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Research Article

Quasigeostrophic Equations for Fractional Powers of Infinitesimal Generators

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In this paper we treat the following partial differential equation, the quasigeostrophic equation: $(\partial/\partial t + u \cdot \nabla) f = -\sigma(-A)^{\alpha} f$, $0 \leq \alpha \leq 1$, where $(A, D(A))$ is the infinitesimal generator of a convolution C_0-semigroup of positive kernel on $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$. Firstly, we give remarkable pointwise and integral inequalities involving the fractional powers $(-A)^{\alpha}$ for $0 \leq \alpha \leq 1$. We use these estimates to obtain L^p-decayment of solutions of the above quasigeostrophic equation. These results extend the case of fractional derivatives (taking $A = \Delta$, the Laplacian), which has been studied in the literature.

1. Introduction

In oceanography and meteorology, the quasigeostrophic equation,

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) f = -\sigma (-\Delta)^{\alpha} f, \quad \text{for } 0 \leq \alpha \leq 1, \tag{1}$$

where f represents the temperature, u the velocity, and σ the viscosity constant, has a great importance (see for example [1, 2]). In the last years, a large number of mathematical papers are dedicated to this equation. For example, in [3, 4], A. Córdoba and D. Córdoba studied regularity and L^p-decay for solutions. In [5] the well-posedness of quasigeostrophic equation was treated on the sphere, on general riemannian manifolds in [6] or the 2D stochastic quasigeostrophic equation on the torus $\mathbb{T}^2$ in [7].

This equation is also denominated as advection-fractional diffusion; see for example [8], or it may be classified as a fractional Fokker-Planck equation [9]. However we follow the usual terminology of quasigeostrophic equation which has appeared in our main references [1–7].

Here we replace the Laplacian operator Δ for an arbitrary infinitesimal generator $(A, D(A))$ of a convolution C_0-semigroup of positive kernel on Lebesgue spaces $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$. The abstract framework of C_0-semigroups of linear bounded operators in Banach spaces was introduced by Hille and Yosida in the last fifties; see for example the monographies [10–13]. Some classical C_0-semigroups, as Gaussian, Poisson, fractional, or the backward semigroups in classical Lebesgue spaces, fit in this approach; see for example [12, Chapter 2]. Note that in particular the Laplacian Δ generates the Gaussian (also called heat or diffusion) semigroup [10, Chapter II, Section 2.13].

The main aim of this paper is to show the decreasing behavior for suitable solutions of

$$\left(\frac{\partial}{\partial t} + u \cdot \nabla\right) f = -\sigma (-A)^{\alpha} f, \quad 0 \leq \alpha \leq 1. \tag{2}$$

Some classical asymptotic behavior of solutions of abstract Cauchy problem,

$$\frac{\partial}{\partial t} f = Af, \quad f \in D(A), \tag{3}$$

is presented in [11, Section 4.4] and for parabolic case of evolution systems in [11, Section 5.8]. Note that for $u = 0$ in (2), we recover the classical Cauchy problem for the fractional power $-\sigma(-A)^{\alpha}$.

We emphasize the key role played by the Balakrishnan integral representation of fractional powers [13, p. 264] in order to get the following pointwise inequalities:

$$f(x)(-A)^{\alpha} f(x) \geq \frac{1}{2}(-A)^{\alpha} f^{2}(x) \quad \text{a.e,} \tag{4}$$

for certain infinitesimal generators of convolution C_0-semigroups on the Lebesgue space $L^p(\mathbb{R}^n)$ (Theorem 1). From such pointwise inequalities, and assuming convolution kernels of real symbol, one gets integral inequalities (Theorem 4 and Lemma 6), which extend [3, Lemma 1] and [4, Lemma 2.4, Lemma 2.5], respectively. For this purpose, we use Fourier transform, obtaining multiplications semigroups from convolution ones. Interesting similar pointwise inequalities have been discussed in [14].

The previous results allow getting a maximum principle for the solutions of (2),

$$\|f(\cdot,t)\|_p \leq \|f(\cdot,0)\|_p, \quad t \geq 0, \tag{5}$$

see Corollary 7. Moreover, one of the most important results along this paper is to estimate the decreasing behavior,

$$\frac{d}{dt}\|f\|_p^p \leq -\sigma \|f\|_p^p D\left(\|f\|_p^p\right), \tag{6}$$

for some suitable solutions $f \in \mathcal{S}(\mathbb{R}^n)$ and nonnegative functions D, see Theorem 8. To prove that, we use some techniques which are based in [15]. In that paper some equivalence between Super-Poincaré and Nash-type inequalities is shown for nonnegative self-adjoint operators. Some of these results were proved in the case of fractional powers of the Laplacian in [3, 4, 16].

In the last section, we apply our results to check estimations about the L^p-decay of some solutions in concrete quasigeostrophic equations. Our main example is to consider subordinated C_0-semigroups to Poisson or Gaussian semigroup. This approach is inspired in [15]. Preliminary versions of these results were included in [17].

Notation. Through this article $(L^p(\mathbb{R}^n), \|\ \|_p)$ with $1 \leq p \leq \infty$ is the usual Lebesgue space and $(L^1(\mathbb{R}^n), \|\ \|_1, *)$ is the Banach algebra where

$$f * g(x) := \int_{\mathbb{R}^n} f(x-y) g(y)\, dy, \quad x \in \mathbb{R}^n. \tag{7}$$

The space $C_0(\mathbb{R}^n)$ is formed by the continuous functions f such that $\lim_{|x|\longrightarrow\infty} f(x) = 0$, and $\|f\|_\infty := \max_{x\in\mathbb{R}^n}|f(x)|$; the set $\mathcal{S}(\mathbb{R}^n)$ is the Schwartz space and Γ is the Gamma function.

2. Pointwise and Integral Estimates for Fractional Powers

Let $(k_t)_{t>0} \subset L^1(\mathbb{R}^n)$ be a one-parameter continuous semigroup in the Banach algebra $L^1(\mathbb{R}^n)$; i.e., $k_t * k_s = k_{t+s}$ for $t, s > 0$; $k_t * f \longrightarrow f$ when $t \longrightarrow 0$ for any $f \in L^1(\mathbb{R}^n)$ and such that $\|k_t\|_1 = 1$ for $t > 0$; see for example [12, Chapter 1]. Then the one-parameter family of linear bounded operators $\mathcal{K} = (K(t))_{t\geq 0}$, defined by

$$\begin{aligned} &K(t) f := k_t * f, \quad f \in L^p(\mathbb{R}^n), \ t > 0; \\ &K(0) = I, \end{aligned} \tag{8}$$

is a convolution C_0-semigroup on $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$. Recall that the infinitesimal generator $(A, D(A))$ of $\mathcal{K}$ is defined by

$$Af = \lim_{t\longrightarrow 0^+} \frac{k_t * f - f}{t}, \quad f \in D(A), \tag{9}$$

that is, the domain of the operator A is the closed and densely defined subspace where the above limit exists on $L^p(\mathbb{R}^n)$, see for example [10, Definition 1.2]. Note that these C_0-semigroups $(K(t))_{t\geq 0}$ are contractive since $\|k_t\|_1 = 1$ for all $t > 0$. We also assume that $(k_t)_{t>0}$ is a positive kernel. Below, there are several examples of convolution C_0-semigroups of positive kernel:

(1) The Gaussian kernel, $g_t(x) = (4\pi t)^{-n/2} e^{-|x|^2/4t}$, whose generator is the Laplacian operator Δ ([12, Theorem 2.15]).

(2) The Poisson kernel, $p_t(x) = (\Gamma((n+1)/2)/\pi^{(n+1)/2})(t/(t^2 + |x|^2)^{(n+1)/2})$, whose infinitesimal generator is $-\sqrt{-\Delta}$ ([12, Theorem 2.17]).

(3) Subordinated semigroups in $L^1(\mathbb{R}^n)$. In [18], new convolution C_0-semigroups are defined by subordination principle, i.e., using the bounded algebra homomorphism $\Theta_a : L^1(\mathbb{R}^+) \longrightarrow L^1(\mathbb{R})$, with

$$\Theta_a(f) = \int_0^\infty f(t) a_t\, dt, \quad f \in L^1(\mathbb{R}^+), \tag{10}$$

where $a = (a_t)_{t>0}$ is an uniformly bounded continuous semigroup on $L^1(\mathbb{R})$; in particular $a_t = g_t$ or p_t for $t > 0$. Now, we take the fractionary semigroup on $L^1(\mathbb{R}^+)$,

$$I_s(t) := \frac{t^{s-1}}{\Gamma(s)} e^{-t}, \quad t > 0, \tag{11}$$

with $s > 0$, and new type kernels are obtained by

$$\begin{aligned} \Theta_a(I_s)(x) &= \int_0^\infty I_s(t) a_t(x)\, dt \\ &= \int_0^\infty \frac{t^{s-1}}{\Gamma(s)} e^{-t} a_t(x)\, dt, \quad x \in \mathbb{R}^n, \end{aligned} \tag{12}$$

see additional details in [18, Theorem 2.1, Corollary 2.2].

In the following, $(-A)^\alpha$ denotes the fractional powers of the infinitesimal generator of these semigroups; see [13, p. 264]:

$$(-A)^\alpha f = \Gamma(-\alpha)^{-1} \int_0^\infty t^{-\alpha-1} (K(t) - I) f\, dt, \tag{13}$$

for all $f \in D(A)$ and $0 < \alpha < 1$. Our first result gives a pointwise inequality for these fractional powers. The main ingredient is to represent the C_0-semigroup $(K(t))_{t\geq 0}$ in terms of the positive kernel functions. Compare with [3, Theorem 1] and [4, Proposition 2.3] in the case of $A = \Delta$.

Theorem 1. *Let $(A, D(A))$ be the infinitesimal generator of a C_0-semigroup $(K(t))_{t\geq 0}$ as above. Then, for all $f \in D(A)$ real-valued with $f^2 \in D(A)$ and $0 \leq \alpha \leq 1$, the inequality*

$$f(x)(-A)^{\alpha} f(x) \geq \frac{1}{2}(-A)^{\alpha} f^2(x) \quad a.e \tag{14}$$

holds.

Proof. We use equality (13), almost everywhere $x \in \mathbb{R}^n$, and $0 < \alpha < 1$ to get

$$\begin{aligned} f(x)(-A)^{\alpha} f(x) &= \Gamma(-\alpha)^{-1} \int_0^{\infty} t^{-\alpha-1} \left(\int_{\mathbb{R}^n} k_t(x-r) \right. \\ &\quad \left. \cdot f(r) f(x)\, dr - f^2(x) \right) dt = \Gamma(-\alpha)^{-1} \\ &\quad \cdot \int_0^{\infty} t^{-\alpha-1} \left(\int_{\mathbb{R}^n} k_t(x-r) \right. \\ &\quad \left. \cdot \left(f(r) f(x) - f^2(x) \right) dr \right) dt. \end{aligned} \tag{15}$$

Note that

$$\begin{aligned} &-\left(f^2(x) - f(r) f(x) \right) \\ &\quad = -\left(\frac{1}{2}\left(f(x) - f(r) \right)^2 + \frac{1}{2}\left(f^2(x) - f^2(r) \right) \right) \\ &\quad \leq -\frac{1}{2}\left(f^2(x) - f^2(r) \right), \end{aligned} \tag{16}$$

since $\Gamma(-\alpha) < 0$ if $0 < \alpha < 1$, and then

$$\begin{aligned} f(x)(-A)^{\alpha} f(x) &\geq \frac{\Gamma(-\alpha)^{-1}}{2} \\ &\quad \cdot \int_0^{\infty} t^{-\alpha-1} \left(\int_{\mathbb{R}^n} k_t(x-r) \left(f^2(r) - f^2(x) \right) dr \right) dt \\ &= \frac{1}{2}(-A)^{\alpha} f^2(x) \quad \text{a.e.} \end{aligned} \tag{17}$$

If $\alpha = 0$, it is trivial, and for $\alpha = 1$ we use the definition of the infinitesimal generator. □

Given $f \in L^1(\mathbb{R}^n)$, the usual Fourier transform is given by

$$\widehat{f}(\eta) := \int_{\mathbb{R}^n} f(x) e^{-2\pi i \eta \cdot x} dx, \quad \eta \in \mathbb{R}^n, \tag{18}$$

and then $\widehat{f} \in C_0(\mathbb{R}^n)$. Let $\mathscr{K} = (K(t))_{t\geq 0}$ be a convolution C_0-semigroup of positive kernel on $L^p(\mathbb{R}^n)$, with kernel $(k_t)_{t>0}$. Note that $\widehat{k_t} \in C_0(\mathbb{R}^n)$, with $\|\widehat{k_t}\|_{\infty} \leq \|k_t\|_1 = 1$. Then, it is well known that $\mathscr{T}_{\mathscr{K}} \ll (T(t))_{t\geq 0}$ with

$$\begin{aligned} T(t) : C_0(\mathbb{R}^n) &\longrightarrow C_0(\mathbb{R}^n) \\ g &\longmapsto \widehat{k_t} g, \quad t > 0, \end{aligned} \tag{19}$$

is a contractive multiplication C_0-semigroup. We obtain the following result as a consequence of [10, p. 28].

Proposition 2. *Let $\mathscr{K} = (K(t))_{t\geq 0}$ be a C_0-semigroup as above. Then there is a $q : \mathbb{R}^n \longrightarrow \mathbb{C}$ continuous function with $\mathfrak{Re}(q(x)) \leq 0$ for all $x \in \mathbb{R}^n$, such that $\widehat{k_t} = e^{tq}$ for $t > 0$ and $(B, D(B))$ is the infinitesimal generator of $\mathscr{T}_{\mathscr{K}}$, with $B = qI$ and*

$$D(B) = \{ f \in C_0(\mathbb{R}^n) \mid qf \in C_0(\mathbb{R}^n) \}. \tag{20}$$

Definition 3. We say that a convolution C_0-semigroup of positive kernel on $L^p(\mathbb{R}^n)$, $\mathscr{K} = (K(t))_{t\geq 0}$, is of real symbol when the infinitesimal generator of the semigroup $\mathscr{T}_{\mathscr{K}}$ is a real function; i.e., $q : \mathbb{R}^n \longrightarrow (-\infty, 0]$.

Theorem 4. *Let $\mathscr{K} = (K(t))_{t\geq 0}$ be a convolution C_0-semigroup of positive kernel and real symbol on $L^p(\mathbb{R}^n)$, with kernel $(k_t)_{t>0}$, and infinitesimal generator $(A, D(A))$ satisfying $\mathscr{S}(\mathbb{R}^n) \subset D(A)$ and for all $h \in \mathscr{S}(\mathbb{R}^n)$, $q\widehat{h} \in L^2(\mathbb{R}^n)$. If $f \in \mathscr{S}(\mathbb{R}^n)$ is a real function, then*

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)(-A)^{\alpha} f(x)\, dx \\ &\quad \geq \frac{1}{p} \int_{\mathbb{R}^n} \left| (-A)^{\alpha/2} f^{p/2}(x) \right|^2 dx, \end{aligned} \tag{21}$$

for $0 \leq \alpha \leq 1$ and $p = 2^j$ with j positive integer.

Proof. We apply equation (14) to get

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)(-A)^{\alpha} f(x)\, dx \\ &\quad \geq \frac{1}{2} \int_{\mathbb{R}^n} |f(x)|^{p-2} (-A)^{\alpha} f^2(x)\, dx \\ &\quad \geq \frac{1}{2^l} \int_{\mathbb{R}^n} |f(x)|^{p-2^l} (-A)^{\alpha} f^{2^l}(x)\, dx, \end{aligned} \tag{22}$$

with $l \in \mathbb{N}_0$. Taking $l = j - 1$, then for $0 \leq \alpha \leq 1$ the following inequality holds:

$$\begin{aligned} &\int_{\mathbb{R}^n} |f(x)|^{p-2} f(x)(-A)^{\alpha} f(x)\, dx \\ &\quad \geq \frac{2}{p} \int_{\mathbb{R}^n} |f(x)|^{p/2} (-A)^{\alpha} f^{p/2}(x)\, dx. \end{aligned} \tag{23}$$

On the other hand, for $0 < \alpha < 1$

$$\begin{aligned}\widehat{(-A)^{\alpha}} f(\eta) &= \int_{\mathbb{R}^n} e^{-2\pi i\eta\cdot x}\Gamma(-\alpha)^{-1} \\ &\cdot\left(\int_0^{\infty} t^{-\alpha-1}\left(k_t * f(x) - f(x)\right)dt\right)dx \\ &= \Gamma(-\alpha)^{-1} \\ &\cdot\int_0^{\infty} t^{-\alpha-1}\left(\int_{\mathbb{R}^n} e^{-2\pi i\eta\cdot x}\left(k_t * f(x) - f(x)\right)dx\right)dt \\ &= \Gamma(-\alpha)^{-1}\int_0^{\infty} t^{-\alpha-1}\left(\widehat{k_t * f}(\eta) - \hat{f}(\eta)\right)dt.\end{aligned} \tag{24}$$

Therefore

$$\begin{aligned}\widehat{(-A)^{\alpha}} f(\eta) &= (-B)^{\alpha}\hat{f}(\eta) = (-q(\eta))^{\alpha}\hat{f}(\eta) \\ &\in L^2(\mathbb{R}^n)\cap C_0(\mathbb{R}^n).\end{aligned} \tag{25}$$

Note that, for $\alpha = 0$, the previous equality is trivial, and, for $\alpha = 1$, it is well known. Finally, by Plancherel and Parseval theorems for Fourier Transform, we obtain

$$\begin{aligned}&\int_{\mathbb{R}^n}\left|(-A)^{\alpha/2} f^{p/2}(x)\right|^2 dx \\ &\quad = \int_{\mathbb{R}^n} f^{p/2}(x)(-A)^{\alpha} f^{p/2}(x)\,dx,\end{aligned} \tag{26}$$

and then

$$\begin{aligned}&\int_{\mathbb{R}^n}|f(x)|^{p-2} f(x)(-A)^{\alpha} f(x)\,dx \\ &\quad \geq \frac{1}{p}\int_{\mathbb{R}^n}\left|(-A)^{\alpha/2} f^{p/2}(x)\right|^2 dx.\end{aligned} \tag{27}$$

Then we conclude the proof. □

In the conditions of the previous theorem, we give the following examples where also the function q is identified:

(1) For the Gaussian semigroup $q(x) = -4\pi^2|x|^2$.

(2) For the Poisson semigroup $q(x) = -2\pi|x|$.

(3) For the subordination semigroups defined in [18], $q(x) = -\log(1 + 4\pi^2|x|^2)$ using the Gaussian kernel and $q(x) = -\log(1 + 2\pi|x|)$ using the Poisson kernel.

Note that all these examples provide kernels and functions q which depend on the norm $|x|$.

3. L^p-Decay of Solutions of Quasigeostrophic Equation

Let $(A, D(A))$ be the infinitesimal generator of a convolution C_0-semigroup of positive and radius dependent kernel of real symbol on $L^p(\mathbb{R}^n)$, with $1 \leq p < \infty$, and $(-A)^{\alpha}$ the fractional power defined by (13) for $0 < \alpha < 1$.

Let f be a solution of the following:

$$\left(\frac{\partial}{\partial t} + u\cdot\nabla\right) f = -\sigma(-A)^{\alpha} f, \tag{28}$$

where $0 \leq \alpha \leq 1$ and u satisfies either $\nabla\cdot u = 0$ or $u_i = G_i(f)$, together with the necessary conditions about regularity and decay at infinity. Existence results on L^p for (28) with smooth initial conditions have been studied in [16] using a functional approach. Note that we use several notations f, $f(x,t)$, $f(\cdot,t)$ through this section.

We want to study the decline in time of the spatial L^p-norm solutions of (28), and, to do this, we will work with its derivatives, as the following lemma shows. Although the next lemma is known, we include it for the sake of completeness.

Lemma 5. *Let $(A, D(A))$ be under the above conditions and f be a solution of (28). If the function u satisfies that $\nabla\cdot u = 0$ or $u_i = G_i(f)$ with $G_i \in \mathcal{S}(\mathbb{R}^n)$ for $1 \leq i \leq n$, then*

$$\frac{d}{dt}\|f\|_p^p = -\sigma p\int_{\mathbb{R}^n}|f|^{p-2} f(-A)^{\alpha} f\,dx. \tag{29}$$

Proof. Note that

$$\begin{aligned}\frac{d}{dt}\|f\|_p^p &= p\int_{\mathbb{R}^n}|f|^{p-2} f\frac{\partial f}{\partial t}dx \\ &= p\int_{\mathbb{R}^n}|f|^{p-2} f\left(-u\cdot\nabla f - \sigma(-A)^{\alpha} f\right)dx.\end{aligned} \tag{30}$$

On the one hand, we suppose that u satisfies that $\nabla\cdot u = 0$. Then

$$\begin{aligned}\int_{\mathbb{R}^n}|f|^{p-2} f(u\cdot\nabla f)\,dx &= \int_{\mathbb{R}^n}\sum_{j=1}^{n}|f|^{p-2} f\frac{\partial f}{\partial x_j}u_j dx \\ &= \sum_{j=1}^{n}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}|f|^{p-1}\frac{\partial|f|}{\partial x_j}u_j dx_j\right)d\hat{x} \\ &= -\sum_{j=1}^{n}\int_{\mathbb{R}^{n-1}}\left(\int_{\mathbb{R}}\frac{|f|^p}{p}\frac{\partial u_j}{\partial x_j}dx_j\right)d\hat{x} \\ &= -\int_{\mathbb{R}^n}\frac{|f|^p}{p}\nabla\cdot u\,dx = 0,\end{aligned} \tag{31}$$

where we have integrated by parts, and $d\hat{x} = dx_1 dx_2 \ldots dx_{j-1}dx_{j+1}\ldots dx_n$.

On the other hand, we suppose that $u_i = G_i(f)$ with $G_i \in \mathcal{S}(\mathbb{R}^n)$ and $1 \leq i \leq n$. Similarly,

$$\begin{aligned}\int_{\mathbb{R}^n}|f|^{p-2} fu\cdot\nabla f\,dx &= \int_{\mathbb{R}^n}\sum_{j=1}^{n}|f|^{p-2} fG_j(f)\frac{\partial f}{\partial x_j}dx \\ &= \sum_{j=1}^{n}\int_{\mathbb{R}^{n-1}}\left[\frac{\left(|f|^{p-2} fG_j(f)\right)^2}{2}\right]_{-\infty}^{\infty}d\hat{x} = 0.\end{aligned} \tag{32}$$

□

The following positivity lemma is a natural extension of [4, Lemma 2.5].

Lemma 6. *Let $(A, D(A))$ be under the above conditions. Then for all $f \in D(A)$ and $0 \leq \alpha \leq 1$ we have*

$$\int_{\mathbb{R}^n}|f|^{p-2} f(-A)^{\alpha} f\,dx \geq 0. \tag{33}$$

Proof. For $0 < \alpha < 1$, a change of variables yields

$$\begin{aligned}\int_{\mathbb{R}^n} |f|^{p-2} f (-A)^{\alpha} f\, dx &= \int_0^{\infty} \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(x)|^{p-2} f(x) k_t(x-r) (f(r)-f(x))\, dr\, dx \right) dt \\ &= -\int_0^{\infty} \frac{t^{-\alpha-1}}{\Gamma(-\alpha)} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |f(r)|^{p-2} f(r) k_t(x-r) (f(r)-f(x))\, dr\, dx \right) dt.\end{aligned} \tag{34}$$

Then, we obtain

$$\begin{aligned}&2\Gamma(-\alpha) \int_{\mathbb{R}^n} |f|^{p-2} f (-A)^{\alpha} f\, dx \\ &\quad = \int_0^{\infty} \frac{1}{t^{\alpha+1}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \left(|f(x)|^{p-2} f(x) - |f(r)|^{p-2} f(r) \right) k_t(x-r) (f(r)-f(x))\, dr\, dx \right) dt \geq 0,\end{aligned} \tag{35}$$

since $(|f(x)|^{p-2}f(x) - |f(r)|^{p-2}f(r))(f(r)-f(x)) \leq 0$ for all $x, r \in \mathbb{R}^n$. For $\alpha = 0$ and $\alpha = 1$ the above inequality is easily checked. □

The previous lemma implies the following maximum principle, which completes similar approaches; see for example [4, Corollary 2.6] and [16, Theorem 1,2].

Corollary 7 (maximum principle). *Let $f \in D(A)$ be a smooth solution of (28). Then for $1 \leq p < \infty$ we have*

$$\|f(\cdot, t)\|_p \leq \|f(\cdot, 0)\|_p, \tag{36}$$

for all $t \geq 0$.

Proof. It is a trivial consequence of Lemma 5 and (33). □

From now on, we focus on study the decay of $(d/dt)\|f\|_p^p$. Applying Theorem 4, we have

$$\frac{d}{dt} \|f\|_p^p \leq -\sigma \int_{\mathbb{R}^n} \left| (-A)^{\alpha/2} f^{p/2} \right|^2 dx, \tag{37}$$

for $p = 2^j$ with j positive integer. For $\alpha = 0$ we have $(d/dt)\|f\|_p^p \leq -\sigma\|f\|_p^p$, then solving this differential inequality we obtain

$$\|f(\cdot, t)\|_p^p \leq e^{-\sigma t} \|f(\cdot, 0)\|_p^p. \tag{38}$$

Below we see what happens to the case $0 < \alpha \leq 1$.

Theorem 8. *Assuming that the symbol $-q$ is an increasing function in the radius, with $\lim_{|x| \to \infty} g(x) = \infty$, then*

$$\frac{d}{dt} \|f\|_p^p \leq -\sigma \|f\|_p^p D\left(\|f\|_p^p\right), \tag{39}$$

for $p = 2^j$ with $j \in \mathbb{N}$, $f \in \mathcal{S}(\mathbb{R}^n)$ real-valued solution of (28), and D a continuous, nonnegative and nondecreasing function.

Proof. For $0 < \alpha \leq 1$, we consider the bijection

$$\begin{aligned}(0, +\infty) &\longrightarrow (0, +\infty) \\ u &\longmapsto u^{\alpha}\end{aligned} \tag{40}$$

with inverse function $u^{1/\alpha}$. Thus for all $t > 0$ and $h \in \mathcal{S}(\mathbb{R}^n)$ one gets

$$\begin{aligned}\|h\|_2^2 &= \|\widehat{h}\|_2^2 \\ &= \int_{\{x \in \mathbb{R}^n : 1 \leq t(-q(x))^{\alpha}\}} |\widehat{h}|^2 (x)\, dx \\ &\quad + \int_{\{x \in \mathbb{R}^n : 1 > t(-q(x))^{\alpha}\}} |\widehat{h}|^2 (x)\, dx \\ &\leq t \int_{\mathbb{R}^n} (-q(x))^{\alpha} |\widehat{h}|^2 (x)\, dx \\ &\quad + \|\widehat{h}\|_{\infty}^2 m_n \left(\left\{ x \in \mathbb{R}^n : \frac{1}{t} > (-q(x))^{\alpha} \right\} \right) \\ &\leq t \left\langle (-q)^{\alpha} \widehat{h}, \widehat{h} \right\rangle \\ &\quad + \|h\|_1^2 m_n \left(\left\{ x \in \mathbb{R}^n : \frac{1}{t^{1/\alpha}} > (-q(x)) \right\} \right),\end{aligned} \tag{41}$$

where m_n denotes the usual Lebesgue measure on $\mathbb{R}^n$. Note that $q(x) = q(|x|)$ is a bijection from $\mathbb{R}^+$ to itself. So

$$\begin{aligned}&m_n \left(\left\{ x \in \mathbb{R}^n : \frac{1}{t^{1/\alpha}} > (-q(x)) \right\} \right) \\ &\quad = \int_{\{|x| < (-q)^{-1}(1/t^{1/\alpha})\}} dx = \left((-q)^{-1} \left(\frac{1}{t^{1/\alpha}} \right) \right)^n w_n\end{aligned} \tag{42}$$

where w_n is the measure of the unit sphere in $\mathbb{R}^n$. We define $\beta(t) := ((-q)^{-1}(1/t^{1/\alpha}))^n w_n$, for $t > 0$, and we rewrite

$$\|h\|_2^2 \leq t \left\langle (-A)^{\alpha} h, h \right\rangle + \|h\|_1^2 \beta(t), \tag{43}$$

where β is a nonnegative and decreasing function.

The operator $(-A)^\alpha$ is a nonnegative and symmetric operator, which satisfies a Super-Poincaré inequality with rate function β, then by [15, Proposition 2.2] this is equivalent to a Nash-type inequality

$$\|h\|_2^2 D\left(\|h\|_2^2\right) \leq \left\langle (-A)^\alpha h, h\right\rangle, \tag{44}$$

with rate function

$$D(s) = \sup_{t>0}\left(t - \frac{t\beta(1/t)}{s}\right), \quad s > 0. \tag{45}$$

Note that the function D is continuous, nonnegative and nondecreasing. So, applying this argument to $f^{p/2}$, with $p = 2^j$, we obtain

$$\left\|(-A)^{\alpha/2} f^{p/2}\right\|_2^2 \geq \left\|f^{p/2}\right\|_2^2 D\left(\left\|f^{p/2}\right\|_2^2\right), \tag{46}$$

and therefore inequality (39) follows from (37). □

4. Examples and Applications

In this last section, we check the L^p-decay of solutions in some concrete examples of quasigeostrophic equations. This approach illustrates our results. To do that, we need to calculate the function D (and also the function β) which appears in Theorem 8 for concrete examples. In [15, section 8], general properties of functions β and D are studied using N-functions; see also [19].

Let $r : [0,\infty) \longrightarrow [0,\infty)$ be a right continuous, monotone increasing function with

(1) $r(0) = 0$;

(2) $\lim_{t\longrightarrow\infty} r(t) = \infty$;

(3) $r(t) > 0$ whenever $t > 0$;

then, the function defined by $R(x) := \int_0^{|x|} r(t)\,dt$ for $x \in \mathbb{R}$ is called an *N-function*. Alternatively, the function $R : \mathbb{R} \longrightarrow [0,\infty)$ is an N-function if and only if R is continuous, even and convex with

(1) $\lim_{x\longrightarrow 0}(R(x)/x) = 0$;

(2) $\lim_{x\longrightarrow\infty}(R(x)/x) = \infty$;

(3) $R(x) > 0$ if $x > 0$.

Given an N-function R, we define the function $G(x) := \int_0^x g(t)\,dt$ for $x > 0$ where g is the right inverse of the right derivative of R, r. The function G is an N-function called the *complement* of R. Furthermore it is straightforward to check that the complement of G is R.

Now suppose that functions β and D are complementary N-functions. Then functions h and h^*, defined by $h(t) := t\beta(1/t)$ for $t > 0$, and $h^*(x) := xD(x)$ for $x > 0$, are also complementary N-functions.

(1) We consider the Laplace operator and $q(x) = -4\pi^2|x|^2$, see Section 2. Then

$$\begin{aligned}(-q)^{-1}(t) &= \frac{t^{1/2}}{2\pi},\\ \text{and } \beta(t) &= \frac{w_n t^{-n/2\alpha}}{(2\pi)^n},\\ &\qquad t > 0,\end{aligned} \tag{47}$$

where w_n is the measure of the unit sphere in $\mathbb{R}^n$. Now we have a couple of N-functions, $h(t) = w_n t^{1+n/2\alpha}/(2\pi)^n$ and $h^*(x) = c_n x^q$, with $1/q + 1/(1 + n/2\alpha) = 1$ and c_n a positive constant; see [15, Section 8]. Then $D(x) = c_n x^{2\alpha/n}$, and we get

$$\frac{d}{dt}\|f\|_p^p \leq -C_n \|f\|_p^{p(1+2\alpha/n)}, \tag{48}$$

for $p = 2^j$. Solving this differential inequality, one obtains

$$\|f(\cdot,t)\|_p^p \leq \frac{\|f(\cdot,0)\|_p^p}{\left(1 + \varepsilon C_n t \|f(\cdot,0)\|_p^{p\varepsilon}\right)^{1/\varepsilon}}, \tag{49}$$

with $\varepsilon = 2\alpha/n$ and $p = 2^j$.

(2) For the subordinated semigroup through Poisson semigroup with $q(x) = -\log(1 + 2\pi|x|)$, we get that

$$\begin{aligned}(-q)^{-1}(t) &= \frac{e^t - 1}{2\pi},\\ \text{and } h(t) &= t\beta\left(\frac{1}{t}\right) = c_n t\left(e^{t^{1/\alpha}} - 1\right)^n,\\ &\qquad t > 0,\end{aligned} \tag{50}$$

with $c_n = w_n/(2\pi)^n$. Then $h(t) = \int_0^t u(s)ds$, with

$$\begin{aligned}u(t) &= c_n\left(e^{t^{1/\alpha}} - 1\right)^n + \frac{n}{\alpha}e^{t^{1/\alpha}}\left(e^{t^{1/\alpha}} - 1\right)^{n-1} t^{1/\alpha},\\ &\qquad t > 0,\end{aligned} \tag{51}$$

so $h^*(x) = \int_0^x u^{-1}(t)dt$. Note that

$$\begin{aligned}u(t) &\leq c_n\left(e^{t^{1/\alpha}} - 1\right)^n + \frac{n}{\alpha}e^{2t^{1/\alpha}}\left(e^{t^{1/\alpha}} - 1\right)^{n-1}\\ &\leq \left(c_n + \frac{n}{\alpha}\right)\left(e^{(n+1)t^{1/\alpha}} - 1\right)\\ &\leq \left(c_n + \frac{n}{\alpha}\right)\left((n+1)^\alpha t\right)^{(1-\alpha)/\alpha} e^{((n+1)^\alpha t)^{1/\alpha}}\\ &=: g(t),\end{aligned} \tag{52}$$

for $t > 0$.

According to [15, Section 8], we consider $h_4(t) = e^{t^p} - 1$, with $p > 1$. If we take $p = 1/\alpha$, then $h_4'(t) = (1/\alpha)t^{(1-\alpha)/\alpha}e^{t^{1/\alpha}}$, and so

$$g^{-1}(t) = \left[\left(c_n + \frac{n}{\alpha}\right)\alpha\right]^{-1}\left(h_4'\right)^{-1}\left((n+1)^\alpha t\right), \quad t > 0. \tag{53}$$

Therefore

$$\begin{aligned} h^*(x) &= \int_0^x u^{-1}(t)\,dt \geq \int_0^x g^{-1}(t)\,dt \\ &\geq c_{n,\alpha} \int_0^{(n+1)^\alpha x} h_4^{-1}(y)\,dy \\ &= c_{n,\alpha} h_4^*\left((n+1)^\alpha x\right), \end{aligned} \tag{54}$$

with $c_{n,\alpha}$ a positive constant.

Now we apply Theorem 8 in the case of $p = 2^j$ with $j \in \mathbb{N}$. If we suppose that the solution of (28) is stable; i.e., $\lim_{t\longrightarrow\infty} \|f(\cdot,t)\|_p^p = 0$, then

$$\frac{d}{dt}\|f\|_p^p \lesssim -C_{n,\alpha}\|f\|_p^{p(1/(1-\alpha))}, \tag{55}$$

for t large enough, where we have used that $h_4^*(x) \sim c_q x^q$, as $x \longrightarrow 0^+$ with $1/q + \alpha = 1$. We conclude that

$$\|f(\cdot,t)\|_p^p \leq \frac{\|f(\cdot,0)\|_p^p}{\left(1 + \varepsilon C_{n,\alpha} t \|f(\cdot,0)\|_p^{p\varepsilon}\right)^{1/\varepsilon}}, \tag{56}$$

for $\varepsilon = \alpha/(1-\alpha)$, and t large enough.

For other $p \neq 2^j$ and $1 < p < \infty$, we obtain the decayment by interpolation property: if $1 \leq p_1 < p < p_2 < \infty$, with $1/p = (1-\theta)/p_1 + \theta/p_2$ and $0 < \theta < 1$, then $\|f\|_p \leq \|f\|_{p_1}^{1-\theta}\|f\|_{p_2}^{\theta}$. When $p > 2$, we have $2^j < p < 2^{j+1}$ for any integer $j \geq 1$, and if $1 < p < 2$ we also use that $\|f(\cdot,t)\|_1 \leq \|f(\cdot,0)\|_1$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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