(D) \). The Toeplitz operator \( T_\varphi \) with symbol \( \varphi \) is defined by

\[ T_\varphi f(z) = \int_D \varphi(w) f(w) \left( 1 - \frac{wz}{w^2} \right)^{2\alpha} dA^\alpha(w), \]  

(6)

where \( dA^\alpha(w) = (1-|w|^2)^\alpha dA(w) \). There are many results related to \( T_\varphi \), see [1] and the references therein. Especially, some characterizations for the operator \( T_\varphi \) on \( L^2_\alpha \) have been obtained by many authors. Since \( \mathcal{B}^\alpha \subseteq A^1_\alpha \), it is nature to ask

\[ T_\varphi f \in \mathcal{B}^\alpha \iff f \in \mathcal{B}^\alpha. \]  

(7)

The following theorem is the first main result in this paper.

**Theorem 1.** Let \( 0 < \alpha < \infty \) and \( \varphi \in L^1(D) \) be harmonic. Then the following statements hold.

1. \( T_\varphi : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1} \) is bounded if and only if \( \varphi \) is bounded.

2. \( T_\varphi : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1} \) is compact if and only if \( \varphi = 0 \).

Given a positive Borel measure \( \mu \), the Toeplitz operator with the symbol \( \mu \) is defined by

\[ T_\mu f(z) = \int_D \frac{f(w)}{(1-\bar{w}z)^{2\alpha}} d\mu(w), \quad f \in L^1(dA^\alpha). \]  

(8)

For the Toeplitz operator \( T_\mu \), we have the following result.
Theorem 2. Let $0 < \alpha, r < \infty$ and $\mu$ be a positive Borel measure. Then the following statements hold. 

1. The inclusion map $I_D : \mathcal{B}^{r+1} \to \mathcal{B}^{r+1}$ is bounded if and only if 
$$\sup_{a \in D} \frac{\mu(D(a, r))}{|1 - |a|^2|^{2+\alpha}} < \infty.$$ 

2. The inclusion map $T_\mu : \mathcal{B}^{r+1} \to \mathcal{B}^{r+1}$ is compact if and only if 
$$\lim_{|a| \to \infty} \frac{\mu(D(a, r))}{|1 - |a|^2|^{2+\alpha}} = 0.$$ 

2. Proofs of Main Results

To prove our main results in this paper, we need some auxiliary results. The following result can be found in [16, Theorem 3.8].

Lemma 4. Let $p \geq 1$, $\alpha > 0$, $-1 + p\alpha < \eta < \infty$, and $c > 0$. Then $f \in \mathcal{B}^{r+1}$ if and only if 
$$\sup_{a \in D} \int_D |f(w) - f(a)|^p \frac{(1 - |z|^2)^c + p\alpha}{|1 - az|^{2+\alpha+\eta}} dA_\eta(w) < \infty.$$ 

From Lemma 4, we can easily deduce the following result.

Lemma 5. Let $p \geq 1$, $\alpha > 0$, $-1 + p\alpha < \eta < \infty$, and $c > 0$. Then $f \in \mathcal{B}^{r+1}$ if and only if 
$$\sup_{a \in D} \int_D |f(z)|^p \frac{(1 - |z|^2)^c + p\alpha}{|1 - az|^{2+\alpha+\eta}} dA_\eta(z) < \infty.$$ 

Proof. First assume that $f \in \mathcal{B}^{r+1}$. It is clear that 
$$|f(z)| \leq \frac{\|f\|_{\mathcal{B}^{r+1}}}{(1 - |z|^2)^c}, \quad z \in D.$$ 

Thus, 
$$\int_D |f(z)|^p \frac{(1 - |z|^2)^c + p\alpha}{|1 - az|^{2+\alpha+\eta}} dA_\eta(z) \leq \int_D |f(z) - f(a)|^p \frac{(1 - |z|^2)^c + p\alpha}{|1 - az|^{2+\alpha+\eta}} dA_\eta(z) + \int_D |f(a)|^p \frac{(1 - |z|^2)^c + p\alpha}{|1 - az|^{2+\alpha+\eta}} dA_\eta(z) \leq \|f\|_{\mathcal{B}^{r+1}} + \|f\|_{\mathcal{B}^{r+1}}^p \int_D \frac{(1 - |z|^2)^c}{|1 - az|^{2+\alpha+\eta}} dA_\eta(z) \leq \|f\|_{\mathcal{B}^{r+1}}.$$ 

The proof of the inverse direction is similar to the above statements we omit the details. The proof is complete.

Proof of Theorem 1. (1) First assume that $\varphi \in L^\infty(D)$. For $f \in \mathcal{B}^{\alpha+1}$, since
$$\|f\|_{\mathcal{B}^{\alpha+1}} = \sup_{z \in D} (1 - |z|^2)^\alpha |f(z)|,$$ 

Throughout this paper, the letter $C$ will denote constants and may differ from one occurrence to the other. The notation $A \leq B$ means that there is a positive constant $C$ such that $A \leq CB$. The notation $A \approx B$ means $A \leq B$ and $B \leq A$.
we obtain
\[
\|T_\varphi f\|_{\mathcal{B}^{\alpha+1}} \leq \sup_{z \in \Delta} (1 - |z|^2)^\alpha |T_\varphi f(z)|
\]
\[
= \sup_{z \in \Delta} (1 - |z|^2)^\alpha \left\{ \int_\Delta \frac{\varphi(w) f(w)}{(1 - wz)^{2\alpha}} dA_\alpha(w) \right\}
\]
\[
\leq \sup_{z \in \Delta} (1 - |z|^2)^\alpha \left\{ \int_\Delta \frac{\varphi(w) |f(w)|}{|1 - wz|^{2\alpha}} dA_\alpha(w) \right\}
\]
\[
\leq \|\varphi\|_{L^2(\Delta)} \|f\|_{\mathcal{B}^{\alpha+1}}.
\] (21)

Hence \(T_\varphi : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1}\) is bounded.

Conversely, assume that \(T_\varphi : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1}\) is bounded.

For \(z \in \Delta\), set
\[
f_z(w) = \frac{(1 - |z|^2)^2}{(1 - zw)^{2\alpha}} \in \mathcal{B}^{\alpha+1}.
\] (22)

It is easy to check that \(\|f_z\|_{\mathcal{B}^{\alpha+1}} \leq 1\). Using Lemma 5 with
\(p = 1, \eta = \alpha\) and \(c = 2 + \alpha\), we get
\[
\sup_{z \in \Delta} (1 - |z|^2)^\alpha \leq \frac{1}{(1 - zw)^{2\alpha}} \leq \frac{1}{\mu(D(a, r))},
\]
which implies that \(\varphi \in L^\infty(\Delta)\), as desired.

Proof of Theorem 2. (1) First suppose that \(T_\mu : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1}\) is bounded. For any \(a \in \Delta\), from the proof of Theorem 1, we obtain that \(f_\alpha \in \mathcal{B}^{\alpha+1}\) and \(\|f_\alpha\|_{\mathcal{B}^{\alpha+1}} \leq 1\). Thus,
\[
\sup_{a \in \Delta} \frac{\mu(D(a, r))}{(1 - |a|^2)^{2\alpha}} < \infty.
\] (27)

Then we can get that \(\mu\) is a Carleson measure for \(L^1(dA_\alpha)\). If \(g \in L^1\) and \(f \in \mathcal{B}^{\alpha+1}\), we can easily obtain that \(fg \in L^1(dA_\alpha)\).

Using Fubini’s Theorem, we obtain
\[
\int_\Delta g(z) T_\mu f(z) (1 - |z|^2)^\alpha dA(a(z)) = \int_\Delta f(\omega) g(z) (1 - |z|^2)^\alpha dA(a(z)) \int_\Delta f(\omega) dA(a(\omega))
\]
\[
= \int_\Delta g(z) (1 - |z|^2)^\alpha dA(a(z)) \int_\Delta f(\omega) d\mu(\omega)
\]
\[
= \int_\Delta g(z) (1 - |z|^2)^\alpha dA(a(z)) \int_\Delta f(\omega) d\mu(\omega)
\]
\[
\leq \int_\Delta \|g\|_{L^1} \|f\|_{\mathcal{B}^{\alpha+1}}.
\] (28)

Hence \(T_\mu : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1}\) is bounded.

(2) Suppose that \(T_\mu : \mathcal{B}^{\alpha+1} \to \mathcal{B}^{\alpha+1}\) is compact. Let \(a_\alpha \in \Delta\).

Set
\[
f_{a_\alpha}(z) = \frac{(1 - |a_\alpha|^2)^2}{(1 - a_\alpha z)^{2\alpha}}, \quad z \in \Delta.
\] (29)

Then \(f_{a_\alpha} \in \mathcal{B}^{\alpha+1}\) and \(f_{a_\alpha} \to 0\) uniformly on compact subset on \(\Delta\) as \(|a_\alpha| \to 1\). Thus,
\[
\|T_\mu f_{a_\alpha}\|_{\mathcal{B}^{\alpha+1}} \leq (1 - |a_\alpha|^2)^\alpha \|T_\mu f_{a_\alpha}(a_\alpha)\|
\]
\[
= \frac{(1 - |a_\alpha|^2)^2}{(1 - a_\alpha z)^{2\alpha}} d\mu(\omega)
\]
Let \(a_n\) be as desired. Then we have
\[
\frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w) - f(b)| d\mu(w)
\]
\[
\leq \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w)| d\mu(w) + \frac{\mu(S(I))}{|I|^{2+2\alpha}}.
\]
Note that
\[
\frac{\mu(S(I))}{|I|^{2+\alpha}} \leq \frac{\mu(S(I))}{|I|^{2+2\alpha}}.
\]
Then \(\mu\) is a Carleson measure for \(L^1(dA_a)\). Since \(\mathcal{B}^{2+\alpha} \subseteq L^1(dA_a)\), combined with Lemma 4, we obtain
\[
\frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w)| d\mu(w)
\]
\[
\leq \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f(w) - f(b)| d\mu(w) + \frac{\mu(S(I))}{|I|^{2+2\alpha}}.
\]
Hence \(I_d : \mathcal{B}^{2+\alpha} \rightarrow \mathcal{T}_{\mu}^{2+\alpha,1}\) is bounded.

(2) Suppose that \(I_d : \mathcal{B}^{2+\alpha} \rightarrow \mathcal{T}_{\mu}^{2+\alpha,1}\) is compact. Let \(a_n \in D\) such that \(|a_n| \rightarrow 1\) as \(n \rightarrow \infty\). We know that \(f_{a_n} \in \mathcal{B}^{2+\alpha}\) and \(f_{a_n} \rightarrow 0\) uniformly on compact subsets of \(D\) as \(n \rightarrow \infty\). By Theorem 5.15 of [1] it follows that \(f_{a_n} \rightarrow 0\) weakly as \(n \rightarrow \infty\). Hence for the compact operator \(I_d : \mathcal{B}^{2+\alpha} \rightarrow \mathcal{T}_{\mu}^{2+\alpha,1}\), we have \(\|f_{a_n}\|_{\mathcal{T}_{\mu}^{2+\alpha,1}} \rightarrow 0\) as \(n \rightarrow \infty\). Thus,
\[
\mu(S(I_{a_n})) \leq \frac{1}{|I_n|^{2+\alpha}} \int_{S(I_{a_n})} |f_{a_n}(z)| d\mu(z)
\]
\[
\leq \frac{1}{|I_n|^{2+\alpha}} \int_{S(I_{a_n})} \left| f_{a_n}(z) - f(b) \right| d\mu(z) + \frac{\mu(S(I_{a_n}))}{|I_n|^{2+2\alpha}}.
\]
As \(n \rightarrow \infty\). Hence \(\mu\) is a vanishing \((2+2\alpha)\)-Carleson measure. Conversely, assume that \(\mu\) is a vanishing \((2+2\alpha)\)-Carleson measure. Let \(f_n \in \mathcal{B}^{2+\alpha}\), \(\|f_n\|_{\mathcal{B}^{2+\alpha}} \leq 1\), and \(f_n \rightarrow 0\) weakly as \(n \rightarrow \infty\). Then we have
\[
\frac{1}{|I_n|^{2+\alpha}} \int_{S(I_n)} |f_n(z)| d\mu(z)
\]
\[
\leq \frac{1}{|I_n|^{2+\alpha}} \int_{S(I_n)} |f_n(z)| d\mu(z) + \frac{\mu(S(I_n))}{|I_n|^{2+2\alpha}}.
\]
(\(n \to \infty\)) uniformly on compact subsets of \(\mathbb{D}\). Then it is easy to get that
\[
\frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f_n(z)| \, d\mu(z) \\
\leq \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f_n(z)| \, d\mu_r(z) \\
+ \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f_n(z)| \, d(\mu - \mu_r)(z) \\
\leq \frac{1}{|I|^{2+\alpha}} \int_{S(I)} |f_n(z)| \, d\mu_r(z) \\
+ \|\mu - \mu_r\|_2 \|f_n\|_{B^\alpha+1}.
\]
Let \(r \to 1^-\) and \(n \to \infty\); we get the desired result. The proof is complete. \(\square\)

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The author declares that they have no conflicts of interest.

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**References**


