Research Article

The Unique Positive Solution for Singular Hadamard Fractional Boundary Value Problems

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In this paper, we investigate singular Hadamard fractional boundary value problems. The existence and uniqueness of the exact iterative solution are established only by using an iterative algorithm. The iterative sequences have been proved to converge uniformly to the exact solution, and estimation of the approximation error and the convergence rate have also been derived.

This paper is dedicated to our advisors

1. Introduction

Fractional differential operators play an important role in describing phenomena in many fields such as physics, chemistry, control, and electromagnetism [1–9]. They have many applications in fractional differential equations and fractional integral equations. In [10], authors investigate a class of fractional integral equations arising from a symmetric transition model

$$\begin{aligned} \frac{d}{dt} \left(\frac{1}{2} {}_0 D_t^{-\beta} (u'(t)) + \frac{1}{2} {}_t D_T^{-\beta} (u'(t)) \right) + \nabla F(t, u(t)) \\ = 0, \quad \text{a.e. } t \in [0, T] \\ u(0) = u(T) = 0 \end{aligned} \quad (1)$$

where ${}_0 D_t^{-\beta}$ and ${}_t D_T^{-\beta}$ are the left and right Riemann-Liouville fractional integrals of order $0 \leq \beta < 1$, respectively, and $\nabla F(t, x)$ is the gradient of F at x .

The fractional order diffusion equation

$$\frac{\partial^\alpha \phi}{\partial t^\alpha} = K \frac{\partial^\beta \phi}{\partial |x|^\beta} \quad (2)$$

where $0 < \alpha \leq 1$ and $1 < \beta \leq 2$, contains fractional differential operators which can describe different diffusion process.

In the past ten years, fractional differential equations have been considered in many papers (see [11–22]). Most of the works on the topic have been based on Riemann-Liouville type and Caputo type fractional differential equations. By means of fixed point theorems and variational methods, authors obtain at least one or multiple positive solutions for boundary value problems of fractional differential equations. Very recently, more studies have been carried out on the boundary value problems of nonlinear Hadamard fractional differential equations. An important characteristic of Hadamard fractional derivative is that it contains logarithmic function of arbitrary exponent. However, there are few results about this topic (see [23–28]).

By using the Krasnoselskii-Zabreiko fixed point theorem, Yang [26] obtained at least one positive solution for the boundary value problem

$$\begin{aligned} D^q u(t) + f(t, u(t)) &= 0, \quad t \in (1, e), \\ u^{(m)}(1) &= 0, \end{aligned}$$

$$u(e) = \int_1^e g(t) u(t) \frac{dt}{t}, \quad (3)$$

where $f \in C([1, e] \times R^+, R)$, $g \in C([1, e], R^+)$. D^q was the Hadamard fractional derivative of order q . $0 \leq m \leq n-2$, $n \in N$, $n \geq 3$, $n-1 < q \leq n$.

In [28], the authors studied the Hadamard fractional differential equation

$${}^H D^q x(t) + \sigma(t) f(t, x(t)) = 0, \quad (4)$$

$$2 < q \leq 3, \quad t \in (1, +\infty),$$

with boundary conditions

$$x(1) = x'(1) = 0, \quad (5)$$

$${}^H D^{q-1} x(\infty) = a {}^H I^\beta x(\xi) + b \sum_{i=1}^{m-2} \alpha_i x(\eta_i),$$

$$1 < \xi < \eta_1 < \eta_2 < \dots < \eta_{m-2} < +\infty,$$

where ${}^H D^q$ denoted Hadamard fractional derivative of order q and ${}^H I^\beta$ denoted Hadamard fractional integral of order β . $f \in C([1, \infty) \times [0, \infty), [0, \infty))$, $\sigma : [1, \infty) \rightarrow [0, \infty)$. By employing the complete continuity of the associated integral operator T and the monotone iterative method, the authors obtained twin positive solutions and the unique positive solution.

Most of the above works required the associated integral operators to be completely continuous because fixed point theorems could be applied. Furthermore, the uniqueness of positive solutions was rarely investigated while the existence and multiplicity of positive solutions were investigated widely.

Inspired by the above results, in this work, we study the existence and uniqueness of positive solutions for the following boundary value problem:

$${}^H D^q x(t) + f(t, x(t)) = 0, \quad 2 < q \leq 3, \quad t \in (1, e),$$

$$x(1) = x'(1) = 0, \quad (6)$$

$$x(e) = 0,$$

where ${}^H D^q$ denotes Hadamard fractional derivative of order q ; $f : (1, e) \times [0, \infty) \rightarrow [0, \infty)$ is continuous.

In this work, only by using the monotone iterative technique, we aim to establish the unique positive solution for problem (6). The main contributions of this work are as follows: (a) the nonlinear term $f(t, x)$ can be singular at $t = 1$ and $t = e$; (b) we do not need the continuity and complete continuity of the associated integral operator; (c) we get the unique positive solution.

Throughout this work, we assume that the following conditions hold without further mention.

$(H_1) : f : (1, e) \times [0, \infty) \rightarrow [0, \infty)$ is continuous.

(H_2) : For $(t, x) \in (1, e) \times [0, \infty)$, $f(t, x)$ is nondecreasing in x and there exists a constant $k \in (0, 1)$ such that, for $\forall \sigma \in (0, 1]$,

$$f(t, \sigma x) \geq \sigma^k f(t, x). \quad (7)$$

It is easy to verify that if $\sigma \in (1, +\infty)$, then $f(t, \sigma x) \leq \sigma^k f(t, x)$.

2. Preliminaries

The way to attack this new problem follows a scheme similar to that used in [21], with the necessary adaptations that Hadamard fractional derivative contains logarithmic function of arbitrary exponent.

In this section, we present some basic concepts and conclusions needed in the proof of our main results.

Definition 1. The Hadamard fractional integral of order $\beta > 0$ of a function $x : (1, \infty) \rightarrow R$ is given by

$${}^H I^\beta x(t) = \frac{1}{\Gamma(\beta)} \int_1^t \left(\log \frac{t}{s} \right)^{\beta-1} \frac{x(s)}{s} ds. \quad (8)$$

Definition 2. The Hadamard fractional derivative of order q is defined by

$${}^H D^q x(t) = \frac{1}{\Gamma(n-q)} \left(t \frac{d}{dt} \right)^n \int_1^t \left(\log \frac{t}{s} \right)^{n-q-1} \frac{x(s)}{s} ds, \quad (9)$$

where $n-1 < q \leq n$, $n = [q] + 1$, $q > 0$.

Specifically, ${}^H D^n x(t) = x^{(n)}(t)$, $n = 1, 2, 3, \dots$

Lemma 3. Suppose that $h(t) \in C(1, e)$, $0 < \int_1^e h(s)(ds/s) < +\infty$. Then the Hadamard type fractional differential equation

$${}^H D^q x(t) + h(t) = 0, \quad 2 < q \leq 3, \quad t \in (1, e),$$

$$x(1) = x'(1) = 0, \quad (10)$$

$$x(e) = 0$$

has a unique solution

$$x(t) = \int_1^e G(t, s) h(s) \frac{ds}{s} \quad (11)$$

where

$$G(t, s) = \frac{1}{\Gamma(q)} \cdot \begin{cases} (\log t)^{q-1} (1 - \log s)^{q-1} - \left(\log \left(\frac{t}{s} \right) \right)^{q-1}, & 1 \leq s \leq t \leq e; \\ (\log t)^{q-1} (1 - \log s)^{q-1}, & 1 \leq t \leq s \leq e. \end{cases} \quad (12)$$

Proof. As argued in [9], the solution of the Hadamard differential equation in (10) can be written as the equivalent integral equation

$$x(t) = c_1 (\log t)^{q-1} + c_2 (\log t)^{q-2} + c_3 (\log t)^{q-3} - \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} h(s) \frac{ds}{s}. \quad (13)$$

From $x(1) = x'(1) = 0$, we have $c_3 = c_2 = 0$. Thus (47) reduces to

$$x(t) = c_1 (\log t)^{q-1} - \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} h(s) \frac{ds}{s}. \quad (14)$$

Using $x(e) = 0$ we obtain

$$c_1 = \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s} \right)^{q-1} h(s) \frac{ds}{s}. \quad (15)$$

Substituting (15) into (14), we obtain

$$\begin{aligned} x(t) &= (\log t)^{q-1} \cdot \frac{1}{\Gamma(q)} \int_1^e \left(\log \frac{e}{s} \right)^{q-1} h(s) \frac{ds}{s} \\ &\quad - \frac{1}{\Gamma(q)} \int_1^t \left(\log \frac{t}{s} \right)^{q-1} h(s) \frac{ds}{s} \\ &= \int_1^e G(t, s) h(s) \frac{ds}{s}. \end{aligned} \quad (16)$$

□

Take $\rho(t) = (\log t)^{q-1}(1 - \log t)$ and $\hat{\rho}(t) = (1 - \log t)^{q-1} \log t$ for $q > 2$, $t \in [1, e]$. We can prove that $G(t, s)$ have the following properties.

Lemma 4 (see [26]). For $t, s \in [1, e]$, Green's function $G(t, s)$ satisfies the following properties:

- (i) $G(t, s)$ is continuous on $[1, e] \times [1, e]$ and $G(t, s) \geq 0$,
 - (ii) $\rho(t)\hat{\rho}(s) \leq \Gamma(q)G(t, s) \leq (q-1)\rho(t)$,
 - (iii) $\rho(t)\hat{\rho}(s) \leq \Gamma(q)G(t, s) \leq (q-1)\hat{\rho}(s)$,
 - (iv) $G(t, s) = G(e/t, e/s)$.
- From $\Gamma(q+1) = q\Gamma(q)$, $q > 0$, and (ii) we get

$$\rho(t) \left[\frac{1}{\Gamma(q)} \hat{\rho}(s) \right] \leq G(t, s) \leq \frac{1}{\Gamma(q-1)} \rho(t). \quad (17)$$

In this paper, we will work in the Banach space $E = C[1, e]$ with the norm $\|x\| = \max_{t \in [1, e]} |x(t)|$.

Define a set $P \subset E$ as follows.

$P = \{x \in E \mid \text{there are constants } 0 < l_x < 1 < L_x \text{ such that } l_x \rho(t) \leq x(t) \leq L_x \rho(t), t \in [1, e]\}$. Evidently $\rho(t) \in P$. Therefore, P is not empty.

3. The Main Results

Theorem 5. Assume (H_1) , (H_2) hold. And

$$0 < \int_1^e f(t, \rho(t)) \frac{dt}{t} < \infty. \quad (18)$$

Then problem (6) has at least one positive solution $x^*(t)$.

Proof. Define the operator $T : E \rightarrow E$ by

$$Tx(t) = \int_1^e G(t, s) f(s, x(s)) \frac{ds}{s}. \quad (19)$$

We can see easily the equivalence between x is a solution of (6) and x is a fixed point of T .

Claim 1. The operator $T : P \rightarrow P$ is nondecreasing.

In fact, for $x \in P$, it is obvious that $Tx \in E$, $Tx(1) = Tx(e) = 0$. For any $x \in P$ and $t \in [1, e]$, from (17),

$$\begin{aligned} Tx(t) &= \int_1^e G(t, s) f(s, x(s)) \frac{ds}{s} \\ &\leq \int_1^e \frac{1}{\Gamma(q-1)} \rho(t) f(s, L_x \rho(s)) \frac{ds}{s} \\ &\leq \rho(t) L_x^k \frac{1}{\Gamma(q-1)} \int_1^e f(s, \rho(s)) \frac{ds}{s} \\ &\leq L_{Tx} \rho(t) \end{aligned} \quad (20)$$

and

$$\begin{aligned} Tx(t) &= \int_1^e G(t, s) f(s, x(s)) \frac{ds}{s} \\ &\geq \int_1^e \frac{1}{\Gamma(q)} \hat{\rho}(s) \rho(t) f(s, l_x \rho(s)) \frac{ds}{s} \\ &\geq \rho(t) l_x^k \frac{1}{\Gamma(q)} \int_1^e \hat{\rho}(s) f(s, \rho(s)) \frac{ds}{s} \\ &= l_{Tx} \rho(t), \end{aligned} \quad (21)$$

where L_{Tx} and l_{Tx} are positive constants satisfying

$$\begin{aligned} L_{Tx} &> \max \left\{ 1, L_x^k \frac{1}{\Gamma(q-1)} \int_1^e f(s, \rho(s)) \frac{ds}{s} \right\}, \\ 0 < l_{Tx} &< \min \left\{ 1, l_x^k \frac{1}{\Gamma(q)} \int_1^e \hat{\rho}(s) f(s, \rho(s)) \frac{ds}{s} \right\}. \end{aligned} \quad (22)$$

Thus, it follows that there are constants $0 < l_{Tx} < 1 < L_{Tx}$ such that, for $t \in [1, e]$,

$$l_{Tx} \rho(t) \leq Tx(t) \leq L_{Tx} \rho(t). \quad (23)$$

Therefore, for any $x \in P$, $Tx \in P$, T is the operator $P \rightarrow P$. From (19), it is easy to see that T is nondecreasing with respect to x . Hence, Claim 1 holds.

Claim 2. There exist a nondecreasing sequence $\{u_n\}$ and a nonincreasing sequence $\{v_n\}$ and there exists $x^* \in P$ such that

$$\begin{aligned} u_n(t) &\rightarrow x^*(t), \\ v_n(t) &\rightarrow x^*(t), \end{aligned} \quad (24)$$

uniformly on $[1, e]$.

First, there exist two constants l_{Tp}, L_{Tp} with $0 < l_{Tp} < 1 < L_{Tp}$ since $T\rho \in P$. Take δ and γ to be fixed numbers satisfying

$$\begin{aligned} 0 < \delta &\leq l_{Tp}^{1/(1-k)}, \\ \gamma &\geq L_{Tp}^{1/(1-k)}. \end{aligned} \quad (25)$$

Obviously, $0 < \delta < 1 < \gamma$.

We construct two iterative sequences as follows:

$$\begin{aligned} u_0(t) &= \delta\rho(t), \\ v_0(t) &= \gamma\rho(t), \end{aligned} \quad (26)$$

$$\begin{aligned} u_n &= Tu_{n-1}, \\ v_n &= Tv_{n-1}, \end{aligned} \quad (27)$$

$n = 1, 2, 3, \dots$

Then

$$u_0 \leq u_1 \leq \dots \leq u_n \leq \dots \leq v_n \leq \dots \leq v_1 \leq v_0. \quad (28)$$

In fact, from (26), we have $u_0, v_0 \in P$ and $u_0 \leq v_0$. Furthermore,

$$\begin{aligned} u_1(t) &= Tu_0(t) = \int_1^e G(t, s) f(s, \delta\rho(s)) \frac{ds}{s} \\ &\geq \delta^k \int_1^e G(t, s) f(s, \rho(s)) \frac{ds}{s} = \delta^k T\rho \\ &\geq \delta^k l_{T\rho} \rho(t) \geq \delta^k \delta^{1-k} \rho(t) = u_0(t), \\ v_1(t) &= Tv_0(t) = \int_1^e G(t, s) f(s, \gamma\rho(s)) \frac{ds}{s} \\ &\leq \gamma^k \int_1^e G(t, s) f(s, \rho(s)) \frac{ds}{s} = \gamma^k T\rho \\ &\leq \gamma^k L_{T\rho} \rho(t) \leq \gamma^k \gamma^{1-k} \rho(t) = v_0(t). \end{aligned} \quad (29)$$

From $u_0 \leq v_0$ and T nondecreasing, (28) holds. Let $c_0 = \delta/\gamma$; then $0 < c_0 < 1$. It follows from (7) that

$$T(c_0 u) \geq c_0^k T u. \quad (30)$$

And, for any natural number n ,

$$\begin{aligned} u_n &= Tu_{n-1} = T^n u_0 = T^n(\delta\rho) = T^n(c_0 \gamma\rho) \\ &\geq c_0^{k^n} T^n(\gamma\rho) = c_0^{k^n} \gamma_n. \end{aligned} \quad (31)$$

Thus, for any natural number n and p^* , we have

$$\begin{aligned} 0 &\leq u_{n+p^*} - u_n \leq v_n - u_n \leq (1 - c_0^{k^n}) v_n \\ &\leq (1 - c_0^{k^n}) \gamma\rho, \end{aligned} \quad (32)$$

which implies that $\{u_n\}$ is a cauchy sequence in $[u_0, v_0]$. So there exists $x^* \in [u_0, v_0] \subset P$ such that

$$u_n(t) \longrightarrow x^*(t), \quad (33)$$

and from (32)

$$v_n(t) \longrightarrow x^*(t). \quad (34)$$

From $x^* \in [u_0, v_0]$, we have $Tx^* \in [u_0, v_0]$. Combining with T nondecreasing on x ,

$$u_n \leq Tx^* \leq v_n. \quad (35)$$

Let $n \longrightarrow \infty$,

$$x^*(t) = Tx^*(t), \quad (36)$$

which implies x^* is a positive solution of problem (6). $\square$

Theorem 6. Assume $(H_1), (H_2)$ hold. Then, we have the following:

(i) Problem (6) has unique positive solution $x^*(t)$ in P and there exist constants l, L with $0 < l < 1 < L$ such that

$$l\rho(t) \leq x^*(t) \leq L\rho(t), \quad t \in [1, e]. \quad (37)$$

(ii) For any initial value $x_0(t) \in P$, there exists a sequence $\{x_n(t)\}$ that uniformly converges to the unique positive solution $x^*(t)$, and we have the error estimation

$$\max |x_n(t) - x^*(t)| = o(1 - \lambda^{k^n}), \quad (38)$$

where λ is a constant with $0 < \lambda < 1$ and determined by x_0 ; $o(1 - \lambda^{k^n})$ represents the same order infinitesimal of $(1 - \lambda^{k^n})$.

Proof. Let u_0, v_0, u_n, v_n be defined in (26) and (27).

(i) It follows from Theorem 5 that problem (6) has a positive solution $x^*(t) \in P$, which implies that there exists constants l and L with $0 < l < L < 1$ such that

$$l\rho(t) \leq x^*(t) \leq L\rho(t), \quad t \in [1, e]. \quad (39)$$

Let $\bar{x}(t) \in P$ be another positive solution of problem (6). Then there exist constants c_1 and c_2 with $0 < c_1 < 1 < c_2$ such that

$$c_1\rho(t) \leq \bar{x}(t) \leq c_2\rho(t), \quad t \in [1, e]. \quad (40)$$

Let δ defined in (25) be small enough so that $\delta < c_1$ and γ defined in (25) be large enough so that $\gamma > c_2$. Then

$$u_0(t) \leq \bar{x}(t) \leq v_0(t), \quad t \in [1, e]. \quad (41)$$

Note that $T\bar{x}(t) = \bar{x}(t)$ and T is nondecreasing; we have

$$u_n(t) \leq \bar{x}(t) \leq v_n(t), \quad t \in [1, e]. \quad (42)$$

Letting $n \longrightarrow \infty$, we obtain that $x^*(t) = \bar{x}(t)$. Hence, the positive solution of problem (6) is unique.

(ii) For any $x_0(t) \in P$, there exist constants l_0 and L_0 with $0 < l_0 < 1 < L_0$ such that

$$l_0\rho(t) \leq x_0(t) \leq L_0\rho(t), \quad t \in [1, e]. \quad (43)$$

Similar to (i), take δ and γ to be defined by (25) satisfying $\delta < l_0$ and $\gamma > L_0$. Then

$$u_0(t) \leq x_0(t) \leq v_0(t), \quad t \in [1, e]. \quad (44)$$

Let

$$\begin{aligned} x_n(t) &= Tx_{n-1}(t) = \int_1^e G(t, s) f(s, x_{n-1}(s)) \frac{ds}{s}, \\ n &= 1, 2, \dots \end{aligned} \quad (45)$$

Note that T is nondecreasing,

$$u_n(t) \leq x_n(t) \leq v_n(t), \quad t \in [1, e]. \quad (46)$$

Letting $n \longrightarrow \infty$, it follows from (33) and (34) that $x_n(t) \longrightarrow x^*(t)$ uniformly on $[1, e]$.

At the same time, (38) follows from (32). $\square$

Remark 7. We just investigate a simple form of boundary value problems for Hadamard differential equations. We can easily apply the monotone iterative technique to multipoint or multistrip boundary value problems.

Remark 8. Suppose that $\beta_i(t) (i = 0, 1, 2, \dots, m)$ are nonnegative continuous functions on $(1, e)$, which may be unbounded at the end points of $(1, e)$. Ω is the set of functions $f(t, x)$ which satisfy the conditions (H_1) and (H_2) . Then we have the following conclusions:

- (1) $\beta_i(t) \in \Omega$, $x^b \in \Omega$, where $0 < b < 1$.
- (2) If $0 < b_i < +\infty (i = 1, 2, \dots, m)$ and $b > \max_{1 \leq i \leq m} \{b_i\}$, then $[\beta_0(t) + \sum_{i=1}^m \beta_i(t)x^{b_i}]^{1/b} \in \Omega$.
- (3) If $f(t, x) \in \Omega$, then $\beta_i(t)f(t, x) \in \Omega$.
- (4) If $f_i(t, x) \in \Omega (i = 1, 2, \dots, m)$, then $\max_{1 \leq i \leq m} \{f_i(t, x)\} \in \Omega$, $\min_{1 \leq i \leq m} \{f_i(t, x)\} \in \Omega$.

The above four facts can be verified directly. This indicates that there are many kinds of functions which satisfy the conditions (H_1) and (H_2) .

4. An Example

Consider the following boundary value problem:

$$\begin{aligned} {}^H D^{5/2} u(t) + a(t)u^{1/4} + b(t)u^{1/3} &= 0, \quad t \in (1, e), \\ u(1) = u'(1) &= 0, \\ u(e) &= 0, \end{aligned} \quad (47)$$

where $q = 5/2$, $f(t, u) = a(t)u^{1/4} + b(t)u^{1/3}$, $a(t), b(t) \in C((1, e), (0, +\infty))$.

Analysis 1. First, $f \in C((1, e) \times [0, \infty), [0, \infty))$ and so (H_1) holds.

For any $\sigma \in (0, 1)$, we take $k = 1/2$ and have

$$f(t, \sigma u) \geq \sigma^k f(t, u). \quad (48)$$

Then (H_2) holds.

Obviously

$$0 < \int_1^e f(t, \rho(t)) \frac{dt}{t} < \infty, \quad (49)$$

where $\rho(t) = (\log t)^{5/2-1}(1 - \log t)$. Hence all conditions of Theorem 5 are satisfied, and consequently we have the following corollary.

Corollary 9. Problem (47) has unique positive solution $u^*(t)$. For any initial value $x_0 \in P$, the successive iterative sequence $\{x_n(t)\}$ generated by

$$\begin{aligned} x_n(t) &= \int_0^1 G(t, s) \left(a(s)u^{1/4} + b(s)u^{1/3} \right) \frac{ds}{s}, \\ n &= 1, 2, \dots \end{aligned} \quad (50)$$

uniformly converges to the unique positive solution $u^*(t)$ on $[1, e]$. We have the error estimation

$$\max |x_n(t) - u^*(t)| = o\left(1 - \lambda^{(1/2)^n}\right), \quad (51)$$

where λ is a constant with $0 < \lambda < 1$ and determined by the initial value x_0 . And there are constants l, L with $0 < l < 1 < L$ such that

$$l\rho(t) \leq u^*(t) \leq L\rho(t), \quad t \in [1, e]. \quad (52)$$

Data Availability

Data sharing is not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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