Research Article

Lyapunov-Type Inequalities for a Conformable Fractional Boundary Value Problem of Order \(3 < \alpha \leq 4\)

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We establish new Lyapunov-type inequalities for the following conformable fractional boundary value problem (BVP):

\[
T_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \quad u(a) = u'(a) = u''(a) = u''(b) = 0,
\]

where \(T_a^\alpha\) is the conformable fractional derivative of order \(\alpha \in (3,4]\) and \(q\) is a real-valued continuous function. Some applications to the corresponding eigenvalue problem are discussed.

1. Introduction

In [1], Lyapunov proved that if the boundary value problem

\[
\begin{align*}
\frac{d^n}{dt^n} u(t) + q(t)u(t) & = 0, \quad a < t < b, \\
u(a) & = u(b) = 0,
\end{align*}
\]

(1)

where \(q : [a,b] \to \mathbb{R}\), has a nontrivial continuous solution, then

\[
\int_a^b |q(r)|\,dr > \frac{4}{b-a}. \quad \text{(2)}
\]

Moreover, the constant 4 in (2) is sharp (see [2]).

We emphasize that the above inequality has been proved to be very useful in the study of various problems related to differential equations; see, for instance, [2–5] and the references therein.

Many researchers have studied generalizations and extensions of Lyapunov’s inequality.

In [6], Wintner improved inequality (2) and obtained the following version:

\[
\int_a^b |q^+(r)|\,dr > \frac{4}{b-a}, \quad \text{(3)}
\]

where \(q^+(r) = \max\{q(r), 0\}\).

In [2], Hartman generalized inequality (2) as follows:

\[
\int_a^b (b-r)(r-a)q^+(r)\,dr > b-a. \quad \text{(4)}
\]

In the frame of fractional differential equations, Ferreira (see [7]) proved a Lyapunov-type inequality for the Caputo fractional BVP

\[
C_D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2
\]

\[
u(a) = u(b) = 0.
\]

(5)

where \(q\) is a real and continuous function.

He showed that if a nontrivial continuous solution to the above problem exists, then

\[
\int_a^b |q(r)|\,dr > \Gamma(\alpha) \left(\frac{4}{b-a}\right)^{\alpha-1}. \quad \text{(6)}
\]

In [8], the same author investigated a Lyapunov-type inequality for the Riemann-Liouville fractional BVP

\[
D_a^\alpha u(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2
\]

\[
u(a) = u(b) = 0.
\]

(7)

He proved that if (7) has a nontrivial continuous solution, then

\[
\int_a^b |q(r)|\,dr > \Gamma(\alpha) \frac{\alpha^n}{[(\alpha-1)(b-a)]^{\alpha-1}}. \quad \text{(8)}
\]
For definitions and properties of Caputo fractional derivatives and Riemann-Liouville fractional, we refer the reader to [9, 10].

Observe that inequalities (6) and (8) lead to Lyapunov’s classical inequality (2) when \( \alpha = 2 \).

Recently, Khalil et al. [11] introduced a new definition of a fractional derivative called conformable fractional derivative (see Definition 1). This derivative is much easier to handle, is well-behaved, and obeys the Leibniz rule and chain rule [12].

In short time, this new fractional derivative definition has attracted many researchers. In [13], Chung used the conformable fractional derivative and integral to discuss fractional Newtonian mechanics.

In [14], the authors proved a generalized Lyapunov-type inequality for a conformable BVP of order \( 1 < \alpha \leq 2 \). They have established that if the BVP

\[
T_\alpha^u(t) + q(t)u(t) = 0, \quad a < t < b, \quad 1 < \alpha \leq 2
\]

where \( T_\alpha^u \) is the conformable derivative of order \( \alpha \in (1, 2] \), has a nontrivial continuous solution, then

\[
\int_a^b |q(r)| \, dr > \frac{\alpha^\alpha}{(\alpha - 1) (b - a)^{\alpha - 1}}
\]

For other generalizations and extensions of the classical Lyapunov’s inequality, we refer the reader to [2, 5, 15–23] and the references therein.

In this paper, we establish new Hartman-type and Lyapunov-type inequalities for the following conformable fractional BVP:

\[
T_\alpha^u(t) + q(t)u(t) = 0, \quad a < t < b, \quad 3 < \alpha \leq 4
\]

where \( T_\alpha^u \) is the conformable derivative starting at \( a \) of order \( 3 < \alpha \leq 4 \) and \( q \) is a real-valued continuous function on \( [a, b] \).

Some applications to the corresponding eigenvalue problem main results. Some applications are discussed. In Section 3, we state and prove our results. Some properties of this Green's function are established. In Section 3, we prove our main results. Some applications are discussed.

2. Preliminaries on Conformable Derivatives

In this section, we recall some basic definitions and lemmas, which will be very useful to state our results.

**Definition 1** (see [11, 12]). For a given function \( h : [a, \infty) \to \mathbb{R} \), the conformable fractional derivative of \( h \) of order \( \alpha \in (0, 1) \) is defined by

\[
T_\alpha^h(t) = \lim_{\varepsilon \to 0} \frac{h(t + \varepsilon (t-a)^{1-\alpha}) - h(t)}{\varepsilon}
\]

for all \( t > a \). If \( a = 0 \), we write \( T_\alpha^h(t) \). If \( T_\alpha^h(t) \) exists on \( (a, b) \), then define \( T_\alpha^h(a) = \lim_{t \to a+} T_\alpha^h(t) \). The geometric and physical interpretation of the conformable fractional derivatives was given in Zhao [24].

**Remark 2.** (i) Let \( 0 < \alpha \leq 1 \) and \( h \) be differentiable function at \( t > a \), and then

\[
T_\alpha^h(t) = (t-a)^{1-\alpha} h'(t) .
\]

(ii) For \( h(t) = 2 \sqrt{t-a} \), we have \( T_{1/2}^h(t) = 1 \), for all \( t > a \). Therefore \( T_{1/2}^h(a) = 1 \), but \( h \) is not differentiable at \( a \).

Some important properties for the conformable fractional derivative given in [11, 12] are as follows.

**Theorem 3.** Let \( \alpha \in (0, 1] \) and \( f, g \) be \( \alpha \)-differentiable at a point \( t \), and then

(i) \( T_\alpha^f(\lambda f + \mu g) = \lambda T_\alpha^h(f) + \mu T_\alpha^h(g) \), for all \( \lambda, \mu \in \mathbb{R} \).

(ii) \( T_\alpha^f((s-a)^\alpha) = \mu((t-a)^\alpha) \), for all \( t > a \) and \( \mu \in \mathbb{R} \).

(iii) \( T_\alpha^f(\sin(\alpha)) \) is defined in the range of \( f \).

(iv) \( T_\alpha^f(\cos(\alpha)) = (T_\alpha^\sin(\alpha))' \).

(v) Assume further that the function \( g \) is defined in the range of \( f \), and then for all \( t \) with \( t \neq a \) and \( g(t) \neq 0 \), one has the following Chain Rule:

\[
T_\alpha^a\left( f \circ g \right)(t) = T_\alpha^a f\left( g(t) \right) \cdot T_\alpha^a g(t) \cdot (g(t))^{\alpha-1}
\]

The following conformable fractional derivatives of certain functions [11] are worth noting:

(i) \( T_\alpha^a((1/\alpha)(t-a)^\alpha) = 1 \).

(ii) \( T_\alpha^a((\sin(1/\alpha)(t-a)^\alpha) = \cos(1/\alpha)(t-a)^\alpha \).

(iii) \( T_\alpha^a((\cos(1/\alpha)(t-a)^\alpha) = -\sin(1/\alpha)(t-a)^\alpha \).

(iv) \( T_\alpha^a(e^{1/(t-a)^\alpha}) = e^{1/(t-a)^\alpha} \).

**Definition 4** (see [11, 12]). Let \( \alpha < \alpha \leq n+1 \) and \( h : [0, \infty) \to \mathbb{R} \) be a function such that \( h^{(n)}(t) \) exists. The conformable fractional derivative of \( h \) of order \( \alpha \) is defined by

\[
T_\alpha^h(t) = \left( T_\alpha^{h^{(n)}} \right)(t), \text{ for which } \gamma = \alpha - n .
\]

**Definition 5** (see [11, 12]). Let \( n < \alpha \leq n+1 \). The fractional integral of a function \( h : [a, \infty) \to \mathbb{R} \) of order \( \alpha \) is defined by

\[
(I_\alpha^a h)(t) = \frac{1}{n!} \int_a^t \left( t-s \right)^n (s-a)^{n-1} h(s) \, ds .
\]

**Lemma 6** (see [11, 12]). Let \( \alpha \in (n, n+1] \).

(i) If \( h \) is continuous on \( [a, \infty) \), then

\[
T_\alpha^a(I_\alpha^a h)(t) = h(t) , \text{ for all } t \geq a .
\]

(ii) \( T_\alpha^a h(t) = 0 \) if and only if \( h(t) = \sum_{k=0}^{n} c_k (t-a)^k \), where \( c_k \in \mathbb{R} \), for \( k = 0, 1, \ldots, n \).

(iii) If \( T_\alpha^a h \) is continuous on \( [a, \infty) \), then, for \( t > a \),
\[ T^\alpha_n h(t) = h(t) + c_0 + c_1 (t-a) + \cdots + c_n (t-a)^n, \quad (18) \]

where \( c_k \in \mathbb{R}, \) for \( k = 0, 1, \ldots, n. \)

**Lemma 7.** Let \( \alpha \in (3, 4] \) and \( h \in C([a, b]). \) Then the BVP
\[
T^\alpha_n u(t) = -h(t), \quad a < t < b, \quad u(a) = u'(a) = u''(a) = u'''(b) = 0, \quad (19)
\]
adsmits a solution \( u \in C([a, b], \mathbb{R}) \) if and only if
\[
u(t) = \int_a^b G(t, s) h(s) ds, \quad (20)
\]
where \( G(t, s) \) is Green's function defined as
\[
G(t, s) = \frac{1}{6} (s-a)^{\alpha-4} \begin{cases} 
(t-a)^3 (b-s) - (t-s)^3, & a \leq s \leq t \leq b, \\
(t-a)^3 (b-s), & a \leq t \leq s \leq b.
\end{cases} \quad (21)
\]

**Proof.** Using Lemma 6 and Definition 5, we deduce that \( u \in C([a, b], \mathbb{R}) \) is a solution of problem (19) if and only if
\[
u(t) = c_0 + c_1 (t-a) + c_2 (t-a)^3 - \frac{1}{6} \int_a^t (t-s)^3 (s-a)^{\alpha-4} h(s) ds, \quad (22)
\]
where \((c_0, c_1, c_2, c_3) \in \mathbb{R}^4.\)

This, together with the boundary conditions, implies \( c_0 = c_1 = c_2 = 0 \) and
\[
c_3 = \frac{1}{6 (b-a)} \int_a^b (b-s) (s-a)^{\alpha-4} h(s) ds. \quad (23)
\]

Hence
\[
u(t) = \frac{1}{6 (b-a)} \int_a^b (t-a)^3 (b-s) (s-a)^{\alpha-4} h(s) ds - \frac{1}{6} \int_a^t (t-s)^3 (s-a)^{\alpha-4} h(s) ds, \quad (24)
\]
where \( G(t, s) \) is given in (21).

**Lemma 8.** Let \( \alpha \in (3, 4]. \) The following property is satisfied by Green's function (21): For any \((t, s) \in [a, b] \times [a, b],\)
\[
0 \leq G(t, s) \leq G(b, s) = \frac{1}{6} (s-a)^{\alpha-3} (b-s) (2b-a-s). \quad (25)
\]

**Proof.** Fix \( s \in [a, b]. \) By differentiating \( G(t, s) \) with respect to \( t, \) we obtain
\[
\partial_t G(t, s) = \frac{1}{2} (s-a)^{\alpha-4} \begin{cases} 
\frac{(t-a)^2 (b-s)}{b-a} - (t-s)^2, & a \leq s \leq t \leq b, \\
\frac{(t-a)^2 (b-s)}{b-a}, & a \leq t \leq s \leq b.
\end{cases} \quad (26)
\]

Hence \( a \leq t \leq s \leq b, \) and we have
\[
\partial_t G(t, s) = \frac{1}{2} (s-a)^{\alpha-4} \left( \frac{(t-a)^2 (b-s)}{b-a} - (t-s)^2 \right) \geq 0, \quad (27)
\]
and for \( a \leq s \leq t \leq b, \) we have
\[
\partial_t G(t, s) = \frac{1}{2} (s-a)^{\alpha-4} \left( \frac{(t-a)^2 (b-s)}{b-a} - (t-s)^2 \right) = \frac{1}{2} (s-a)^{\alpha-4} \left( \frac{(t-a)^2 (b-s)}{b-a} - \left( \frac{b-s}{b-a} \left( \frac{b-a}{(b-s)(t-a)} \right) \right)^2 \right). \quad (28)
\]

Using the fact that \((b-s)/(b-a))\) and \((b-a)(t-s)/(b-s)(t-a)\) are in \([0, 1],\) we deduce that
\[
\partial_t G(t, s) \geq 0, \quad \text{for } a \leq s \leq t \leq b. \quad (29)
\]

So, the function \( t \mapsto G(t, s) \) is nondecreasing on \([a, b].\) This implies that
\[
0 = G(a, s) \leq G(t, s) \leq G(b, s), \quad (20)
\]
where \((t, s) \in [a, b] \times [a, b].\)

The proof is completed.

**3. Main Results**

**Theorem 9** (Hartman-type inequality). Assume that the BVP (11) has a nontrivial continuous solution; then
\[
\int_a^b (s-a)^{\alpha-3} (b-s) (2b-a-s) q(s) ds \geq 6. \quad (31)
\]

**Proof.** Let \( \alpha \in (3, 4], \) \( a, b \in \mathbb{R} \) with \( a < b \) and \( q \in C([a, b], \mathbb{R}).\)

Consider the Banach space \( C([a, b], \mathbb{R}), \) equipped with the uniform norm \( \| u \|_{\infty} = \sup_{t \in [a, b]} |u(t)|. \)

Assume that problem (11) has a nontrivial solution \( u \in C([a, b], \mathbb{R}).\)

By (20), we have
\[
u(t) = \int_a^b G(t, s) q(s) u(s) ds, \quad t \in [a, b]. \quad (32)
\]
Note that since \( u \) is nontrivial, then \( q \) cannot be the zero function on \([a, b]\). This, with Lemma 8, implies, for all \( t \in [a, b] \),
\[
|u(t)| \leq \int_a^b G(t, s)|q(s)||u(s)|
\]
\[
\leq \|u\|_{\infty} \left( \int_a^b G(b, s)|q(s)|
\right),
\]
(33)
Therefore
\[
\|u\|_{\infty} \leq \|u\|_{\infty} \left( \int_a^b G(b, s)|q(s)|
\right).
\]
(34)
Now, since \( \|u\|_{\infty} \neq 0 \), then we deduce that
\[
1 \leq \left( \int_a^b G(b, s)|q(s)|
\right),
\]
(35)
where
\[
G(b, s) = \frac{1}{6}(s-a)^{\alpha-3}(b-s)(2b-a-s),
\]
for \( s \in [a, b] \).
(36)
The proof is completed.

**Corollary 10.** Assume that the BVP (11) has a nontrivial continuous solution; then
\[
\int_a^b (s-a)^{\alpha-3}(b-s)|q(s)|
\]
\[
\geq \frac{3}{b-a},
\]
(37)
Proof. The property follows from Theorem 9 and the fact
\[
(2b-a-s) \leq 2(b-a), \quad \text{for } s \in [a, b].
\]
(38)

**Theorem 11** (Lyapunov-type inequality). Assume that the BVP (11) has a nontrivial continuous solution; then
\[
\int_a^b |q(s)|
\]
\[
\geq \frac{3(\alpha-2)^{\alpha-2}}{(\alpha-3)^{\alpha-3}(b-a)^{\alpha-1}}.
\]
(39)
Proof. From Corollary 10, we have
\[
\int_a^b \varphi(s)|q(s)|
\]
\[
\geq \frac{3}{b-a},
\]
(40)
where \( \varphi(s) = (s-a)^{\alpha-3}(b-s), \quad s \in [a, b] \).

By simple computation, one can check that
\[
\max_{s \in [a, b]} \varphi(s) = \varphi\left(\frac{a + (\alpha - 3)b}{\alpha - 2}\right)
\]
\[
= (\alpha - 3)^{\alpha-3}\left(\frac{b-a}{\alpha - 2}\right)^{\alpha-2}.
\]
(41)
This fact with (40) gives the required result.

**Corollary 12.** If \( \lambda \) is an eigenvalue to the fractional BVP
\[
T_\alpha u(t) + \lambda u(t) = 0, \quad 0 < t < 1,
\]
\[
u(0) = u'(0) = u''(0) = u''(1) = 0,
\]
(42)
then
\[
|\lambda| \geq \frac{6\alpha(\alpha-1)(\alpha-2)}{(\alpha+2)(\alpha-1)(\alpha-2)}.
\]
(43)
Proof. By using Theorem 9, we obtain
\[
|\lambda| \geq \frac{1}{\int_a^b s^{\alpha-3}(1-s)(2-s)}
\]
\[
\geq \frac{6}{b-a}.
\]
(44)
Now, by simple computation, we have
\[
\int_a^b s^{\alpha-3}(1-s)(2-s)ds = \frac{\alpha + 2}{(\alpha - 1)(\alpha - 2)}.
\]
(45)
This gives inequality (43).

**Corollary 13.** Let \( 3 < \alpha \leq 4 \) and \( q \in C([0,1], \mathbb{R}) \), such that
\[
\int_0^1 |q(r)|dr < \frac{3(\alpha-2)^{\alpha-2}}{(\alpha-3)^{\alpha-3}}.
\]
(46)
Then the fractional BVP
\[
T_\alpha u(t) + q(t)u(t) = 0, \quad 0 < t < 1,
\]
\[
u(0) = u'(0) = u''(0) = u''(1) = 0,
\]
(47)
has no nontrivial solution.
Proof. The assertion follows from Theorem 11.

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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