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Research Article

Inequalities of Lyapunov and Stolarsky Type for Choquet-Like Integrals with respect to Nonmonotonic Fuzzy Measures

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The aim of this paper is to generalize the Choquet-like integral with respect to a nonmonotonic fuzzy measure for generalized real-valued functions and set-valued functions, which is based on the generalized pseudo-operations and σ-$\oplus$-measures. Furthermore, the characterization theorem and transformation theorem for the integral are given. Finally, we study the Lyapunov type inequality and Stolarsky type inequality for the Choquet-like integral.

1. Introduction

The Choquet integral with respect to a fuzzy measure λ, which is monotone, does not require continuity and was proposed by Murofushi and Sugeno [1]. It was introduced by Choquet [2] in potential theory with the concept of capacity. The Choquet integral of a nonnegative single-valued measurable function is defined as

$$(C)\int_A f d\lambda = \int_0^{+\infty} \lambda\left(f_\alpha \cap A\right) d\alpha, \tag{1}$$

where $f_\alpha = \{x \in X \mid f(x) \geqslant \alpha\}$. To generate the Choquet integral to the generalized real valued measurable function, the symmetric Choquet integral, which was most early proposed by Šipoš [3] in 1979, and the asymmetric Choquet integral were introduced and later in [4, 5] had been given specific discussions. Schmeidler [6] established an integral representation theorem through the Choquet integral for functionals satisfying monotonicity and a weaker condition than additivity, namely, comonotonic additivity. However, violations of monotonicity in multiperiod models occur frequently, and nonmonotone set functions seem to be better suited [7, 8]. Furthermore, from the mathematical point of view, the monotonicity is inessential. We can construct measure theory without monotonicity [1, 9]. Aumann and Shapely [10] had investigated nonmonotonic fuzzy measures as games and this issue had been addressed by Murofushi et al. in [9], where a complete characterization of nonmonotonic Choquet integral was achieved; that is, they generalized the representation to the case of bounded variation functionals omitting the monotonicity condition.

Sugeno introduced another integral for any fuzzy measure λ and any nonnegative single-valued measurable function f, nowadays called a Sugeno integral, as follows:

$$\int_A f d\lambda = \sup_{\alpha\in[0,+\infty]} \inf\left\{\alpha, \lambda\left(\mathrm{f}_\alpha \cap A\right)\right\}, \tag{2}$$

where $f_\alpha = \{x \in X \mid f(x) \geqslant \alpha\}$. Notice that when the fuzzy measure is with the usual additive, the Choquet integral is coincident with the Lebesgue integral. However, the Sugeno integral is not with the usual additive; thus it is not an extension of the Lebesgue integral.

Recently, pseudo-analysis is a research hotspot, and it presents a contemporary mathematical theory that is being successfully applied in many different areas of mathematics as well as in various practical problems [5, 11–14]. In fact, in many problems with uncertainty as in the theory of probabilistic metric spaces, fuzzy logics, fuzzy sets, and fuzzy measures, we often work with many operations different from the usual addition and multiplication of reals, e.g., triangular norms, triangular conorms, pseudo-additions, and pseudo-multiplications. The triangular conorm decomposable measure was first introduced by Dubois and Prade [15] as a special important class of fuzzy measures. Furthermore, it could be transferred into the corresponding results of reals [5, 11, 16–19], such as the addition operator, multiplication operator, differentiability, and integrability, by using Aczel's representation [20, 21]. Gong and Xie [22] coincided the definition of g-integrability with the definition of pseudo-integrability with respect to a decomposable measure in different papers, obtained Newton-Leibniz formula, and directly applied the results to the discussion of nonlinear differential equations. Sugeno and Murofushi [23] introduced an integral (briefly, SM integral) with respect to a pseudo-additive measure based on pseudo-operations. Note that the Choquet integral and the SM integral are extensions of the Lebesgue integral but not of the Sugeno integral and the SM integral does not cover some well-known integrals such as the Sugeno integral and the Choquet integral, in general. Mesiar [18] characterized the operations of pseudo-addition and pseudo-multiplication leading to integrals with properties similar to those of the Choquet and the Sugeno integral, respectively, and developed a type of integral based on the SM integral, the so-called Choquet-like integral, which generalized the concepts of some well-known integrals including both the Sugeno integral and the Choquet integral. However, as the basis for the pseudo-integrals, the definitions of the pseudo-operations and the relative measures have some differences. In fact, the pseudo-operations need to be continuous and valued in $[0, +\infty]$ and the relative measures need to be continuous from below introduced in [18, 23, 24], while the pseudo-operations need not to be continuous and valued in $[-\infty, +\infty]$, the relative measures need not to be continuous from below and the measurable function, and f need not to be nonnegative in the relative integral in [5, 25]. In this paper, we generalized the Choquet-like integrals with respect to nonmonotonic measures based on the generalized definitions of pseudo-operations.

As is well known, the set-valued function, besides being an important mathematical notion, has become an essential tool in several practical areas, especially in economic analysis [26]. The integration of set-valued functions has roots in Aumann's research [27] based on the classical Lebesgue integral. By using the approach of Aumann, Jang et al. [28] defined Choquet integrals of set-valued mappings as

$$(C)\int_A F d\lambda = \left\{(C)\int_A f d\lambda \mid f \in S(F)\right\}, \tag{3}$$

where F is a measurable set-valued mapping and $S(F)$ denotes the family of Choquet measurable selection of F. In the field of the pseudo-analysis, an approach to the problem of integration of set-valued functions from the pseudo-analysis' point of view has been introduced in [29]. Similarly, we introduce the Choquet-like integrals for set-valued functions.

On the other hand, integral inequalities are an important aspect of the classical mathematical analysis [30]. Generally, any integral inequality can be a very strong tool for applications. For example, when we think of an integral operator as a predictive tool, then an integral inequality can be very important in measuring and dimensioning such process. Recently, Flores, Agahi, Pap, and Mesiar et al. generalized several classical integral inequalities to Sugeno integral and choquet integral, including Chebyshev type inequality [31, 32], Jensen type inequality [33, 34], Stolarsky type inequality [35, 36], Hölder type inequality [37], Minkowski type inequalities [38], Carlson type inequality [39], and Liapunov type inequality [40]. Pseudo-analysis would be an interesting topic to generalize an inequality from the frame work of the classical analysis to that of some integrals which contain the classical analysis as special cases. In fact, Jensen inequality was generalized into pseudo-integrals by Pap and Štrboja [41], where two cases of real semirings defined by pseudo-operations were considered. In the first case, the pseudo-operations (pseudo-addition and pseudo-multiplication) are defined by the monotone and continuous function g. In this case, the pseudo-integral reduces to the g-integral. In the second case, the semiring $([a, b], \sup, \odot)$ is used, where the pseudo-addition is the idempotent operation sup and $\odot$ is generated, as in the first case. Chebyshev type inequalities for pseudo-integrals were investigated in [42] and Chebyshev's inequality for Choquet-like integral was subsequently introduced in [43]. Daraby [44] obtained generalization of the Stolarsky type inequality for pseudo-integrals. Li et al. [45] investigated generalization of the Lyapunov type inequality for pseudo-integrals. Jensen and Chebyshev inequalities for pseudo-integrals of set-valued functions were proved in [46]. In 2015, Agahi and Mesiar [47] introduced Cauchy-Schwarz's inequality for Choquet-like integrals. In 2017, Mihailović and Štrboja [48] proposed the generalized Minkowski type inequality for pseudo-integrals. Abbaszadeh et al. established a refinement of the Hadamard integral inequality [49] for g-integrals in 2018 and Hölder type integral inequalities [50] for pseudo-integrals by means of the above two cases of real semirings in 2019. As a further study, we generalize some of these inequalities to the frame of the Choquet-like integral presented in this paper and prove the Lyapunov type inequality and Stolarsky type inequality for the Choquet-like integral.

To make our analysis possible, we recall some basic results of the pseudo-analysis and the Choquet integral in Section 2. Section 3 defines the Choquet-like integral with respect to a nonmonotonic fuzzy measure and gives the characterization theorem and transformation theorem for the integral. In addition, the Choquet-like integral of set-valued functions is also obtained. The Lyapunov type inequality and Stolarsky type inequality for the Choquet-like integral are investigated in Sections 4 and 5, respectively.

2. Preliminaries

In the paper, the following concepts and notations will be used. R denotes the set of all real numbers, $\overline{R} = R \cup \{-\infty, \infty\}$ denotes the set of generalized real numbers, $\overline{R}^{+} = [0, \infty]$ denotes the set of extended nonnegative real numbers, $P(\overline{R})$ denotes the class of all the subsets of $\overline{R}$. X denotes a nonempty set, $\mathscr{A}$ is a σ-algebra on X, and $(X, \mathscr{A})$ is a measurable space. Let $\lambda : \mathscr{A} \longrightarrow [0, \infty]$ be a set function; then λ is called a fuzzy measure ([51]) or a pre-measure ([3, 18]), if

(1) $\lambda(\emptyset) = 0$,
(2) $\lambda(A) \leqslant \lambda(B)$ whenever $A \subset B$, $A, B \in \mathscr{A}$.

The triplet $(X, \mathscr{A}, \lambda)$ is called a fuzzy measure space. We say that λ is finite if $\lambda(X) < +\infty$. When λ is finite, we define the conjugate λ^c of λ by $\lambda^c(A) = \lambda(X) - \lambda(A^c)$ for all $A \in \mathscr{A}$.

For measurable functions $f : X \longrightarrow \overline{R}$, $f^+ = f \vee 0$, and $f^- = -(f \wedge 0)$, we have ([4, 5]) the following:

(i) The integral

$$\int_A f d\lambda = (C) \int_A f^+ d\lambda - (C) \int_A f^- d\lambda \tag{4}$$

is called a symmetric Choquet integral, also called Šipoš integral.

(ii) Suppose $\lambda(X) < \infty$. The integral

$$\int f d\lambda = (C) \int f^+ d\lambda - (C) \int f^- d\lambda^d \tag{5}$$

is called a asymmetric Choquet integral.

In the case that, the right-hand side is $\infty - \infty$ and the Choquet integral is not defined.

Definition 1 (see [9]). A nonmonotonic fuzzy measure on $(X, \mathscr{A})$ is a real-valued set function $m : \mathscr{A} \longrightarrow R$ satisfying $m(\emptyset) = 0$.

We can represent the relation between the fuzzy measure λ in the original monotonic version and the nonmonotonic fuzzy measure μ in the nonmonotonic version as $\lambda(A) = \max\{\mu(B) \mid B \subseteq A\}, \forall A \subseteq X$; if $\lambda(A) = \mu(B)$, the members of $A \setminus B$ are turned out. Generally, for each nonmonotonic fuzzy measure μ on $(X, \mathscr{A})$, if we define a set function λ on $(X, \mathscr{A})$ by $\lambda(A) = \sup\{\mu(B) \mid B \subseteq A, B \in \mathscr{A}\}, \forall A \in \mathscr{A}$, then λ is a monotonic fuzzy measure. We denote the set of monotonic fuzzy measures on $(X, \mathscr{A})$ by $\mathrm{FM}(X, \mathscr{A})$, and the set of nonmonotonic fuzzy measures of bounded variation on $(X, \mathscr{A})$ by $\mathrm{BV}(X, \mathscr{A})$.

Definition 2 (see [10]). For a given real-valued set function $m : \mathscr{A} \longrightarrow R$, the total variation $V(m)$ of m on X is defined by

$$V(m) = \sup \left\{ \sum_{i=1}^{n} \left| m(A_i) - m(A_{i-1}) \right| \mid \emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = X,\ \{A_i\}_{i=0}^{n} \subseteq \mathscr{A} \right\} \tag{6}$$

A real-valued set function m is said to be of bounded variation if $V(m) < +\infty$.

A finite monotonic fuzzy measure λ is of bounded variation since $V(\lambda) = \lambda(X) < +\infty$.

For every $m \in (X, \mathscr{A})$, we define ([9, 10])

$$|m|(A) = \sup \left\{ \sum_{i=1}^{n} \left| m(A_i) - m(A_{i-1}) \right| \mid \emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A,\ \{A_i\}_{i=0}^{n} \subseteq \mathscr{A} \right\},$$

$$m^+(A) = \sup \left\{ \sum_{i=1}^{n} \left[m(A_i) - m(A_{i-1}) \right]^+ \mid \emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A,\ \{A_i\}_{i=0}^{n} \subseteq \mathscr{A} \right\},$$

$$m^-(A) = \sup \left\{ \sum_{i=1}^{n} \left[m(A_i) - m(A_{i-1}) \right]^- \mid \emptyset = A_0 \subseteq A_1 \subseteq \cdots \subseteq A_n = A,\ \{A_i\}_{i=0}^{n} \subseteq \mathscr{A} \right\}, \tag{7}$$

where $[r]^+ = r \vee 0$ and $[r]^- = -(r \wedge 0)$. We call $|m|, m^+, m^-$ the total variation, positive (or upper) variation, and negative (or lower) variation of m on A, respectively.

Lemma 3 (see [10]). *Let $m \in BV(X, \mathscr{A})$. We have*

(1) $m^+, m^- \in FM(X, \mathscr{A})$.
(2) $m = m^+ - m^-$.
(3) $V(m) = m^+(X) + m^-(X)$.

The Choquet integral of a measurable function $f : X \longrightarrow R$ with respect to a nonmonotonic fuzzy measure m is defined by ([9])

$$(C) \int f dm = \int_{-\infty}^{+\infty} m(f_\alpha)\, d\alpha \tag{8}$$

whenever the integral in the right-hand side exists, where

$$m(f_\alpha) = \begin{cases} m(\{x \mid f(x) \geqslant \alpha\}), & \alpha \geqslant 0, \\ m(\{x \mid f(x) \geqslant \alpha\}) - m(X), & \alpha < 0, \end{cases} \tag{9}$$

A measurable function f is called integrable if the Choquet integral of f exists and its value is finite.

Sugeno-Murofushi ([23]) introduced an integral (briefly, SM integral) with respect to a pseudo-additive measure based on pseudo-operations, where the pseudo-operations and pseudo-additive measures were defined as follows, respectively:

(1) A binary operation $\tilde{\oplus} : [0, \infty] \times [0, \infty] \longrightarrow [0, \infty]$ is called a pseudo-addition if it satisfies (P1) $a \tilde{\oplus} 0 = 0 \tilde{\oplus} a = a$ (neutral element), (P2)$(a \tilde{\oplus} b) \tilde{\oplus} c = a \tilde{\oplus} (b \tilde{\oplus} c)$ (associativity), (P3) $a \leqslant c$ and $b \leqslant d \Longrightarrow a \tilde{\oplus} b \leqslant c \tilde{\oplus} d$ (monotonicity), and (P4) $a_n \longrightarrow a$ and $b_n \longrightarrow b \Longrightarrow a_n \tilde{\oplus} b_n \longrightarrow a \tilde{\oplus} b$ (continuity).

Another binary operation $\tilde{\odot}$ on $\overline{R}^+$ is called a pseudo-multiplication corresponding to $\tilde{\odot}$ if it satisfies (M1) $a \tilde{\odot} (x \tilde{\oplus} y) = (a \tilde{\odot} x) \tilde{\oplus} (a \tilde{\odot} y)$, (M2) $a \leqslant b \Longrightarrow (a \tilde{\odot} x) \leqslant (b \tilde{\odot} x)$, (M3) $a \tilde{\odot} x = 0 \Longrightarrow a = 0$ or $x = 0$, (M4) $\exists e \in [0, \infty]$ such that $e \tilde{\odot} x = x$ for any $x \in [0, \infty]$, (M5) $a_n \longrightarrow a \in (0, \infty)$ and $x_n \longrightarrow x \Longrightarrow (a_n \tilde{\odot} x_n) \longrightarrow (a \tilde{\odot} x)$ and $\infty \tilde{\odot} x = \lim_{a \longrightarrow \infty} (a \tilde{\odot} x)$, (M6) $a \tilde{\odot} x = x \tilde{\odot} a$, and (M7) $(a \tilde{\odot} b) \tilde{\odot} c = a \tilde{\odot} (b \tilde{\odot} c)$.

(2) A set function $\tilde{\mu} : \mathscr{A} \longrightarrow [0, \infty]$ is said to be a pseudo-additive measure with respect to $\tilde{\oplus}$ ($\tilde{\oplus}$-additive measure, for

short) ([23]) or $\widetilde{\oplus}$-decomposable measure in Klement and Weber's paper ([24]) if $\widetilde{\mu}$ satisfies the following conditions:

(i) $\widetilde{\mu}(\emptyset) = 0$,

(ii) $A, B \in \mathscr{A}$ and $A \cap B = \emptyset \Longrightarrow \widetilde{\mu}(A \cup B) = \widetilde{\mu}(A) \widetilde{\oplus} \widetilde{\mu}(B)$,

(iii) $A_n \subset \mathscr{A}$ and $A_n \uparrow A \Longrightarrow \widetilde{\mu}(A_n) \uparrow \widetilde{\mu}(A)$.

The triple $(X, \mathscr{A}, \widetilde{\mu})$ is called a $\widetilde{\oplus}$-measure space.

Later, Mesiar ([18]) developed Choquet-like integrals based on the SM integrals. More on Choquet-like integrals can be found in ([43, 47, 52]). H. Ichihashi and E. Pap et al. ([5, 25]) generalized the above pseudo-operations from $\overline{R}^+$ to $\overline{R}$ and introduced the relative measures and integrals based on the generalized pseudo-operations. Let $[a, b]$ be a closed real interval of $\overline{R}$ and $\preceq$ be a total order on $[a, b]$.

Definition 4 (see [25]). A 2-place function $\oplus : [a, b] \times [a, b] \longrightarrow [a, b]$ is called a pseudo-addition if it satisfies the following conditions

(i) $\oplus$ is commutative,

(ii) $\oplus$ is nondecreasing in each place (with respect to $\preceq$),

(iii) $\oplus$ is associative,

(iv) There exist a zero element, denoted by $\mathbf{0} \in [a, b]$, i.e., $x \oplus \mathbf{0} = x$ for all $x \in [a, b]$.

A pseudo-addition is said to be continuous if it is a continuous function in $[a, b]^2$; a pseudo-addition $\oplus$ is called strict if it is continuous and strictly monotone. Pseudo-addition $\oplus$ is idempotent if for any $x \in [a, b]$, $x \oplus x = x$ holds.

Definition 5 (see [25]). A 2-place function $\odot : [a, b] \times [a, b] \longrightarrow [a, b]$ is called a pseudo-multiplication if it satisfies the following conditions

(i) $\odot$ is commutative,

(ii) $\odot$ is nondecreasing in each place (with respect to $\preceq$),

(iii) $\odot$ is associative,

(iv) There exists a unit element, denoted by $\mathbf{1} \in [a, b]$, i.e., $x \odot \mathbf{1} = x$ for all $x \in [a, b]$.

A pseudo-multiplication is said to be continuous if it is a continuous function in $[a, b]^2$.

For example, the usual addition $+$, $\vee$, and t-conorm are pseudo-additions; the usual multiplication $\times$, $\wedge$, and t-norm are pseudo-multiplications.

The structure $([a, b], \oplus, \odot)$ is called a semiring, where $\odot$ is a distributive pseudo-multiplication (corresponding to $\oplus$); i.e., it is positively nondecreasing ($x \preceq y \Longrightarrow x \odot z \preceq y \odot z$, $z \in [a, b]_+ = \{x : x \in [a, b], \mathbf{0} \preceq x\}$) and $a \odot (x \oplus y) = (a \odot x) \oplus (a \odot y)$. There are three basic classes of semirings with continuous (up to some points) pseudo-operations. The first class contains semirings with idempotent pseudo-addition and nonidempotent pseudo-multiplication. Semirings with strict pseudo-operations defined by monotone and continuous generator function $g : [a, b] \longrightarrow [0, \infty]$, i.e., g-semirings, form the second class. In this paper, we generalize g-semirings to generalized g-semirings. Semirings with both idempotent operations belong to the third class. More on this structure can be found in [5, 11–14].

Total order $\preceq$ on $[a, b]$ is closely connected to the choice of the pseudo-addition. If $\oplus$ is an idempotent operation, total order is induced by $x \preceq y \Longleftrightarrow x \oplus y = y$; if $\oplus$ is given by generalized generator g, total order is given by $x \preceq y \Longleftrightarrow g(x) \leq g(y)$. Additionally, $x \prec y \Longleftrightarrow x \preceq y$ and $x \neq y$.

Let us suppose that the interval $[a, b]$ is endowed with metric. A function $d : [a, b] \times [a, b] \longrightarrow [a, b]$ is a pseudo-metric if it satisfies the conditions (i) $d(x, x) = 0$ for all $x \in X$, (ii) $d(x, y) = d(y, x)$ all $x, y \in X$, and (iii) $d(x, y) \leqslant d(x, z) \oplus d(z, y)$ for all $x, y, z \in X$.

Definition 6 (see [5]). Let $\mathscr{A}$ be a σ-algebra of subsets of X. A set function $\mu : \mathscr{A} \longrightarrow [a, b]_+$ is said to be a σ-$\oplus$-measure if

(i) $\mu(\emptyset) = \mathbf{0}$,

(ii) $\mu(\bigcup_{i=1}^{\infty} A_i) = \bigoplus_{i=1}^{\infty} A_i = \lim_{n \longrightarrow \infty} \bigoplus_{i=1}^{n} \mu(A_i)$, where $(A_i)_{i \in \mathbb{N}}$ is a sequence of pairwise disjoint sets from $\mathscr{A}$.

A pseudo-integral based on σ-$\oplus$-measure μ is defined as ([5])

(i) for elementary function $e = \bigoplus_{i=1}^{\infty} a_i \odot \chi_{A_i}$, where $a_i \in [a, b]$, $A_i \in \mathscr{A}$, and χ_A is the pseudo-characteristic function of A given by $\chi_A(x) = 1_A(x) = \begin{cases} \mathbf{1}, & x \in \mathscr{A}, \\ \mathbf{0}, & x \notin \mathscr{A}, \end{cases}$

$$\int_X^{\oplus, \odot} e d\mu = \bigoplus_{i=1}^{\infty} a_i \odot \mu(A_i). \tag{10}$$

(ii) for bounded measurable function $f : X \longrightarrow [a, b]$,

$$\int_X^{\oplus, \odot} f d\mu = \lim_{n \longrightarrow \infty} \int_X^{\oplus, \odot} \varphi_n d\mu, \tag{11}$$

where φ_n is a sequence of elementary functions such that $d(\varphi_n, f(x)) \longrightarrow 0$ uniformly while $n \longrightarrow \infty$ and d is previously mentioned metric.

(iii) for function f on some arbitrary subset A of X is given by

$$\int_A^{\oplus, \odot} f d\mu = \int_X^{\oplus, \odot} (f \odot \chi_A) d\mu. \tag{12}$$

Notice that $\int_A^{\oplus, \odot} f d\mu$ is also denoted by $\int_A^{\oplus} f \odot d\mu$.

Lemma 7 (Aczel's theorem, [20, 21]). *If $\oplus$ is continuous and strictly increasing in $(a, b) \times (a, b)$, then there exists a monotone function $g : [a, b] \longrightarrow [0, \infty]$ such that $g(\mathbf{0}) = 0$ and*

$$x \oplus y = g^{-1}(g(x) + g(y)), \tag{13}$$

where g is called a generator of $\oplus$.

Obviously, the pseudo-addition $\oplus$ is strict. And the pseudo-multiplication with the generator g of strict pseudo-addition $\oplus$ is defined as

$$x \odot y = g^{-1}(g(x) \cdot g(y)). \tag{14}$$

The pseudo-operations with the generator are also called g-operations.

It is not difficult to obtain the g-power operation

$$x_{\odot}^{p} = \underbrace{x \odot x \odot \cdots \odot x}_{p} = g^{-1}((g(x))^p). \tag{15}$$

Corollary 8. *Let μ be a σ-$\oplus$-measure; $\oplus$ is continuous and strictly increasing in $(a,b)\times(a,b)$, and then there exists a monotone function $g:[a,b]\longrightarrow[0,\infty]$ such that $g(\mathbf{0})=0$ and*

$$g\left(\bigcup_{i=1}^{\infty}A_i\right)=g\left(\bigoplus_{i=1}^{\infty}\mu(A_i)\right)=\sum_{i=1}^{\infty}g(\mu(A_i)). \qquad (16)$$

Proof. According to Lemma 7 and by induction, it is not difficult to obtain

$$g\left(\bigcup_{i=1}^{n}A_i\right)=g\left(\bigoplus_{i=1}^{n}\mu(A_i)\right)=\sum_{i=1}^{n}g(\mu(A_i)), \quad n\geqslant 2. \qquad (17)$$

Let $n\longrightarrow\infty$, we have (16). □

Example 9. Let $m:\mathscr{A}\longrightarrow[0,\infty)$; m is said to be a Sugeno measure ([53]), denoted by g_λ, if

(i) m is normal, i.e., $m(X)=1$,

(ii) m satisfies the σ-λ rule, i.e.,

$$m\left(\bigcup_{i=1}^{\infty}A_i\right)=\begin{cases}\dfrac{1}{\lambda}\left\{\prod_{i=1}^{\infty}[1+\lambda m(A_i)]-1\right\}, & \lambda\neq 0,\\ \sum_{i=1}^{\infty}m(A_i), & \lambda=0,\end{cases} \qquad (18)$$

where $\lambda\in(-1/\sup m,+\infty)\cup\{0\}$, $\{A_i\}\subseteq\mathscr{A}$, and $A_i\cap A_j=\emptyset$, $i\neq j$ $(i,j=1,2,\cdots)$.

Then the sugeno measure g_λ is a σ-$\oplus$-measure, and the generating function for pseudo-addition $\oplus$ is

$$g(x)=\begin{cases}\dfrac{\ln(1+\lambda x)}{\ln(1+\lambda)}, & \lambda\neq 0,\\ x, & \lambda=0,\end{cases} \qquad (19)$$

then

$$g^{-1}(x)=\begin{cases}\dfrac{(1+\lambda)^x-1}{\lambda}, & \lambda\neq 0,\\ x, & \lambda=0.\end{cases} \qquad (20)$$

Obviously, if $\lambda=0$, then we have $\bigoplus_{i=1}^{\infty}g_\lambda(A_i)=g^{-1}(\sum_{i=1}^{\infty}g(g_\lambda(A_i)))$; if $\lambda\neq 0$, then we have

$$\begin{aligned}&g^{-1}(g(g_\lambda(A_1))+g(g_\lambda(A_2)))\\&=g^{-1}\left(\frac{\ln(1+\lambda g_\lambda(A_1))}{\ln(1+\lambda)}+\frac{\ln(1+\lambda g_\lambda(A_2))}{\ln(1+\lambda)}\right)\\&=\frac{(1+\lambda)^{\ln(1+\lambda g_\lambda(A_1))(1+\lambda g_\lambda(A_2))/\ln(1+\lambda)}-1}{\lambda}\\&=\frac{\ln(1+\lambda g_\lambda(A_1))(1+\lambda g_\lambda(A_2))-1}{\lambda}\\&=g_\lambda(A_1)\oplus g_\lambda(A_2).\end{aligned} \qquad (21)$$

By induction, we obtain

$$\bigoplus_{i=1}^{\infty}g_\lambda(A_i)=g^{-1}\left(\sum_{i=1}^{\infty}g(g_\lambda(A_i))\right). \qquad (22)$$

Moreover, notice that if $\lambda=0$, then the σ-λ rule is σ-+-additive, i.e., σ-additive, and g_λ is the probability measure; if $\lambda\neq 0$ and $i=2$, we have

$$\begin{aligned}g_\lambda(A\cup B)&=g_\lambda(A)\oplus g_\lambda(B)\\&=g_\lambda(A)+g_\lambda(B)+\lambda g_\lambda(A)g_\lambda(B),\end{aligned} \qquad (23)$$

when $\lambda\neq -1$, g_λ is said to be λ-additive; when $\lambda=-1$, it is the addition formula of Probability.

Definition 10 (see [54]). Let g be a strictly monotone real-valued function defined on $[a,b]\subseteq\overline{R}$ such that $0\in\operatorname{ran}(g)$. The generalized generated pseudo-addition $\oplus$ and the generalized generated pseudo-multiplication $\odot$ are given by

$$x\oplus y=g^{(-1)}(g(x)+g(y)), \qquad (24)$$

$$x\odot y=g^{(-1)}(g(x)\cdot g(y)), \qquad (25)$$

where $g^{(-1)}$ is pseudo-inverse function for function g.

Remark 11. For nondecreasing function $f:[a,b]\longrightarrow[a_1,b_1]$, where $[a,b]$ and $[a_1,b_1]$ are closed subintervals of the generalized real line $\overline{R}$, the pseudo-inverse is $f^{-1}(y)=\sup\{x\in[a,b]\mid f(x)<y\}$. If f is nonincreasing function, its pseudo-inverse is $f^{-1}(y)=\sup\{x\in[a,b]\mid f(x)>y\}$. More on this subject can be found in ([55]).

The semiring $([a,b],\oplus,\odot)$ is called a generalized g-semiring, where pseudo-operations defined by the generalized generator function $g:[a,b]\longrightarrow[-\infty,\infty]$.

Proposition 12. *Let μ be a σ-$\oplus$-measure and $\oplus$ be generalized generated by a generator g.*

(i) If the generating function $g:[a,b]\longrightarrow[-\infty,\infty]$ is either strictly increasing right-continuous or strictly decreasing left-continuous function such that $\infty\in Rag(g)$, then

$$\bigoplus_{i=1}^{\infty}\mu(A_i)\geq g^{(-1)}\left(\sum_{i=1}^{\infty}g(\mu(A_i))\right), \quad i\in\mathbb{N}. \qquad (26)$$

(ii) If the generating function $g:[a,b]\longrightarrow[-\infty,\infty]$ is either strictly increasing left-continuous or strictly decreasing right-continuous function such that $-\infty\in Rag(g)$, then

$$\bigoplus_{i=1}^{\infty}\mu(A_i)\leq g^{(-1)}\left(\sum_{i=1}^{\infty}g(\mu(A_i))\right), \quad i\in\mathbb{N}. \qquad (27)$$

(iii) If the generating function $g:[a,b]\longrightarrow[-\infty,\infty]$ is a monotone bijection, then

$$\bigoplus_{i=1}^{\infty}\mu(A_i)=g^{(-1)}\left(\sum_{i=1}^{\infty}g(\mu(A_i))\right), \quad i\in\mathbb{N}. \qquad (28)$$

Proof. For g strictly increasing right-continuous or strictly decreasing left-continuous generating function that fulfills condition $g(b)=\infty$ or $g(a)=\infty$, respectively, holds $gg^{(-1)}\geq x$, for all $x\in[-\infty,\infty]$. According to Definition 10 and by induction, we have

$$\bigoplus_{i=1}^{\infty}\mu(A_i)\geq g^{(-1)}\left(\sum_{i=1}^{\infty}g(\mu(A_i))\right),\quad i\in\mathbb{N}. \tag{29}$$

Proof for (ii) is similar and based on $gg^{(-1)}\leq x$, for all $x\in[-\infty,\infty]$. In (iii) pseudo-inverse coincides with inverse, which gives us (28). □

Remark 13. If the generalized generator is a monotone bijection, then the pseudo-inverse coincides with the inverse. We have

$$\int_A^{\oplus,\odot} f d\mu = g^{-1}\left(\int_A gfdg\mu\right)=g^{(-1)}\left(\int_A gfdg\mu\right). \tag{30}$$

The integral is said to be g-integral ([11, 16, 17, 19, 22]). For the sake of brevity, we denote $d(g\mu)=dg\mu$.

In addition, the generalized g-power operation is

$$x_{\odot}^{p}=\underbrace{x\odot x\odot\cdots\odot x}_{p}=g^{(-1)}\left((g(x))^{p}\right). \tag{31}$$

We give the definition of comonotonic, which is similar to the definition of comonotonic [6] or compatible ([51]) in real-analysis. Let f and g be generalized real-valued bounded measurable functions on X. We say f and g are comonotonic, denoted by $f\sim g$, if $f(x)\prec f(x')\Longrightarrow g(x)\preceq g(x')$ for $x,x'\in X$. We denote by $B(X,\mathscr{A})$ the set of bounded measurable functions on $(X,\mathscr{A})$. Let I be a functional defined on $B(X,\mathscr{A})$.

Definition 14. (1) I is said to be comonotonically $\oplus$-additive if $f\sim g\Longrightarrow I(f\oplus g)=I(f)\oplus I(g)$.

(2) I is said to be positively $\odot$-homogeneous if $I(a\odot f)=a\odot I(f)$, $a\succ\mathbf{0}$.

(3) I is said to be monotonic if $f\preceq g\Longrightarrow I(f)\preceq I(g)$.

The total variation $V(I)$ of I is defined by ([9])

$$\begin{aligned}V(I)=\sup\Bigg\{&\sum_{i=1}^{n}\left|I(f_i)-I(f_{i-1})\right|\mid 0=f_0\leq f_1\leq\cdots\\&\leq f_n=1,\ \{f_i\}_{i=0}^{n}\subseteq B(X,\mathscr{A})\Bigg\}.\end{aligned} \tag{32}$$

I is said to be of bounded variation if $V(I)<\infty$. Note that if I is monotonic, then $V(1)=I(1)$ and hence I is of bounded variation.

For every pair of f and g of functions in $B(X,\mathscr{A})$ for which $f\leq g$, we define $V_I(f,g)$ by

$$\begin{aligned}V_I(f,g)=\sup\Bigg\{&\sum_{i=1}^{n}\left|I(f_i)-I(f_{i-1})\right|\mid f=f_0\leq f_1\\&\leq\cdots\leq f_n=g,\ \{f_i\}_{i=0}^{n}\subseteq B(X,\mathscr{A})\Bigg\}.\end{aligned} \tag{33}$$

3. Choquet-Like Integral with respect to a Nonmonotonic Fuzzy Measure

In this section, we introduce the Choquet-like integrals based on $\oplus$ and $\odot$ with respect to (w.r.t) nonmonotonic fuzzy measures for generalized real-valued functions and set-valued functions. In addition, the characterization theorem and transformation theorem for the integrals are given.

3.1. Choquet-Like Integral with respect to a Nonmonotonic Fuzzy Measure for Generalized Real-Valued Functions

Definition 15. Let m be a nonmonotonic fuzzy measure, μ be a σ-$\oplus$-measure satisfying $\mu([\mathbf{0},\alpha])=\alpha$ for $\alpha\in[\mathbf{0},\infty]$ and $\mu([\alpha,\mathbf{0}])=-\alpha$ for $\alpha\in[-\infty,\mathbf{0}]$, and $\oplus$ be a given pseudo-addition and corresponding a pseudo-multiplication $\odot$. Let $f: X\longrightarrow\overline{R}$ be a $\mathscr{A}$-measurable function and $A\in\mathscr{A}$ be a measurable set. Then the integral of f with respect to the nonmonotonic fuzzy measure m over A defined by

$$(Cl)\int_A^{\oplus,\odot} fdm=\int_A^{\oplus,\odot} m(f_a\cap A)\,d\mu, \tag{34}$$

where

$$\begin{aligned}&m(f_a\cap A)\\&\quad=\begin{cases}m(\{x\mid f(x)\geqslant\alpha\}\cap A), & \alpha\geqslant 0,\\ m(\{x\mid f(x)\geqslant\alpha\}\cap A)-m(X), & \alpha<0,\end{cases}\end{aligned} \tag{35}$$

will be called a Choquet-like integral if it is

(1) monotone; i.e., $f\preceq g$ implies $(Cl)\int_A^{\oplus,\odot} fdm\preceq(Cl)\int_A^{\oplus,\odot} gdm$,

(2) comonotone $\oplus$-additive; i.e., $f\sim g$ implies $(Cl)\int_A^{\oplus,\odot} f\oplus gdm=(Cl)\int_A^{\oplus,\odot} fdm\oplus(Cl)\int_A^{\oplus,\odot} gdm$,

(3) positively $\odot$-homogeneous; i.e., $(Cl)\int_A^{\oplus,\odot} c\odot fdm=c\odot(Cl)\int_A^{\oplus,\odot} fdm$ for $c\succ\mathbf{0}$,

(4) coincident; i.e., $(Cl)\int_A^{\oplus,\odot} fdm$ is $\oplus$-additive if and only if m is $\oplus$-additive; if μ is continuous from below, $\oplus$-additive and nonnegative, and f is nonnegative, then $(Cl)\int_A^{\oplus,\odot} fdm=(SM)\int_A^{\oplus,\odot} fdm$.

Instead of $(Cl)\int_A^{\oplus,\odot} fdm$, we shall write $(Cl)\int^{\oplus,\odot} fdm$. If the Choquet-like integral of a measurable function f exists and its value is finite, we say f is Choquet-like integrable, denoted by $I_{m,\mu}^{\oplus,\odot}(f)$, i.e., $I_{m,\mu}^{\oplus,\odot}(f)=(Cl)\int_A^{\oplus,\odot} fdm$.

Remark 16. If $\oplus=+$, $\odot=\cdot$, the Choquet-like integral coincides with the Choquet integral w.r.t. a nonmonotonic fuzzy measure introduced in ([9]); if $\oplus=\vee$, $\odot=\wedge$, m is monotone, and f is nonnegative, the Choquet-like integral coincides with the Sugeno integral; if $\oplus,\odot$ are continuous, μ is continuous from below and nonnegative, and f is nonnegative, then the Choquet-like integral is coincident with the Choquet-like integral introduced by Mesiar ([18]).

Proposition 17. *Let f, g be Choquet-like integrable. If $f \sim g$, then for any real numbers $a, b \succ \mathbf{0}$, we have*

$$(Cl)\int^{\oplus,\odot}(a \odot f \oplus b \odot g)\,dm = a \odot (Cl)\int^{\oplus,\odot} fdm \oplus b \odot (Cl)\int^{\oplus,\odot} gdm. \tag{36}$$

Proof. Since f, g is Choquet-like integrable and $f \sim g$, according to Definition 15, $(Cl)\int^{\oplus,\odot} fdm$ and $(Cl)\int^{\oplus,\odot} gdm$ are comonotone $\oplus$-additive and positively $\odot$-homogeneous; that is, for $a, b > \mathbf{0}$, we have

$$\begin{aligned}(Cl)\int^{\oplus,\odot}(a \odot f \oplus b \odot g)\,dm &= (Cl)\int^{\oplus,\odot}(a \odot f)\,dm \oplus (Cl)\int^{\oplus,\odot}(b \odot g)\,dm \\ &= a \odot (Cl)\int^{\oplus,\odot} fdm \oplus b \odot (Cl)\int^{\oplus,\odot} gdm.\end{aligned} \tag{37}$$

□

Corollary 18. *Let f, g be Choquet-like integrable. If $f \sim g$, then for every fuzzy measure λ on $\mathscr{A}$, we have*

$$(C)\int (f+g)\,d\lambda = (C)\int fd\lambda + (C)\int gd\lambda, \tag{38}$$

where $(C)\int$ is the ordinary Choquet integral w.r.t fuzzy measure λ.

This result was proved with use of the representation theory of fuzzy measures by Murofushi-Sugeno in [51].

Theorem 19. *Let I be a functional defined on $B(X, \mathscr{A})$, $m \in BV(X, \mathscr{A})$, and $I = I^{\oplus,\odot}_{m,\mu}$. If $m(A) = I(1_A)$, $\forall A \in \mathscr{A}$, then I is comonotonically $\oplus$-additive, positively $\odot$-homogeneous, and of bounded variation.*

Proof. Since $I = I^{\oplus,\odot}_{m,\mu}$, according to Definition 15, I is comonotonically $\oplus$-additive, positively $\odot$-homogeneous. Then we prove $I^{\oplus,\odot}_{m,\mu}$ is of bounded variation. It follows from the definition of the total variation that $V(I+I') \le V(I) + V(I')$ and $V(-I) = V(I)$. Since $m \in \mathrm{BV}(X, \mathscr{A})$, we have $m = m^+ - m^-$, and $m^+, m^- \in \mathrm{FM}(X, \mathscr{A})$, thus, we obtain

$$\begin{aligned}V\left(I^{\oplus,\odot}_{m,\mu}\right) &= V\left(I^{\oplus,\odot}_{m^+,\mu} - I^{\oplus,\odot}_{m^-,\mu}\right) \le V\left(I^{\oplus,\odot}_{m^+,\mu}\right) + V\left(I^{\oplus,\odot}_{m^-,\mu}\right) \\ &< \infty;\end{aligned} \tag{39}$$

that is, $I^{\oplus,\odot}_{m,\mu}$ is of bounded variation. Therefore, I is of bounded variation. □

Theorem 20. *Let $\oplus$ and $\odot$ be generalized generated by a generator g. If g is a monotone bijection, then the Choquet-like integral of a measurable function $f : X \longrightarrow \overline{R}$ over a measurable set $A \in \mathscr{A}$ w.r.t. a nonmonotonic fuzzy measure m can be represented as*

$$\begin{aligned}(Cl)\int_A^{\oplus,\odot} fdm &= g^{(-1)}\left((C)\int_A gfdgm\right) \\ &= g^{(-1)}\left(\int_{-\infty}^{\infty} g\left(m\left((gf)_y \cap A\right)\right)dy\right),\end{aligned} \tag{40}$$

where $(gf)_y = \{x \in X \mid (gf)(x) \geqslant y\}$, $g^{(-1)}$ is pseudo-inverse function for function g, $(C)\int$ denotes the Choquet integral w.r.t a nonmonotonic fuzzy measure, and the right-hand side integral is the Lebesgue integral.

Proof. Let $a \oplus b = g^{(-1)}(g(a) + g(b))$ and μ be a σ-$\oplus$-measure. Since g is a monotone bijection, $gg^{(-1)}(x) = x$, for disjoint sets $A, B \in \mathscr{A}$, we have

$$\begin{aligned}g\mu(A \cup B) &= g(\mu(A) \oplus \mu(B)) \\ &= g\left(g^{(-1)}(g(\mu(A)) + g(\mu(B)))\right) \\ &= g(\mu(A)) + g(\mu(B)) \\ &= g\mu(A) + g\mu(B).\end{aligned} \tag{41}$$

By induction, it is easy to see that $g\mu$, $g\mu(A) = g(\mu(A))$, $A \in \mathscr{A}$, is a $+$-decomposable measure, i.e., a σ-additive measure. Consequently, $g\mu$ is a σ-additive measure on Borel subsets of $[-\infty, \infty]$ such that $g\mu([\mathbf{0}, \alpha]) = g(\alpha)$ for each $\alpha \in [\mathbf{0}, \infty]$ and $g\mu([\alpha, \mathbf{0}]) = g(-\alpha)$ for each $\alpha \in [-\infty, \mathbf{0}]$. According to Definition 15, we have

$$(Cl)\int_A^{\oplus,\odot} fdm = \int_A^{\oplus,\odot} m(f_a \cap A)\,d\mu. \tag{42}$$

Since g is a monotone bijection, we see that

$$(Cl)\int_A^{\oplus,\odot} fdm = g^{(-1)}\left(\int_{[-\infty,+\infty]} gm(f_\alpha \cap A)\,dg\mu\right), \tag{43}$$

where the integral on the right-hand side is the Lebesgue integral on $[-\infty, \infty]$. Let g' be a function on $[-\infty, \infty]$ identical with the first derivative of the generalized generator g in those points, where this derivative exists (recall that g is a strictly monotone function). Then

$$\begin{aligned}&\int_{[-\infty,\infty]} gm(f_\alpha \cap A)\,dg\mu \\ &\quad= \int_{[-\infty,\infty]} gm(f_\alpha \cap A) \cdot g'd\mu \\ &\quad= \int_{-\infty}^{\infty} gm(f_x \cap A) \cdot g'(x)\,dx \\ &\quad= \int_{-\infty}^{\infty} gm\left((gf)_y \cap A\right)dy = (C)\int_A gfdgm,\end{aligned} \tag{44}$$

where μ is the common Lebesgue measure on $[-\infty, \infty]$ and the substitution $y = g(x)$ is used. Therefore,

$$(Cl)\int_A^{\oplus,\odot} fdm = g^{(-1)}\left((C)\int_A gfdgm\right) = g^{(-1)}\left(\int_{-\infty}^{\infty} g\left(m\left((gf)_y \cap A\right)\right)dy\right). \tag{45}$$

□

Remark 21. If $g(x) = x$ and m is monotone, then the integral coincides with the symmetric Choquet integral. Moreover, when f is nonnegative, the integral coincides with the original Choquet integral.

The theorem shows that Choquet-like integral w.r.t a nonmonotonic fuzzy measure can be transformed into the Choquet integral w.r.t a nonmonotonic fuzzy measure and the Lebesgue integral.

Note that the Choquet-like based on the g-operations will be called a g-Choquet-like integral and we denote $I_m^g = (Cl)\int_A^{\oplus,\odot} fdm$.

Proposition 22. *Let m and n be nonmonotonic fuzzy measures and $\oplus$ and $\odot$ be generalized generated by a generator g. If g is a monotone bijection, then for any real numbers a and b, we have*

$$(Cl)\int^{\oplus,\odot} fd(a\odot m\oplus b\odot n) = a\odot(Cl)\int^{\oplus,\odot} fdm\oplus b\odot(Cl)\int^{\oplus,\odot} fdn. \tag{46}$$

Proof. Since g is a monotone bijection, $gg^{(-1)}(x) = x$, according to Theorem 20, we have

$$\begin{aligned}(Cl)&\int^{\oplus,\odot} fd(a\odot m\oplus b\odot n)\\ &= g^{(-1)}\left((C)\int gfdg(a\odot m\oplus b\odot n)\right)\\ &= g^{(-1)}\left((C)\int gfd(g(a)g(m)+g(b)g(n))\right)\\ &= g^{(-1)}\left((C)\int gfdg(a)g(m)\right.\\ &\quad\left.+(C)\int gfdg(b)g(n)\right).\end{aligned} \tag{47}$$

Since the Choquet integral is linear with respect to the nonmonotonic fuzzy measure, we obtain

$$\begin{aligned}(Cl)&\int^{\oplus,\odot} fd(a\odot m\oplus b\odot n)\\ &= g^{(-1)}\left(g(a)(C)\int gfdgm+g(b)(C)\int gfdgn\right)\\ &= g^{(-1)}\left(g(a)\left(gg^{(-1)}\left((C)\int gfdgm\right)\right)\right.\\ &\quad\left.+g(b)\left(gg^{(-1)}\left((C)\int gfdgn\right)\right)\right)\\ &= g^{(-1)}\left(g(a)g\left((Cl)\int^{\oplus,\odot} fdm\right)\right.\\ &\quad\left.+g(b)g\left((Cl)\int^{\oplus,\odot} fdn\right)\right) = a\odot(Cl)\int^{\oplus,\odot} fdm\\ &\quad\oplus b\odot(Cl)\int^{\oplus,\odot} fdn.\end{aligned} \tag{48}$$

□

Lemma 23 (see [6]). *If I is a continuous, comonotonically additive functional on $B(X,\mathscr{A})$ and if $m(A) = I(1_A)$, $\forall A\in\mathscr{A}$, then I is positively homogeneous and $I = I_m$.*

Theorem 24. *Let I be a functional defined on $B(X,\mathscr{A})$. Then the following conditions are equivalent to one another:*

(a) If $m(A) = I(1_A)$, $\forall A\in\mathscr{A}$, then $m\in BV(X,\mathscr{A})$ and $I = I_m^g$.

(b) I is comonotonically $\oplus$-additive, positively $\odot$-homogeneous, and of bounded variation.

(c) I is comonotonically $\oplus$-additive and uniformly continuous.

Proof. (a)$\Longrightarrow$(b). It can be easily obtained by Theorem 19.

(b)$\Longrightarrow$(c). Let $I : B(X,\mathscr{A}) \longrightarrow R$ be a positively $\odot$-homogeneous and comonotonically $\oplus$-additive functional of bounded variation and $f, g\in B(X,\mathscr{A})$. If we write $a = \|f-g\|$, then obviously $f-a\le g\le f+a$, and hence it follows that

$$\begin{aligned}|I(f)-I(g)| &\le |I(f)-I(g)|+|I(f)-I(g)|\\ &\le |I(a)|+V_I(f-a,f+a)\\ &\le a|I(1)|+2aV(I)\\ &= (|I(1)|+2V(I))\cdot\|f-g\|.\end{aligned} \tag{49}$$

Therefore, I is uniformly continuous.

(c)$\Longrightarrow$(a). Let $m(A) = I(1_A)$, $\forall A\in\mathscr{A}$. Since the uniform continuity implies the continuity, it follows from Theorem 20 and Lemma 23 that $I = I_m^g$. We shall prove that $m\in BV(X,\mathscr{A})$. Assume that m is not of bounded variation. Then for each positive integer t there is a finite sequence $\{A_{t,i}\}_{i=1}^{s(t)}\subseteq\mathscr{A}$ such that $\emptyset = A_{t,()}\subseteq\cdots\subseteq A_{t,s(t)} = X$ and

$$\sum_{i=1}^{s(t)}\left|m(A_{t,i})-m(A_{t,i-1})\right| > t. \tag{50}$$

We now put $f_t = (1/t)\sum_{i=1}^{s(t)} 1_{B_{t,i}}$ and $g_t = (1/t)\sum_{i=1}^{s(t)} 1_{C_{t,i}}$, where

$$\begin{aligned}B_{t,i} &= \begin{cases}A_{t,i}, & m(A_{t,i})\ge m(A_{t,i-1}),\\ A_{t,i-1}, & \text{otherwise},\end{cases}\\ C_{t,i} &= \begin{cases}A_{t,i-1}, & m(A_{t,i})\ge m(A_{t,i-1}),\\ A_{t,i}, & \text{otherwise},\end{cases}\end{aligned} \tag{51}$$

and then it is easy to see that $\|f_t - g_t\| = 1/t$ but

$$\begin{aligned} \left|I(f_t) - I(g_t)\right| &= \frac{1}{t}\left|\sum_{i=1}^{s(t)}\left[m(B_{t,i}) - m(C_{t,i})\right]\right| \\ &= \frac{1}{t}\sum_{i=1}^{s(t)}\left|m(A_{t,i}) - m(A_{t,i-1})\right| > 1, \end{aligned} \tag{52}$$

and thus, I is not uniformly continuous. This contradicts the fact that I is uniformly continuous. Therefore, m is of bounded variation. □

Corollary 25. *If m is monotonic, then a functional I on $B(X,\mathscr{A})$ is represented as a g-Choquet-like integral with respect to the monotonic fuzzy measure m if and only if I is a monotonic and comonotonically $\oplus$-additive.*

3.2. Choquet-Like Integral with respect to a Nonmonotonic Fuzzy Measure for Set-Valued Functions. A set-valued mapping is a mapping $F : X \longrightarrow \mathscr{P}(\overline{R}) \setminus \{\emptyset\}$, and it is said to be measurable if $F^{-1}(B) = \{x \in X \mid F(x) \cap B \neq \emptyset\} \in \mathscr{A}$ for every $B \in \mathscr{B}(\overline{R})$, where $\mathscr{B}(\overline{R})$ is the Borel algebra of $\overline{R}$. Let $F, G : X \longrightarrow \mathscr{P}(\overline{R}) \setminus \{\emptyset\}$ be measurable set-valued mappings and m be a nonmonotonic fuzzy measure on $(X,\mathscr{A})$. If $m(\{x \mid F(x) = G(x)\}) = 0$, then we say F equals G almost everywhere, denoted by $F = G$ a.e.

Definition 26. Let F be a set-valued function and $A \in \mathscr{A}$. Then the Choquet-like integral of F on A is defined by

$$(Cl)\int_A^{\oplus,\odot} Fdm = \left\{(Cl)\int_A^{\oplus,\odot} fdm : f \in S_{Cl}(F)\right\}, \tag{53}$$

where $S_{Cl}(F)$ is the family of Choquet-like integrable selections of F, i.e.,

$$\begin{aligned} &S_{Cl}(F) \\ &\quad = \left\{f \in L^1_{Cl}(m) : f(x) \in F(x) \ \text{on } X \ m-a.e.\right\}, \\ &L^1_{Cl}(m) = \left\{f : X \longrightarrow [a,b]_+ \mid (Cl)\int_X^{\oplus,\odot} fdm < \infty\right\}. \end{aligned} \tag{54}$$

Specially, when $\int^{\oplus,\odot}$ coincides with Lebesgue integral and F is nonnegative, the set-valued Choquet-like integral is the classical Aumann's integral.

For a set-valued function $F : X \longrightarrow \mathscr{P}(\overline{R}) \setminus \{\emptyset\}$, we say that it is pseudo-integrable on some $A \in \mathscr{A}$ if $\int_A^{\oplus,\odot} Fdm \neq \emptyset$.

Let $A, B \in \mathscr{P}(\overline{R}) \setminus \{\emptyset\}$; we say $A \preceq B$ if for all $x \in A$ there exists $y \in B$ such that $x \preceq y$ and for all $y \in B$ there exists $x \in A$ such that $x \preceq y$.

Definition 27. A set-valued function F is said to be Choquet-like integrably bounded if there is a function $h \in L^1_{Cl}(m)$ such that

(i) $\bigoplus_{\omega \in F(x)} \omega \preceq h(x)$, for the idempotent pseudo-addition,

(ii) $\sup_{\omega \in F(x)} \omega \preceq h(x)$, for the pseudo-addition given by an increasing generalized generator g,

(iii) $\inf_{\omega \in F(x)} \omega \preceq h(x)$, for the pseudo-addition given by a decreasing generalized generator g.

Note that if $\oplus = \sup$ and nonnegative, i.e., $\oplus$ is idempotent, then the definition of Choquet-like integrably bounded is coincident with the definition of Choquet integrably bounded proposed in [28].

Theorem 28. *If F is a Choquet-like integrably bounded set-valued function, then F is Choquet-like integrable.*

Proof. Let F be a Choquet-like integrably bounded set-valued function; that is, let us suppose that the function h from Definition 27 exists. If $h(x) \in F(x)$ on X m-a.e., the set (53) is obviously not empty. If $h(x) \notin F(x)$, let f be a selection of F, i.e., $f(x) \in F(x)$ m-a.e. on X. It can be easily shown that $f \preceq h$ holds almost everywhere. According to Definition 15, we have

$$(Cl)\int_X^{\oplus,\odot} fdm \preceq (Cl)\int_X^{\oplus,\odot} hdm. \tag{55}$$

Since $h \in L^1_{Cl}$, h is a Choquet-like integrable function, thus, the function f is also Choquet-like integrable and the set (53) is not empty.

For example, if $\oplus$ and $\odot$ be generalized generated by a generator g, then the Choquet-like integral of some set-valued function F is

$$\begin{aligned} &(Cl)\int_X^{\oplus,\odot} Fdm \\ &\quad = \left\{g^{(-1)}\left(\int gm\left((gf)_y\right)dy\right) \mid f \in S_{Cl}(F)\right\}. \end{aligned} \tag{56}$$

□

Proposition 29. *Let F be pseudo-integrable set-valued function, F_1 and F_2 Choquet-like integrably bounded set-valued functions and let $A, B \in \mathscr{A}$.*

(1) If $A \subseteq B$, $A, B \in \mathscr{A}$, then $(Cl)\int_A^{\oplus,\odot} Fdm \preceq (Cl)\int_B^{\oplus,\odot} Fdm$.

(2) If $m(A) = \mathbf{0}$, then $(Cl)\int_A^{\oplus,\odot} Fdm = \{\mathbf{0}\}$.

(3) If $a \succ \mathbf{0}$, then $(Cl)\int_A^{\oplus,\odot} a \odot Fdm = a \odot (Cl)\int_A^{\oplus,\odot} Fdm$.

Proof. (1) Suppose that $x \in (Cl)\int_A^{\oplus,\odot} Fdm$. By Definition 27, there exists $f \in S_{Cl}(F)$ such that $x = (Cl)\int_A^{\oplus,\odot} fdm$. According to the definition of pseudo-integral, $\int_A^{\oplus,\odot} fd\mu = \int_X^{\oplus,\odot}(f \odot \chi_A)d\mu$. Since $A \subseteq B$, we have $f \odot \chi_A \preceq f \odot \chi_B$, thus, $\int_X^{\oplus,\odot}(f \odot \chi_A)d\mu \preceq \int_X^{\oplus,\odot}(f \odot \chi_B)d\mu$; according to Definition 15, we obtain $(Cl)\int_A^{\oplus,\odot} fdm \preceq (Cl)\int_B^{\oplus,\odot} fdm \in (Cl)\int_B^{\oplus,\odot} Fdm$. Thus, there exists $y = (Cl)\int_B^{\oplus,\odot} fdm \in (Cl)\int_B^{\oplus,\odot} Fdm$, such that $x \preceq y$. Similarly, we can prove that for $y \in (Cl)\int_B^{\oplus,\odot} Fdm$, there exists $x \in (Cl)\int_A^{\oplus,\odot} Fdm$, such that $x \preceq y$.

The statements (2) and (3) follow directly from Definition 15. □

Proposition 30. *Let $F : X \longrightarrow \mathscr{P}(\overline{R}) \setminus \{\emptyset\}$ be a measurable set-valued mapping and $A \in \mathscr{A}$; then*

$$(Cl)\int_A^{\oplus,\odot} Fdm = (Cl)\int_X^{\oplus,\odot} \chi_A \odot Fdm, \tag{57}$$

where $(\chi_A \odot F)(x) = \begin{cases} F(x), & x \in A, \\ \{\mathbf{0}\}, & x \notin A. \end{cases}$

Proof. By the definition of pseudo-integral, we have $\int_A^{\oplus,\odot} fd\mu = \int_X^{\oplus,\odot}(f \odot \chi_A)d\mu$. Thus, according to Definition 15, this statement holds. □

Definition 31 (see [29]). Set $A \subseteq [a,b]$ is pseudo-convex if $A = \emptyset$ or if for all $x, y \in A$ and $a, b \in [a,b]_+$, where $a \oplus b = \mathbf{1}$, $a \odot x \oplus b \odot y \in A$ holds.

Theorem 32. *Let F be a set-valued function such that the sets $F(x)$ are pseudo-convex for all $x \in X$. Then $(Cl)\int_X^{\oplus,\odot} Fdm$ is a pseudo-convex subset of $[a,b]_+$.*

Proof. If F is not Choquet-like integrable, i.e., the set (53) is empty, then this claim trivially holds.

If (53) is not an empty set, let us suppose that there are some values x and y from $(Cl)\int_X^{\oplus,\odot} Fdm$ such that $x = (Cl)\int_X^{\oplus,\odot} fdm$, $y = (Cl)\int_X^{\oplus,\odot} hdm$, and $f, h \in S_{Cl}(F)$. Since the sets F are pseudo-convex, $a \odot f \oplus b \odot h \in S_{Cl}(F)$ for $a, b \in [a,b]_+$ fulfilling $a \oplus b = \mathbf{1}$. Thus, according to Proposition 17, we have

$$\begin{aligned} a \odot x \oplus b \odot y &= a \odot (Cl)\int_X^{\oplus,\odot} fdm \oplus b \\ &\quad \odot (Cl)\int_X^{\oplus,\odot} hdm \\ &= (Cl)\int_X^{\oplus,\odot} (a \odot f \oplus b \odot h)\, dm \\ &\in (Cl)\int_X^{\oplus,\odot} Fdm. \end{aligned} \tag{58}$$

Therefore, $(Cl)\int_X^{\oplus,\odot} Fdm$ is pseudo-convex. □

4. Lyapunov Type Inequality for the Choquet-Like Integral

In this section, we discuss the Lyapunov and Stolarsky type inequality for the Choquet-like integral based on the semiring $([0,1], \oplus, \odot)$. Without loss generality, suppose that $\mathbf{0} \prec \mathbf{1}$.

According to probability theory, the classical Liapunov inequality provides the inequality ([56])

$$\begin{aligned} &\left(\int_0^1 f(x)^s\, dx\right)^{r-t} \\ &\quad \le \left(\int_0^1 f(x)^t\, dx\right)^{r-s} \left(\int_0^1 f(x)^r\, dx\right)^{s-t}, \end{aligned} \tag{59}$$

where $0 < t < s < r$, $f : [0,1] \longrightarrow [0,\infty)$ is an integrable function, and x is the Lebesgue measure on R. Inequality (59) shows an interesting upper bound for the Lebesgue integral of the product of two functions.

Theorem 33. *Let the generalized generator $g : [\mathbf{0},\mathbf{1}] \longrightarrow [0,1]$ of $\oplus, \odot$ be a monotone bijection, and $f : [0,1] \longrightarrow [\mathbf{0},\mathbf{1}]$ be a measurable function. Then for $\mathbf{0} < t < s < r$, the inequality holds:*

$$\begin{aligned} &\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_\odot^s dm\right)_\odot^{r-t} \\ &\quad \preceq \left((Cl)\int_{[0,1]}^{\oplus,\odot} f_\odot^t dm\right)_\odot^{r-s} \odot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f_\odot^r dm\right)_\odot^{s-t}. \end{aligned} \tag{60}$$

Proof. Using the classical Lyapunov inequality, then we obtain

$$\begin{aligned} &\left(\int_0^1 gm\left(\left((gf)^s\right)_y\right) dy\right)^{r-t} \\ &\quad \le \left(\int_0^1 gm\left(\left((gf)^t\right)_y\right) dy\right)^{r-s} \\ &\quad \cdot \left(\int_0^1 gm\left(\left((gf)^r\right)_y\right) dy\right)^{s-t}. \end{aligned} \tag{61}$$

Since the function g is a monotone bijection, $gg^{(-1)}(x) = x$, then there is

$$\begin{aligned} &\left(\int_0^1 gm\left(\left(gg^{(-1)}(gf)^s\right)_y\right) dy\right)^{r-t} \\ &\quad \le \left(\int_0^1 gm\left(\left(gg^{(-1)}(gf)^t\right)_y\right) dy\right)^{r-s} \\ &\quad \cdot \left(\int_0^1 gm\left(\left(gg^{(-1)}(gf)^r\right)_y\right) dy\right)^{s-t}, \end{aligned} \tag{62}$$

and thus, we have

$$\begin{aligned} &\left(g\left(g^{(-1)}\left(\int_0^1 gm\left(\left(gg^{(-1)}(gf)^s\right)_y\right) dy\right)\right)\right)^{r-t} \\ &\quad \le \left(g\left(g^{(-1)}\left(\int_0^1 gm\left(\left(gg^{(-1)}(gf)^t\right)_y\right) dy\right)\right)\right)^{r-s} \\ &\quad \cdot \left(g\left(g^{(-1)}\left(\int_0^1 gm\left(\left(gg^{(-1)}(gf)^r\right)_y\right) dy\right)\right)\right)^{s-t}, \end{aligned} \tag{63}$$

According to Theorem 20, we have

$$\begin{aligned} &\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} g^{(-1)}\left((gf)^s\right) dm\right)\right)^{r-t} \\ &\quad \le \left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} g^{(-1)}\left((gf)^t\right) dm\right)\right)^{r-s} \\ &\quad \cdot \left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} g^{(-1)}\left((gf)^r\right) dm\right)\right)^{s-t}, \end{aligned} \tag{64}$$

that is,

$$\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{s}dm\right)\right)^{r-t} \leq \left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{t}dm\right)\right)^{r-s} \cdot \left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{r}dm\right)\right)^{s-t}. \quad (65)$$

Therefore,

$$g\left(g^{(-1)}\left(\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{s}dm\right)\right)^{r-t}\right)\right) \leq g\left(g^{(-1)}\left(\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{t}dm\right)\right)^{r-s}\right)\right) \cdot g\left(g^{(-1)}\left(\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{r}dm\right)\right)^{s-t}\right)\right), \quad (66)$$

and it implies that

$$g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{s}dm\right)_{\odot}^{r-t}\right) \leq g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{t}dm\right)_{\odot}^{r-s}\right) \cdot g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{r}dm\right)_{\odot}^{s-t}\right). \quad (67)$$

Thus, we obtain

$$g\left(g^{(-1)}\left(g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{s}dm\right)_{\odot}^{r-t}\right)\right)\right) \leq g\left(g^{(-1)}\left(g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{t}dm\right)_{\odot}^{r-s}\right) \cdot g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{r}dm\right)_{\odot}^{s-t}\right)\right)\right), \quad (68)$$

and it follows that

$$g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{s}dm\right)_{\odot}^{r-t}\right) \leq g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{t}dm\right)_{\odot}^{r-s} \odot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{r}dm\right)_{\odot}^{s-t}\right), \quad (69)$$

therefore,

$$\left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{s}dm\right)_{\odot}^{r-t} \preceq \left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{t}dm\right)_{\odot}^{r-s} \odot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f_{\odot}^{r}dm\right)_{\odot}^{s-t}. \quad (70)$$

□

Example 34. If $g(x) = \ln x$, then $x \oplus y = xy$, $x \odot y = e^{\ln x \cdot \ln y}$, and $\mathbf{0} = 1$, $\mathbf{1} = e$. Let $x \in [\mathbf{0}, \mathbf{1}] = [1, e]$; then we have $g : [\mathbf{0}, \mathbf{1}] = [1, e] \longrightarrow [0, 1]$, such that $g(\mathbf{0}) = g(1) = 0$ and $g(\mathbf{1}) = g(e) = 1$.

Suppose that $f : [0, 1] \longrightarrow [\mathbf{0}, \mathbf{1}]$ is a measurable function. By Theorem 33, for $1 = \mathbf{0} < t < s < r$, the following inequality holds:

$$\left((Cl)\int_{[0,1]}^{\oplus,\odot} e^{(\ln f(x))^{s}}dm\right)^{r-t} \leq \left((Cl)\int_{[0,1]}^{\oplus,\odot} e^{(\ln f(x))^{t}}dm\right)^{r-s} \cdot \left((Cl)\int_{[0,1]}^{\oplus,\odot} e^{(\ln f(x))^{r}}dm\right)^{s-t}, \quad x \in [0, 1]. \quad (71)$$

According to Theorem 20, we obtain

$$e^{(r-t)(\int_0^1 \ln[m(([\ln f(x)]^{s})_y)]dy)} \leq e^{(r-s)(\int_0^1 \ln[m(([\ln f(x)]^{t})_y)]dy)+(s-t)(\int_0^1 \ln[m(([\ln f(x)]^{r})_y)]dy)}, \quad (72)$$

that is,

$$(r-t)\left(\int_0^1 \ln\left[m\left(\left([\ln f(x)]^{s}\right)_y\right)\right]dy\right) \leq (r-s)\left(\int_0^1 \ln\left[m\left(\left([\ln f(x)]^{t}\right)_y\right)\right]dy\right) + (s-t)\left(\int_0^1 \ln\left[m\left(\left([\ln f(x)]^{r}\right)_y\right)\right]dy\right), \quad (73)$$

where m is a nonmonotonic fuzzy measure and y is the Lebesgue measure.

5. Stolarsky Type Inequality for the Choquet-Like Integral

The classical Stolarsky integral inequality provides the inequality ([57])

$$\int_0^1 f\left(x^{1/(a+b)}\right)dx \geq \left(\int_0^1 f\left(x^{1/a}\right)dx\right)\left(\int_0^1 f\left(x^{1/b}\right)dx\right), \quad (74)$$

where $a, b > 0$, $f : [0, 1] \longrightarrow [0, 1]$ is a nonincreasing function, and x is the Lebesgue measure. This result was obtained by Stolarsky in ([57]). Later, Maligranda et al. ([58]) proved another version of Stolarsky inequality using the Chebyshev's inequality.

Theorem 35. *Let the generalized generator $g : [\mathbf{0}, \mathbf{1}] \longrightarrow [0, 1]$ of the pseudo-operations be a monotone bijection and*

$f : [0,1] \longrightarrow [\mathbf{0},\mathbf{1}]$ *be a nonincreasing function. Then for* $a,b \succ \mathbf{0}$, *the inequality*

$$\begin{aligned}(Cl)&\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/(a+b)}\right) dm \\ &\geq \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right) \\ &\odot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right) dm\right)\end{aligned} \tag{75}$$

holds.

Proof. Using the classical Stolarsky inequality and then we obtain

$$\begin{aligned}&\int_0^1 gm\left(\left(gf\left(x^{1/(a+b)}\right)\right)_y\right) dy \\ &\geq \left(\int_0^1 gm\left(\left(gf\left(x^{1/a}\right)\right)_y\right) dy\right) \\ &\cdot \left(\int_0^1 gm\left(\left(gf\left(x^{1/b}\right)\right)_y\right) dy\right), \quad x \in [0,1].\end{aligned} \tag{76}$$

Since the function g is a monotone bijection, $gg^{(-1)}(x) = x$, then there is

$$\begin{aligned}&g\left(g^{(-1)}\left(\int_0^1 gm\left(\left(gf\left(x^{1/(a+b)}\right)\right)_y\right) dy\right)\right) \\ &\geq g\left(g^{(-1)}\left(\int_0^1 gm\left(\left(gf\left(x^{1/a}\right)\right)_y\right) dy\right)\right) \\ &\cdot g\left(g^{(-1)}\left(\int_0^1 gm\left(\left(gf\left(x^{1/b}\right)\right)_y\right) dy\right)\right).\end{aligned} \tag{77}$$

According to Theorem 20, we have

$$\begin{aligned}g&\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/(a+b)}\right) dm\right) \\ &\geq g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right) \\ &\cdot g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right) dm\right).\end{aligned} \tag{78}$$

Thus, we have

$$\begin{aligned}&g\left(g^{(-1)}\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/(a+b)}\right) dm\right)\right)\right) \\ &\geq g\left(g^{(-1)}\left(g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right)\right.\right. \\ &\left.\left.\cdot g\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right) dm\right)\right)\right),\end{aligned} \tag{79}$$

that is,

$$\begin{aligned}g&\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/(a+b)}\right) dm\right) \\ &\geq g\left(\left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right)\right. \\ &\left.\odot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right) dm\right)\right).\end{aligned} \tag{80}$$

Therefore,

$$\begin{aligned}(Cl)&\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/(a+b)}\right) dm \\ &\geq \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right) \\ &\odot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right) dm\right).\end{aligned} \tag{81}$$

□

Example 36. Let $g(x) = 1 - x$; then the corresponding pseudo-operations are $x \oplus y = x + y - 1$, $x \odot y = x + y - xy$, and $\mathbf{0} = 1$, $\mathbf{1} = 0$. Let $x \in [\mathbf{1},\mathbf{0}] = [0,1]$; then we have $g : [\mathbf{1},\mathbf{0}] = [0,1] \longrightarrow [0,1]$, such that $g(\mathbf{0}) = g(1) = 0$ and $g(\mathbf{1}) = g(0) = 1$.

Suppose that $f : [0,1] \longrightarrow [\mathbf{1},\mathbf{0}]$ is a nonincreasing function. According to Theorem 35, for $a,b \succ \mathbf{0} = 1$, the following inequality holds:

$$\begin{aligned}(Cl)&\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/(a+b)}\right) dm \\ &\leq \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right) \\ &+ \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right)\right) \\ &- \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/a}\right) dm\right) \\ &\cdot \left((Cl)\int_{[0,1]}^{\oplus,\odot} f\left(x^{1/b}\right)\right),\end{aligned} \tag{82}$$

where $x \in [0,1]$. According to Theorem 20, we obtain

$$\begin{aligned}1 &- \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/(a+b)}\right)\right)_y\right)\right] dy\right) \leq \left(1\right. \\ &\left.- \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/a}\right)\right)_y\right)\right] dy\right)\right) + \left(1\right. \\ &\left.- \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/b}\right)\right)_y\right)\right] dy\right)\right) \\ &- \left(\left(1 - \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/a}\right)\right)_y\right)\right] dy\right)\right)\right. \\ &\left.\cdot \left(1 - \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/b}\right)\right)_y\right)\right] dy\right)\right)\right),\end{aligned} \tag{83}$$

that is,

$$\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/(a+b)}\right)\right)_y\right)\right] dy \geq \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/a}\right)\right)_y\right)\right] dy\right) \cdot \left(\int_0^1 \left[1 - m\left(\left(1 - f\left(x^{1/b}\right)\right)_y\right)\right] dy\right), \quad (84)$$

where m is a nonmonotonic fuzzy measure.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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