Research Article

Weighted Endpoint Estimates for Commutators of Singular Integral Operators on Orlicz-Morrey Spaces

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In this paper, we obtain the weighted endpoint estimates for the commutators of the singular integral operators with the \( \text{BMO} \) functions and the associated maximal operators on Orlicz-Morrey spaces. We also get the similar results for the commutators of the fractional integral operators with the \( \text{BMO} \) functions and the associated maximal operators.

1. Introduction and Main Results

The Morrey spaces were introduced by Morrey in [1] to investigate the local behavior of solutions to second-order elliptic partial differential equations. Chiarenza and Frasca [2] showed the boundedness of the Hardy-Littlewood maximal operator, singular integral operators, and the fractional integral operators on the Morrey spaces. Komori and Shirai [3] introduced the weighted Morrey spaces and proved that, for \( 1 < p < \infty \) and \( w \in A_p \), \( T \) and \([b, T]\) are bounded on \( L^{p, \infty}(w) \), and if \( p = 1 \) and \( w \in A_1 \), then for all \( t > 0 \) and any cube \( Q \),

\[
\omega\left(\{x \in Q : |Tf(x)| > t\}\right) \leq C \frac{1}{t} \|f\|_{L^{1, \infty}(w)} w(Q)^{\kappa}. \tag{1}
\]

In this paper, we obtain the weighted endpoint estimates for the commutators of the singular integral operators with \( \text{BMO} \) functions and associated maximal operators. We also obtain the similar results for the commutators of the fractional integral operators with \( \text{BMO} \) functions and associated maximal operators.

Let \( f \) be a measurable function on \( \mathbb{R}^n \) and \( 1 \leq p < \infty \), \( 0 < \kappa < 1 \), for two weights \( w \) and \( u \), and the weighted Morrey space is defined by

\[
L^{p, \infty}(w, u) = \left\{ f \in L^{p, \infty}(w) : \|f\|_{L^{p, \infty}(w, u)} < \infty \right\}, \tag{2}
\]

where

\[
\|f\|_{L^{p, \infty}(w, u)} = \sup_Q \left(\frac{1}{u(Q)} \int_Q |f(x)|^p w(x) \, dx\right)^{1/p}, \tag{3}
\]

and the supremum is taken over all cubes \( Q \) in \( \mathbb{R}^n \). When \( w = u \), write \( L^{p, \infty}(w, u) \) as \( L^{p, \infty}(w) \).

We say that \( T \) is a singular integral operator if there exists a function \( K \) which satisfies the following conditions:

\[
Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} K(x - y) f(y) \, dy,
\]

\[
|K(x)| \leq \frac{C}{|x|}, \tag{4}
\]

\[
|\nabla K(x)| \leq \frac{C}{|x|^{n+1}}.
\]

\( x \neq 0 \).

The \( \text{BMO}(\mathbb{R}^n) \) space is defined by

\[
\text{BMO}(\mathbb{R}^n) = \left\{ b \in L^{\text{loc}}(\mathbb{R}^n) : \|b\|_{\text{BMO}} < \infty \right\}, \tag{5}
\]

\[
= \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx < \infty,
\]

where \( b_Q = (1/|Q|) \int_Q b(y) \, dy \).
notations and facts about the Young functions and Orlicz increasing, and if 
\[ w(x) \in A \] for any weight \( w \) and any \( t > 0 \), \( \Phi(\tau) = \tau \log(e + \tau) \).

The following theorems are our main results.

**Theorem 1.** Let \( w \in A_1 \) and \( \Phi(t) = t \log(e + t) \), then there exists a positive constant \( C \) such that, for any cube \( Q \) and any \( t > 0 \),
\[
\frac{w(\{x \in Q : |[b, T] f(x)| > t\})}{t} \leq C \frac{\|f\|_{L^{\Phi,q}(w)}}{\|f\|_{L^{\Phi,q}(w)}} w(Q)^{\frac{1}{q}}.
\]

**Theorem 2.** Let \( T \) be any singular integral operator, \( w \in A_1 \), \( \Phi(t) = t \log(e + t) \), and \( b \in BMO \). Then there exists a positive constant \( C \) such that, for any cube \( Q \) and any \( t > 0 \),
\[
\frac{w(\{x \in Q : |[b, T] f(x)| > t\})}{t} \leq C \frac{\|f\|_{L^{\Phi,q}(w)}}{\|f\|_{L^{\Phi,q}(w)}} w(Q)^{\frac{1}{q}}.
\]

**Theorem 3.** Let \( 0 < \alpha < n \), \( w \in A_1 \), \( 1/\alpha > 1 - \alpha/n \), \( 0 < \kappa < 1/\alpha \), \( \Phi(t) = t \log(e + t) \), \( \Psi(t) = t^{1/\alpha} \log(e + t)^{-1} \), and \( \Theta(t) = t^{1/\alpha} \log(e + t^{-1}) \). Then there exists a positive constant \( C \) such that, for any cube \( Q \) and any \( t > 0 \),
\[
\frac{\Psi(w(\{x \in Q : |M_{\alpha,\kappa}\{b, I_{\alpha}\} f(x)| > t\}))}{t} \leq C \frac{\|f\|_{L^{\Phi,q}(w)}}{\|f\|_{L^{\Phi,q}(w)}} w(Q)^{\frac{1}{q}}.
\]

**Theorem 4.** Let \( 0 < \alpha < n \), \( w \in A_1 \), \( b \in BMO \), \( 1/\alpha > 1 - \alpha/n \), \( 0 < \kappa < 1/\alpha \), \( \Phi(t) = t \log(e + t) \), \( \Psi(t) = t^{1/\alpha} \log(e + t)^{-1} \), and \( \Theta(t) = t^{1/\alpha} \log(e + t^{-1}) \). Then there exists a positive constant \( C \) such that, for any cube \( Q \) and any \( t > 0 \),
\[
\frac{\Psi(w(\{x \in Q : |M_{\alpha,\kappa}\{b, I_{\alpha}\} f(x)| > t\}))}{t} \leq C \frac{\|f\|_{L^{\Phi,q}(w)}}{\|f\|_{L^{\Phi,q}(w)}} w(Q)^{\frac{1}{q}}.
\]

**2. Proof of Theorems 1 and 2**

**Lemma 5** (see [5]). Let \( \Phi(t) = t \log(e + t) \), then there exists a positive constant \( C \) such that, for any weight \( w \) and all \( t > 0 \),
\[
\frac{w(\{x \in \mathbb{R}^n : M_{\alpha,\kappa}\{b, I_{\alpha}\} f(x)| > t\})}{t} \leq C \int_{\mathbb{R}^n} \Phi\left(\frac{\|f(x)|}{t}\right) M(w(x)) \, dx
\]
for every locally integrable function \( f \).

**Lemma 6** (see [6]). Let \( w \in A_1 \), then there exist a constant \( C > 0 \) and \( \eta > 0 \) such that, for any cube \( Q \) and a measurable subset \( E \subset Q \),
\[
\frac{w(E)}{w(Q)} \leq C \left(\frac{|E|}{|Q|}\right)^{\eta}.
\]
Proof of Theorem 1. Fix a cube $Q$ centered at $x_0$. By Lemma 5, we have
\[
 w \left( \{ x \in Q : M_{L_\log L} f(x) > t \} \right) \\
 = \int_{\{ x \in \mathbb{R}^n : M_{L_\log L} f(x) > t \}} \chi_Q w(x) \, dx \\
 \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{t} \right) M(\chi_Q w)(x) \, dx \\
 \leq C \left( \int_Q + \int_{(3Q)^c} \right) \Phi \left( \frac{|f(x)|}{t} \right) M(\chi_Q w)(x) \, dx \\
 \leq I + II.
\]

To estimate term $I$, since $w \in A_1$, we have
\begin{align*}
 I & \leq C \int_Q \Phi \left( \frac{|f(x)|}{t} \right) w(x) \, dx \\
 & \leq C \left\| \frac{f}{t} \right\|_{L^{\phi_n}(w)} w(Q)^\kappa.
\end{align*}

For term $II$, observe that, for $x \in (3Q)^c$, $x \in R$ and $R \cap Q \neq \emptyset$. We have
\begin{align*}
 \frac{1}{|R|} \int_R \chi_Q(y) w(y) \, dy & = \frac{1}{|R|} \int_{R \cap Q} w(y) \, dy \\
 & \leq C \frac{1}{|x - x_0|^n} \int_Q w(y) \, dy \\
 & = C \frac{1}{|x - x_0|^n} w(Q).
\end{align*}

Therefore we obtain
\[ M(\chi_Q w)(x) \leq C |x - x_0|^{-n} w(Q). \tag{22} \]

Since $w \in A_1$, using Lemma 6, we get
\begin{align*}
 II & \leq C \int_{(3Q)^c} \Phi \left( \frac{|f(x)|}{t} \right) |x - x_0|^{-n} w(Q) \, dx \\
 & \leq C w(Q) \sum_{j=1}^\infty \int_{3^{j+1}Q \setminus 3^jQ} \Phi \left( \frac{|f(x)|}{t} \right) |x - x_0|^{-n} \, dx \\
 & \leq C w(Q) \sum_{j=1}^\infty \frac{1}{|3^jQ|} \int_{3^jQ} \Phi \left( \frac{|f(x)|}{t} \right) \, dx \\
 & \leq C w(Q)^\kappa \\
 & \cdot \sum_{j=1}^\infty \frac{w(Q)^{1-x}}{w(3^{j+1}Q)^{1-x}} \int_{3^jQ} \Phi \left( \frac{|f(x)|}{t} \right) \, dx \\
 & \cdot w(x) \, dx \\
 & \leq C w(Q)^\kappa \left\| \frac{f}{t} \right\|_{L^{\phi_n}(w)} \sum_{j=1}^\infty \frac{1}{3^{j\eta(1-x)}}.
\end{align*}

This ends the proof. \qed

Lemma 7 (see [7]). Let $T$ be any Calderón-Zygmund singular integral operator, $\Phi(t) = t \log(e + t)$, $\varepsilon > 0$, and $b \in BMO$. Then there exists a positive constant $C$ such that, for all weights $w$,
\begin{align*}
 w \left( \{ x \in \mathbb{R}^n : |[b, T] f(x)| > t \} \right) \\
 \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{t} \right) M_{L_\log L} w(x) \, dx.
\end{align*}

Lemma 8 (see [61]). Let $w \in A_1$, then there exist a constant $C > 0$ and $\theta > 0$ such that, for any cube $Q$,
\begin{align*}
 \left( \frac{1}{|Q|} \int_Q w(y)^{1+\theta} \, dy \right)^{1/(1+\theta)} & \leq C \frac{1}{|Q|} \int_Q w(y) \, dy.
\end{align*}

Proof of Theorem 2. Fix a cube $Q$ centered at $x_0$. By Lemma 7, we have
\[ w \left( \{ x \in Q : |[b, T] f(x)| > t \} \right) \]
\begin{align*}
 & = \int_{\{ x \in \mathbb{R}^n : |[b, T] f(x)| > t \}} w(x) \chi_Q(x) \, dx \\
 & \leq C \int_{\mathbb{R}^n} \Phi \left( \frac{|f(x)|}{t} \right) M_{L_\log L} w(Q) \, dx \\
 & \leq C \left( \int_Q + \int_{(3Q)^c} \right) \Phi \left( \frac{|f(x)|}{t} \right) M_{L_\log L} w(Q) \\
 & \cdot w(x) \, dx \leq I + II.
\end{align*}

To estimate term $I$, since $w \in A_1$, it is easy to prove that $M_{L_\log L} w(Q)(x) \leq C w(x), x \in 3Q,$ and we have
\begin{align*}
 I & \leq C \int_Q \Phi \left( \frac{|f(x)|}{t} \right) w(x) \, dx \\
 & \leq C \left\| \frac{f}{t} \right\|_{L^{\phi_n}(w)} w(Q)^\kappa.
\end{align*}

For term $II$, observe that, for $x \in (3Q)^c$, $x \in R$ is a cube, and $R \cap Q \neq \emptyset$, by Lemma 8, for any $\delta : 0 < \delta \leq \theta$, we have
\begin{align*}
 \left( \frac{1}{|R|} \int_R (w(x) \chi_Q(y))^{1+\delta} \, dy \right)^{1/(1+\delta)} \\
 & \leq \left( \frac{1}{|R|} \int_Q w(y)^{1+\delta} \, dy \right)^{1/(1+\delta)} \\
 & \leq \left( \frac{|Q|}{|R|} \right)^{1/(1+\delta)} \left( \frac{1}{|Q|} \int_Q w(y)^{1+\delta} \, dy \right)^{1/(1+\delta)} \\
 & \leq C \left( \frac{|Q|}{|R|} \right)^{1/(1+\delta)} \frac{1}{|Q|} \int_Q w(y) \, dy \\
 & \leq C \left( \frac{|Q|}{|R|} \right)^{1/(1+\delta)} w(Q).
\end{align*}
Noticing the definition of the maximal function \( M \), we obtain
\[
M_{\log L} \leq \frac{1}{|x - x_0|^{\beta}} \frac{w(Q)}{|Q|} \leq \frac{M \left( \frac{w^\beta}{3^{\gamma/3}} \right)(x)}{|x - x_0|^{\beta}}.
\]
By Lemma 6, we get
\[
in which we take
\[
\delta > 0 \text{ small enough such that } \eta(1 - \kappa) - \delta/(1 + \delta) > 0. \text{ This ends the proof.} \]

3. Proof of Theorems 3 and 4

Given an increasing function \( \varphi : [0, \infty) \rightarrow [0, \infty) \), as in [8], we define the function \( h_\varphi \) by
\[
h_\varphi(s) = \sup_{t > 0} \frac{\varphi(st)}{\varphi(t)}, \quad 0 \leq s < \infty.
\]
If \( \varphi \) is submultiplicative, then \( h_\varphi \leq \varphi \). Also, for all \( s, t > 0 \), \( \varphi(st) \leq h_\varphi(s)\varphi(t) \).

In this section, we set \( \Phi(t) = t \log(e + t) \), it is submultiplicative, and so \( h_\varphi \approx \Phi \). Let \( 0 < \alpha < n \), and \( q \) be a number \( 1/q = 1 - \alpha/n \). Denote
\[
\Psi(t) = \begin{cases} 0, & t = 0, \\ t^{1/q} \log(e + t), & t > 0. \end{cases}
\]
So
\[
\Psi(t) \approx t^{1/q} \log(e + t)^{-1}.
\]
The function \( \Psi \) is invertible with
\[
\Psi^{-1}(t) = \Gamma(t) = \left[ t^{1/q} \log(e + t) \right]^q = \Phi(t)^q.
\]

Lemma 9 (see [8]). If \( \varphi(t)/t \) is decreasing, then, for any positive sequence \( \{t_j\} \),
\[
\varphi \left( \sum_{j} t_j \right) \leq \sum_{j} \varphi(t_j).
\]

Lemma 10. Let \( 0 < \alpha < n, 1/q = 1 - \alpha/n \). Then there exists a constant \( C > 0 \) such that, for any \( t > 0 \), for any weight \( w \), we have
\[
\Psi \left( \frac{w(Q)}{|Q|} > r \right) \leq C h_\varphi \left( M w(y) \right) dy.
\]

Proof. By homogeneity, we may assume that \( t = 1 \). Define the set
\[
\Omega = \left\{ x \in \mathbb{R}^n : M_{\log L} \left( f \right)(x) > r \right\}.
\]
It is easy to see that \( \Omega \) is open and we may assume that it is not empty. To estimate the size of \( \Omega \), it is enough to estimate the size of every compact set \( F \) contained in \( \Omega \). We can cover \( F \) by a finite family of cubes \( \{Q_j\} \) for which
\[
\left| Q_j \right|^{\alpha/n} \left\| f \right\|_{L^q(\mathbb{R}^n)} > 1.
\]

Using Vitali’s covering lemma, we can extract a subfamily of disjoint cubes \( \{Q_k\} \) such that
\[
F \subset \bigcup_k 3Q_k.
\]

For each \( k \), by homogeneity and the properties of the norm \( \left\| \cdot \right\|_{L^q(\mathbb{R}^n)} \), we have
\[
1 < \frac{1}{\left| Q_k \right|^{\alpha/n}} \int_{Q_k} \Phi \left( f(y) \right) dy \leq C \frac{\Phi \left( \left| Q_k \right|^{\alpha/n} \right)}{\left| Q_k \right|} \int_{Q_k} \Phi \left( f(y) \right) dy \leq C \frac{C}{\Psi \left( \left| Q_k \right| \right)} \int_{Q_k} \Phi \left( f(y) \right) dy.
\]

For each \( k \), we have
\[
\Psi \left( w(Q_k) \right) \leq C \Psi \left( w(Q_k) \right) \int_{Q_k} \Phi \left( f(y) \right) dy \leq C h_\varphi \left( \frac{w(Q_k)}{|Q_k|} \right) \int_{Q_k} \Phi \left( f(y) \right) dy \leq C \int_{Q_k} \Phi \left( f(y) \right) h_\varphi \left( M w(y) \right) dy.
\]
It is easy to see that $\Psi(t)/t$ is decreasing; by Lemma 9, we have

$$
\Psi(w(F)) \leq \sum_k \Psi(w(Q_k))
$$

$$
\leq C \sum_k \int_{Q_k} \Phi(f(y)) h_\Psi(Mw(y)) dy
$$

$$
\leq C \int_{\mathbb{R}^n} \Phi(f(y)) h_\Psi(Mw(y)) dy.
$$

This ends the proof. \qed

Proof of Theorem 3. Fix a cube $Q$ centered at $x_0$. By Lemma 10, we have

$$
\Psi\left(\left\{ x \in \mathbb{R}^n : M_{\alpha,L}(\log L)f(x) > t \right\}\right)
$$

$$
= \Psi\left(\int_{\{x \in \mathbb{R}^n : M_{\alpha,L}(\log L)f(x) > t\}} w(x) \chi_Q(x) \, dx\right)
$$

$$
\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{f(x)}{t}\right) h_\Psi(Mw(x)) \, dx
$$

$$
\leq C \left(\int_{3Q} \int_{3Q^c} + \int_{3Q^c^c}\right) \Phi\left(\frac{f(x)}{t}\right) h_\Psi(Mw(x))
$$

$$
\cdot (x) \, dx \leq I + II.
$$

Now we estimate term I. Noticing that, for $s > 0$, we have

$$
h_\Psi(s) = \sup_{t > 0} \Psi(s) = s \sup_{t > 0} \Phi\left(\frac{e^{\alpha/n}}{t}\right) \leq C \Theta(s).
$$

Since $w \in A_1$, we get

$$
I \leq C \int_{3Q} \Phi\left(\frac{f(x)}{t}\right) h_\Psi(w(x)) \, dx
$$

$$
\leq C \int_{3Q} \Phi\left(\frac{f(x)}{t}\right) \Theta(w(x)) \, dx
$$

$$
\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{f(y)}{t}\right) \Theta(w(y)) \, dy.
$$

For term II, observe that, for $x \in (3Q)^c$, $x \in R$ and $R \cap Q \neq \emptyset$. As in the proof of Theorem 1, we have

$$
M(\chi_Qw)(x) \leq C |x - x_0|^{-n} w(Q).
$$

(46)

Since $w \in A_1$, $\Theta$ is submultiplicative, and using Lemma 6, we get

$$
II \leq C \int_{(3Q)^c} \Phi\left(\frac{|f(x)|}{t}\right) h_\Psi\left(|x - x_0|^{-n} w(Q)\right) dx
$$

$$
\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q/3^j} \Phi\left(\frac{|f(x)|}{t}\right) \Theta\left(\frac{w(Q)}{3^{j+1}Q}\right) dx
$$

$$
\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right)
$$

This ends the proof. \qed

Lemma 11 (see [9]). Let $0 < \alpha < n$, $1/q = 1 - \alpha/n$, $w \in A_1$, and $b \in \text{BMO}$. Then there exists a constant $C > 0$ such that, for any $t > 0$,

$$
\Psi\left(\left\{ x \in \mathbb{R}^n : [b, I_{\alpha}]f(x) > t \right\}\right)
$$

$$
\leq C \int_{\mathbb{R}^n} \Phi\left(\frac{f(y)}{t}\right) \Theta(w(y)) \, dy.
$$

(48)

Lemma 12 (see [6]). Let $f(x) \geq 0$, $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, and $0 < \delta < 1$, then $M(f)^\delta \in A_1$.

Proof of Theorem 4. Fix a cube $Q$ centered at $x_0$, for any $w \in A_1$ and $\delta : 0 < \delta \leq \theta$, and by Lemma 12, we have $M(w^{1+\delta})^{1/(1+\delta)} \in A_1$. By Lemma II, we obtain

$$
\Psi\left(\left\{ x \in \mathbb{R}^n : [b, I_{\alpha}] f(x) > t \right\}\right)
$$

$$
= \Psi\left(\int_{\left\{ x \in \mathbb{R}^n : [b, I_{\alpha}] f(x) > t \right\}} w(x) \chi_Q(x) \, dx\right)
$$

$$
\leq C \Psi\left(\int_{\left\{ x \in \mathbb{R}^n : [b, I_{\alpha}] f(x) > t \right\}} M(w \chi_Q)(x) \, dx\right)
$$

$$
\leq C \Psi\left(\int_{\left\{ x \in \mathbb{R}^n : [b, I_{\alpha}] f(x) > t \right\}} (M(w^{1+\delta})^{1/(1+\delta)}) \, dx\right)
$$

\cdot \Theta\left(\frac{w(3^{j+1}Q)}{3^{j+1}Q}\right) dx
$$

$$
\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right)
$$

\cdot \Theta\left(\frac{w(Q)}{w(3^{j+1}Q)}\right) dx
$$

\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right) \Theta(w(x)) \, dx
$$

\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right) \Theta((M(\chi_Qw)(x))^{1/(1+\delta)}) \, dx
$$

\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right) \Theta((M(\chi_Qw)(x))^{1/(1+\delta)}) \, dx
$$

\leq C \sum_{j=1}^\infty \int_{3^{j+1}Q} \Phi\left(\frac{|f(x)|}{t}\right) \Theta((M(\chi_Qw)(x))^{1/(1+\delta)}) \, dx
$$
\[ \leq C \left( \int_{3Q} + \int_{(3Q)^c} \right) \Phi \left( \frac{|f(x)|}{t} \right) \cdot \Theta \left( \left( M \left( w^{1+\delta} \chi_Q \right)(x) \right)^{1/(1+\delta)} \right) dx \leq I + II. \] (49)

Now we estimate term I. Noticing that \( w \in A_1 \), Lemma 8, we have
\[ \Theta \left( \left( M \left( w^{1+\delta} \chi_Q \right)(x) \right)^{1/(1+\delta)} \right) \leq C \Theta(w(x)). \]
Then
\[ I \leq C \int_{3Q} \Phi \left( \frac{|f(x)|}{t} \right) \Theta \left( w(x) \right) dx \leq C \left\| \frac{|f|}{t} \right\|_{L^{6\delta}(w, \delta w)} w(Q)^\kappa. \] (50)

For term II, as the proof of Theorem 2, for \( x \in (3Q)^c \),
\[ \left( M \left( w^{1+\delta} \chi_Q \right)(x) \right)^{1/(1+\delta)} \leq C \left( \frac{|Q|}{|x-x_0|^p} \right)^{1/(1+\delta)} w(Q). \] (51)

By Lemma 6, we get
\[ II \leq C \int_{(3Q)^c} \Phi \left( \frac{|f(x)|}{t} \right) \Theta \left( \left( \frac{|Q|}{|x-x_0|^p} \right)^{1/(1+\delta)} \right) dx \leq C \sum_{j=1}^{\infty} \int_{3^{j-1}Q} \Phi \left( \frac{|f(x)|}{t} \right) \cdot \Theta \left( \left( \frac{|Q|}{3^{j+1}Q} \right)^{\eta/2} \right) w(x) dx \leq Cw(Q)^\kappa \]
\[ \cdot \sum_{j=1}^{\infty} \left( \frac{|Q|}{3^{j+1}Q} \right)^{\eta(1/q-\kappa) - \delta(1/1+\delta)} \cdot \log \left( 1 + \left( \frac{3^{j+1}Q}{|Q|} \right)^{\eta-\delta/(1+\delta)} \right) \cdot \frac{1}{w(3^{j+1}Q)^\kappa} \]
\[ \cdot \int_{3^{j+1}Q} \Phi \left( \frac{|f(x)|}{t} \right) \Theta \left( w(x) \right) dx \leq Cw(Q)^\kappa \]
\[ \cdot \left\| \frac{|f|}{t} \right\|_{L^{6\delta}(w, \delta w)} \sum_{j=1}^{\infty} \left( \frac{1}{3^n} \right)^{\eta(1/q-\kappa) - \delta(1/1+\delta)} \]
\[ \cdot \log \left( e + 3^{jn(\eta-\delta/(1+\delta))} \right) \leq Cw(Q)^\kappa \left\| \frac{|f|}{t} \right\|_{L^{6\delta}(w, \delta w)} , \]
in which we take \( \delta > 0 \) small enough such that \( \eta(1/q-\kappa) - \delta(1/1+\delta) > 0 \) and \( \eta - \delta/(1+\delta) > 0 \). This ends the proof. \( \Box \)

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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