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*Research Article*

# Asymptotic Behavior of Almost Quartic $*$-Derivations on Banach $*$-Algebras

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The purpose of this paper is to obtain the stability theorems of quartic $*$-derivations associated with the quartic functional equation $f(3x-y)+f(x+y)+6f(x-y)=4f(2x-y)+4f(y)+24f(x)$ on Banach $*$-algebras.

## 1. Introduction

In 1940, Ulam [1] gave a wide ranging talk before the Mathematics Club of the University of Wisconsin in which he discussed a number of unsolved problems. In the next year, Hyers [2] gave a clear answer to this problem for additive mappings between Banach spaces. Then this theorem [2] was generalized by Aoki [3] for additive mappings and by Rassias [4] for linear mappings by considering an unbounded Cauchy difference. Since then, many mathematicians have come to deal with this problem and also there are many interesting results concerning this problem [5–8].

First, we recall definition of $*$-derivation.

*Definition 1.* Let $B$ be a Banach $*$-algebra and let $A$ be a Banach $*$-subalgebra of $B$. A $\mathbb{C}$-linear mapping $D : A \longrightarrow B$ is said to be derivation on $A$ if $D(ab) = D(a)b + aD(b)$ for all $a, b \in A$. Moreover, if $D$ satisfies the additional condition $D(a^*) = D(a)^*$ for all $a \in A$, then it is called a $*$-derivation.

The stability of $*$-derivations and of quadratic $*$-derivations with the Cauchy functional equation and the Jensen functional equation on Banach $*$-algebras was investigated in [9]. Jang and Park [9] proved the superstability of $*$-derivations and of quadratic $*$-derivations on $C^*$-algebras. The stability of $*$-derivations on Banach $*$-algebras by using fixed-point alternative was proven by Park and Bodaghi [10]. Thereafter, Yang et al. [11] obtained the stability results of cubic $*$-derivations on $*$-algebras and superstability of cubic $*$-derivations on Banach $*$-algebras which are left approximately unital. Recently, Koh and Kang [12] proved the stability of a generalized cubic $*$-derivations on Banach $*$-algebras.

In 1999, Rassias [13] treated the stability of the following quartic equation:

$$\begin{aligned} &f(x+2y)+f(x-2y)+6f(x) \\ &\quad = 4f(x+y)+4f(x-y)+24f(y) \end{aligned} \tag{1}$$

for a mapping $f : X \longrightarrow Y$ where $X$ is a linear space and $Y$ is a Banach space. Thereafter, Lee and Chung [14] studied the general solution and stability theorem of generalized quartic functional equations in the spaces of generalized functions between real vector spaces. Kang [15] has then extended the stability theorems of the following generalized quartic functional equation:

$$\begin{aligned} &f(ax+by)+f(ax-by)+2b^2\left(a^2-b^2\right)f(y) \\ &\quad = 2a^2\left(a^2-b^2\right)f(x) \\ &\qquad + (ab)^2\left[f(x+y)+f(x-y)\right], \end{aligned} \tag{2}$$

where $a, b \neq 0, a \pm b \neq 0$ in quasi-$\beta$-normed spaces. Recently, Bodaghi [16] obtained the general solution of the generalized quartic functional equation

$$\begin{aligned} &f(x+my)+f(x-my)+(m-1)^2 f(2x) \\ &\quad = 2(7m-9)(m-1) f(x)+2m^2\left(m^2-1\right) f(y) \\ &\qquad + m^2[f(x+y)+f(x-y)] \end{aligned} \tag{3}$$

for a fixed positive integer $m$ and proved the Hyers-Ulam stability for this quartic functional equation by the direct method and the fixed-point method on real Banach spaces and non-Archimedean spaces. For more information about the stability of quartic functional equations, we refer to [17–20].

Hyer's direct method used in [2] has been widely applied for studying the generalized Hyers-Ulam stability of various functional equations. Nevertheless, there exist also other approaches proving the Hyers-Ulam stability of functional equations. The most popular technique of proving stability of functional equations except for direct method is the fixed-point method. Although fixed-point method was used for the first time by J.A. Baker [21], most authors follow the alternative fixed-point approach [22, 23] using a theorem of Diaz and Magolis [24].

In this paper, we deal with the following quartic functional equation:

$$\begin{aligned} &f(3x-y)+f(x+y)+6f(x-y) \\ &\quad = 4f(2x-y)+4f(y)+24f(x) \end{aligned} \tag{4}$$

in Banach $*$-algebras. First of all, we show that (4) is equivalent to (1) and then the mapping satisfying (4) on the punctured domain at zero is quartic. In the sequel, we investigate the stability of quartic $*$-derivations associated with the given functional equation on Banach $*$-algebras by using direct method and fixed-point method, respectively.

## 2. Approximate Quartic $*$-Derivations

First of all, we find out the general solution of (4) in the class of mappings between vector spaces.

**Lemma 2.** *Let $U$ and $V$ be vector spaces. A mapping $f: U \longrightarrow V$ satisfies the functional equation (4) if and only if the mapping $f: U \longrightarrow V$ satisfies (1).*

*Proof.* The proof is obvious by taking $(x, y) := (x-y, x)$ in (1) and $(x, y) := (y, x+y)$ in (4) on the basis of evenness of $f$, respectively. □

Throughout this section, let $B$ be a Banach $*$-algebra and let $A$ be a Banach $*$-subalgebra of $B$. For a given mapping $f: A \longrightarrow B$, we define

$$\begin{aligned} \mathbf{Q}_\mu f(x,y) &= f(3\mu x-\mu y)+f(\mu x+\mu y) \\ &\quad +6f(\mu x-\mu y)-4\mu^4 f(2x-y) \\ &\quad -4\mu^4 f(y)-24\mu^4 f(x) \\ \mathbf{D} f(a,b) &= f(ab)-a^4 f(b)-f(a)b^4 \end{aligned} \tag{5}$$

for all $a, b, c, x, y \in A$ and all $\mu \in \mathbb{T}^1_{1/n_0} := \{e^{i\theta} : 0 \le \theta \le 2\pi/n_0, n_0 \in \mathbb{N}\}$.

The following proposition provides a solution of the functional equation (4) on the punctured domain at zero.

**Proposition 3.** *If $f: A \longrightarrow B$ is a mapping satisfying the equality $\mathbf{Q}_1 f(x,y) = 0$ for all $x, y \in A-\{0\}$ and $f(0) = 0$, then $\mathbf{Q}_1 f(x,y) = 0$ for all $x, y \in A$ and hence $f$ is quartic.*

*Proof.* Since $f(0) = 0$, it is trivial that $\mathbf{Q}_1 f(0,0) = 0$. We obtain the equalities $\mathbf{Q}_1 f(x,x) = 2f(2x) - 32f(x) = 0$ so $f(2x) = 16f(x)$ for $x \in A-\{0\}$. And we get $\mathbf{Q}_1 f(x,3x) - \mathbf{Q}_1 f(x,-x) = 6f(-2x) - 6f(2x) = 0$; then $f(-x) = f(x)$ for $x \in A-\{0\}$.

By using above properties, we can show that

$$\begin{aligned} &\mathbf{Q}_1 f(x,0) = \mathbf{Q}_1 f(x,2x) = 0, \\ &\mathbf{Q}_1 f(0,y) = 7f(-y) - 7f(y) = 0 \end{aligned} \tag{6}$$

for $x, y \in A-\{0\}$. Thus, $\mathbf{Q}_1 f(x,y) = 0$ for all $x, y \in A$ and so $f$ is quartic. □

*Definition 4.* A mapping $\delta: A \longrightarrow B$ is called a quartic homogeneous mapping if $\delta$ satisfies (1) and $\delta(\mu x) = \mu^4 \delta(x)$ for all $x \in A$ and $\mu \in \mathbb{C}$. A quartic homogeneous mapping $\delta: A \longrightarrow B$ is said to be a quartic derivation if $\delta(ab) = a^4\delta(b) + \delta(a)b^4$ for all $a, b \in A$. In addition, if $\delta(a^*) = \delta(a)^*$ for all $a \in A$, then it is called a quartic $*$-derivation.

Now we present a main theorem, which is a stability of quartic functional equation (4) in Banach $*$-algebras.

**Theorem 5.** *Let $f: A \longrightarrow B$ be a mapping with $f(0) = 0$ and let $\psi: A^5 \longrightarrow [0,\infty)$ be a function such that*

$$\begin{aligned} &\sum_{i=0}^{\infty} \frac{1}{2^{4i}} \psi\left(2^i x, 2^i y, 2^i z, 0, 0\right) < \infty \\ &\& \lim_{n\longrightarrow\infty} \frac{1}{2^{8n}} \psi\left(0,0,0,2^n a, 2^n b\right) = 0, \\ &\left(\sum_{i=1}^{\infty} 2^{4i} \psi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, 0, 0\right) < \infty\right. \\ &\qquad \left.\& \lim_{n\longrightarrow\infty} 2^{8n} \psi\left(0,0,0,\frac{a}{2^n}, \frac{b}{2^n}\right) = 0, \textit{ respectively}\right) \end{aligned} \tag{7}$$

$$\left\|\mathbf{Q}_\mu f(x,y) + f(z^*) - f(z)^*\right\| \le \psi(x,y,z,0,0), \tag{8}$$

$$\left\|\mathbf{D} f(a,b)\right\| \le \psi(0,0,0,a,b) \tag{9}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and all $x, y, z, a, b \in A$. Assume that the mapping $t \longmapsto f(ta)$ from $\mathbb{R}$ to $B$ is continuous for each*

*fixed $a \in A$. Then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying*

$$\|f(x) - \delta(x)\| \le \frac{1}{32}\sum_{i=0}^{\infty}\frac{1}{2^{4i}}\psi\left(2^i x, 2^i x, 0, 0, 0\right)$$
$$\left(\|f(x) - \delta(x)\| \le \frac{1}{32}\sum_{i=1}^{\infty}2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{10}$$

*for all $x \in A$.*

*Proof.* Taking $y = x$, $z = 0$, and $\mu = 1$ in inequality (8), we get

$$\left\|\frac{1}{16}f(2x) - f(x)\right\| \le \frac{1}{32}\psi(x, x, 0, 0, 0) \tag{11}$$

for all $x \in A$. By using induction, it is implied from inequality (11) that

$$\left\|\frac{1}{2^{4m}}f(2^m x) - \frac{1}{2^{4n}}f(2^n x)\right\| \le \frac{1}{32} \cdot \sum_{i=n}^{m-1}\frac{\psi\left(2^i x, 2^i x, 0, 0, 0\right)}{2^{4i}}$$
$$\left(\left\|2^{4m}f\left(\frac{x}{2^m}\right) - 2^{4n}f\left(\frac{x}{2^n}\right)\right\| \le \frac{1}{32}\sum_{i=n+1}^{m}2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{12}$$

for $m > n \ge 0$ and $x \in A$. By (7) and (12), the sequence $\{f(2^n x)/2^{4n}\}$ $(\{2^{4n}f(x/2^n)\}, \text{respectively})$ is a Cauchy sequence. So define a mapping $\delta : A \longrightarrow B$ by

$$\lim_{n\longrightarrow\infty}\frac{f(2^n x)}{2^{4n}} = \delta(x)$$
$$\left(\lim_{n\longrightarrow\infty}2^{4n}f\left(\frac{x}{2^n}\right) = \delta(x), \text{ respectively}\right) \tag{13}$$

for all $x \in A$. And letting $n = 0$ in inequality (12), we get

$$\left\|\frac{1}{2^{4m}}f(2^m x) - f(x)\right\| \le \frac{1}{32}\sum_{i=0}^{m-1}\frac{\psi\left(2^i x, 2^i x, 0, 0, 0\right)}{2^{4i}}$$
$$\left(\left\|2^{4m}f\left(\frac{x}{2^m}\right) - f(x)\right\| \le \frac{1}{32}\sum_{i=1}^{m}2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{14}$$

for $m > 0$ and $x \in A$. Hence (7) and (14) show that approximate inequality (10) holds.

Next, we have to show that the mapping $\delta$ is a quartic $*$-derivation such that inequality (10) holds for all $x \in A$. Replacing $x, y$ by $2^i x, 2^i y$ in (8), respectively, and putting $z = 0$, we have

$$\frac{1}{2^{4i}}\left\|\mathbf{Q}_\mu f\left(2^i x, 2^i y\right)\right\| \le \frac{\psi\left(2^i x, 2^i y, 0, 0, 0\right)}{2^{4i}}$$
$$\left(2^{4i}\left\|\mathbf{Q}_\mu f\left(\frac{x}{2^i}, \frac{y}{2^i}\right)\right\| \le 2^{4i}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}, 0, 0, 0\right), \text{ respectively}\right) \tag{15}$$

and so it follows from (7) and (13) that

$$\left\|\mathbf{Q}_\mu \delta(x, y)\right\| = 0 \tag{16}$$

for all $x, y \in A$ and $\mu \in \mathbb{T}^1_{1/n_0}$. Thus, by the same argument in the proof of Theorem 3.2 in [18], $\mathbf{Q}_1\delta(x, y) = 0$ and $\delta(\mu x) = \mu^4\delta(x)$ for all $x, y \in A$ and $\mu \in \mathbb{C}$, which implies that the mapping $\delta$ is quartic homogeneous by Lemma 2.

Next, replacing $a, b$ by $2^n a, 2^n b$ in inequality (9), we get

$$\frac{1}{2^{8n}}\|\mathbf{D}f(2^n a, 2^n b)\| \le \frac{1}{2^{8n}}\psi(0, 0, 0, 2^n a, 2^n b)$$
$$\left(2^{8n}\left\|\mathbf{D}f\left(\frac{a}{2^n}, \frac{b}{2^n}\right)\right\| \le 2^{8n}\psi\left(0, 0, 0, \frac{a}{2^n}, \frac{b}{2^n}\right), \text{ respectively}\right) \tag{17}$$

for all $a, b \in A$. By (7), we have $\mathbf{D}\delta(a, b) = 0$ for all $a, b \in A$. Letting $x = y = 0$ and replacing $z$ by $2^n z$ in inequality (8), we have

$$\left\|\frac{f(2^n z^*)}{2^{4n}} - \frac{f(2^n z)^*}{2^{4n}}\right\| \le \frac{1}{2^{4n}}\psi(0, 0, 2^n z, 0, 0)$$
$$\left(\left\|2^{4n}f\left(\frac{z^*}{2^n}\right) - 2^{4n}f\left(\frac{z}{2^n}\right)^*\right\| \le 2^{4n}\psi\left(0, 0, \frac{z}{2^n}, 0, 0\right), \text{ respectively}\right) \tag{18}$$

for all $z \in A$. Also by (7), we have $\delta(z^*) = \delta(z)^*$ for all $z \in A$. Therefore $\delta$ is a quartic $*$-derivation.

Lastly, we should show that $\delta$ is unique. Suppose that $\delta' : A \longrightarrow B$ is another quartic $*$-derivation satisfying approximate inequality (10). So

$$\begin{aligned}
&\left\|\delta(x)-\delta'(x)\right\| = \frac{1}{2^{4n}}\left\|\delta(2^n x)-\delta'(2^n x)\right\| \\
&\quad \le \frac{1}{2^{4n}}\left[\left\|\delta(2^n x)-f(2^n x)\right\| + \left\|f(2^n x)\right.\right. \\
&\quad \left.\left. -\delta'(2^n x)\right\|\right] \le \frac{1}{2^{4(n+1)}} \\
&\quad \cdot \sum_{i=0}^{\infty}\frac{1}{2^{4i}}\psi\left(2^{i+n}x, 2^{i+n}x, 0, 0, 0\right), \\
&\left(\left\|\delta(x)-\delta'(x)\right\| = 2^{4n}\left\|\delta\left(\frac{x}{2^n}\right)-\delta'\left(\frac{x}{2^n}\right)\right\|\right. \\
&\quad \le 2^{4n}\left[\left\|\delta\left(\frac{x}{2^n}\right)-f\left(\frac{x}{2^n}\right)\right\|\right. \\
&\quad \left.\left. + \left\|f\left(\frac{x}{2^n}\right)-\delta'\left(\frac{x}{2^n}\right)\right\|\right] \le \frac{1}{16} \right. \\
&\quad \left. \cdot \sum_{i=n+1}^{\infty} 2^{4i}\psi\left(\frac{x}{2^i}, \frac{x}{2^i}, 0, 0, 0\right), \textit{ respectively}\right)
\end{aligned} \tag{19}$$

which tends to zero as $n \longrightarrow \infty$ for all $x \in A$. Hence $\delta(x) = \delta'(x)$ for all $x \in A$. □

**Corollary 6.** *Let $\theta_j, p_i, q_j$ $(i = 1, 2, 3, j = 1, 2)$ be nonnegative real constants with either $0 \le p_i < 4, q_j < 8$ or $p_i > 4, q_j > 8$ $(i = 1, 2, 3, j = 1, 2)$ and let $f : A \longrightarrow B$ be a mapping with $f(0) = 0$ such that*

$$\begin{aligned}
&\left\|\mathbf{Q}_\mu f(x,y) + f(z^*) - f(z)^*\right\| \\
&\quad \le \theta_1\left(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}\right) \\
&\left\|\mathbf{D}f(a,b)\right\| \le \theta_2\left(\|a\|^{q_1} + \|b\|^{q_2}\right)
\end{aligned} \tag{20}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Assume that the mapping $t \longmapsto f(ta)$ from $\mathbb{R}$ to $B$ is continuous for each fixed $a \in A$. Then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying*

$$\left\|f(x) - \delta(x)\right\| \le \frac{\theta_1\|x\|^{p_1}}{2\left|2^{p_1}-16\right|} + \frac{\theta_1\|x\|^{p_2}}{2\left|2^{p_2}-16\right|} \tag{21}$$

*for all $x \in A$.*

*Proof.* Letting $\psi(x, y, z, a, b) := \theta_1(\|x\|^{p_1} + \|y\|^{p_2} + \|z\|^{p_3}) + \theta_2(\|a\|^{q_1} + \|b\|^{q_2})$ and applying Theorem 5, we obtain the desired result. □

Concerning the stability of quartic homogeneous function, the following example presents that the stability of functional equation $\mathbf{Q}_\mu f(x, y) = 0$ in Corollary 6 with $p_i = 4(i = 1, 2)$ does not hold.

*Example 7.* Let $\phi : \mathbb{C} \longrightarrow \mathbb{C}$ be defined by

$$\phi(x) = \begin{cases} x^4, & \text{if } |x| < 1, \\ 1, & \text{if } |x| \ge 1. \end{cases} \tag{22}$$

Consider the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ defined by

$$f(x) = \sum_{m=0}^{\infty}\alpha^{-4m}\phi(\alpha^m x) \tag{23}$$

for all $x \in \mathbb{C}$, where $\alpha \ge 2$. Then, using similar way to that in [25], $f$ satisfies

$$\left|\mathbf{Q}_\mu f(x,y)\right| \le \theta_1\left(|x|^4 + |y|^4\right), \quad \theta_1 := \frac{40\alpha^8}{\alpha^4 - 1} \tag{24}$$

for all $x, y \in \mathbb{C}$ and $\mu \in \mathbb{T}^1_{1/n_0}$, but there do not exist a quartic mapping $\delta : \mathbb{C} \longrightarrow \mathbb{C}$ and a constant $\gamma > 0$ such that $|f(x) - \delta(x)| \le \gamma|x|^4$ for all $x \in \mathbb{C}$.

However, the stability problem of $p_i = 4(i = 1, 2, 3)$ and $q_j = 8(j = 1, 2)$ is open in Corollary 6 concerning the stability of quartic $*$-derivations.

**Corollary 8.** *Let $\theta_j, p_i, q_j$ $(i = 1, 2, 3, j = 1, 2)$ be nonnegative real constants with either $0 \le p_i < 2, 0 \le q_j < 4$ or $p_i > 2, q_j > 4$ $(i = 1, 2, 3, j = 1, 2)$ and let $f : A \longrightarrow B$ be a mapping with $f(0) = 0$ such that*

$$\begin{aligned}
&\left\|\mathbf{Q}_\mu f(x,y) + f(z^*) - f(z)^*\right\| \\
&\quad \le \theta_1\left(\|x\|^{p_1}\|y\|^{p_2} + \|y\|^{p_2}\|z\|^{p_3} + \|z\|^{p_3}\|x\|^{p_1}\right), \\
&\left\|\mathbf{D}f(a,b)\right\| \le \theta_2\left(\|a\|^{q_1}\|b\|^{q_2}\right)
\end{aligned} \tag{25}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Assume that the mapping $t \longmapsto f(ta)$ from $\mathbb{R}$ to $B$ is continuous for each fixed $a \in A$. Then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying*

$$\left\|f(x) - \delta(x)\right\| \le \frac{\theta_1\|x\|^{p_1+p_2}}{2\left|16 - 2^{p_1+p_2}\right|} \tag{26}$$

*for all $x \in A$.*

*Proof.* Letting $\psi(x, y, z, a, b) := \theta_1(\|x\|^{p_1}\|y\|^{p_2} + \|y\|^{p_2}\|z\|^{p_3} + \|x\|^{p_1}\|z\|^{p_3}) + \theta_2(\|a\|^{q_1}\|b\|^{q_2})$, as well as applying Theorem 5, we obtain the desired result. □

Now, we investigate the stability using the alternative fixed-point method. Before proceeding to the main result, we state the following definition and theorem which are useful for our purpose.

*Definition 9.* Let $X$ be a set. A function $d : X \times X \longrightarrow [0, \infty]$ is called a generalized metric on $X$ if $d$ satisfies the following:

(i) $d(x, y) = 0$, if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, z) \le d(x, y) + d(y, z)$ for all $x, y, z \in X$.

**Theorem 10** (see [24]). *Let $(\Omega, d)$ be a complete generalized metric space and let $T : \Omega \longrightarrow \Omega$ be a mapping with Lipschitz constant $L < 1$. Then, for each element $\alpha \in \Omega$, either $d(T^n\alpha, T^{n+1}\alpha) = \infty$ for all $n \geq 0$ or there exists a natural number $n_0$ such that*

(i) *$d(T^n\alpha, T^{n+1}\alpha) < \infty$ for all $n \geq n_0$,*

(ii) *the sequence $\{T^n\alpha\}$ is convergent to a fixed point $\beta^*$ of $T$,*

(iii) *$\beta^*$ is the unique fixed point of $T$ in the set $\Lambda = \{\beta \in \Omega : d(T^{n_0}\alpha, \beta)\}$,*

(iv) *$d(\beta, \beta^*) \leq (1/(1-L))d(\beta, T\beta)$ for all $\beta \in \Lambda$.*

**Theorem 11.** *Let $f : A \longrightarrow B$ be a continuous mapping with $f(0) = 0$ and let $\psi : A^5 \longrightarrow [0, \infty)$ be a function such that*

$$\left\|\mathbf{Q}_\mu f(x, y) + f(z^*) - f(z)^*\right\| \leq \psi(x, y, z, 0, 0), \tag{27}$$

$$\|\mathbf{D} f(a, b)\| \leq \psi(0, 0, 0, a, b) \tag{28}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $a, b, x, y, z \in A$. If there exist constants $l_1, l_2 \in (0, 1)$ such that*

$$\begin{gathered}\psi(2x, 2y, 2z, 0, 0) \leq 2^4 l_1 \psi(x, y, z, 0, 0) \\ \& \ \psi(0, 0, 0, 2a, 2b) \leq 2^8 l_2 \psi(0, 0, 0, a, b) \\ \left(\psi\left(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}, 0, 0\right) \leq \frac{l_1}{2^4}\psi(x, y, z, 0, 0)\right. \\ \& \ \psi(0, 0, 0, a, b) \\ \left.\leq \frac{l_2}{2^8}\psi(0, 0, 0, a, b), \text{ respectively}\right)\end{gathered} \tag{29}$$

*for all $x, y, z, a, b \in A$, then there exists a unique quartic $*$-derivation $\delta : A \longrightarrow B$ satisfying*

$$\|f(x) - \delta(x)\| \leq \frac{1}{32(1 - l_1)}\widetilde{\psi}(x) \tag{30}$$

$$\left(\|f(x) - \delta(x)\| \leq \frac{l_1}{32(1 - l_1)}\widetilde{\psi}(x), \text{ respectively}\right) \tag{31}$$

*for all $x \in A$, where $\widetilde{\psi}(x) = \psi(x, x, 0, 0, 0)$.*

*Proof.* First, we consider a set

$$\Omega = \{g : A \longrightarrow B : g(0) = 0\} \tag{32}$$

and define a mapping $d$ on $\Omega \times \Omega$ as follows:

$$\begin{gathered}d(g_1, g_2) := \inf\{k \in (0, \infty) : \|g_1(x) - g_2(x)\| \\ \leq k\psi(x, x, 0, 0, 0)\}\end{gathered} \tag{33}$$

if there exists such constant $k$ and $d(g_1, g_2) \equiv \infty$, if not. Then we can easily show that $d$ is a generalized metric on $\Omega$ and the metric space $(\Omega, d)$ is complete. We define a mapping $\Psi : \Omega \longrightarrow \Omega$ by

$$\begin{gathered}\Psi g(x) = \frac{1}{2^4} g(2x) \\ \left(\Psi g(x) = 2^4 g\left(\frac{x}{2}\right), \text{ respectively}\right)\end{gathered} \tag{34}$$

where $g \in \Omega$ and for all $x \in A$.

Now we remark that $\Psi$ is a strictly contractive mapping on $\Omega$ with the Lipschitz constant $l_1$ [18].

On the other hand, letting $\mu = 1, y = x, z = 0$ in inequality (27), we get

$$\begin{gathered}\left\|\frac{1}{16} f(2x) - f(x)\right\| \leq \frac{1}{32}\psi(x, x, 0, 0, 0) \\ \left(\left\|16 f\left(\frac{x}{2}\right) - f(x)\right\| \leq \frac{1}{2}\psi\left(\frac{x}{2}, \frac{x}{2}, 0, 0, 0\right)\right. \\ \left.\leq \frac{l_1}{32}\psi(x, x, 0, 0, 0), \text{ respectively}\right)\end{gathered} \tag{35}$$

for all $x \in A$. This implies that

$$\begin{gathered}d(\Psi f, f) \leq \frac{1}{32} \\ \left(d(\Psi f, f) \leq \frac{l_1}{32}, \text{ respectively}\right).\end{gathered} \tag{36}$$

It follows from Theorem 10 that $d(\Psi^n f, \Psi^{n+1} f) < \infty$ for all $n \geq 0$. So parts (iii) and (iv) of Theorem 10 hold on the whole $\Omega$. Therefore, there exists a unique mapping $\delta : A \longrightarrow B$ such that $\delta$ is a fixed point of $\Psi$ and

$$\begin{gathered}\delta(x) = \lim_{n\longrightarrow\infty} \frac{f(2^n x)}{2^{4n}} \\ \left(\delta(x) = \lim_{n\longrightarrow\infty} 2^{4n} f\left(\frac{x}{2^n}\right), \text{ respectively}\right)\end{gathered} \tag{37}$$

for all $x \in A$ and

$$\begin{gathered}d(f, \delta) \leq \frac{1}{1 - l_1} d(\Psi f, f) \leq \frac{1}{32(1 - l_1)} \\ \left(d(f, \delta) \leq \frac{l_1}{1 - l_1} d(\Psi f, f)\right. \\ \left.\leq \frac{l_1}{32(1 - l_1)}, \text{ respectively}\right)\end{gathered} \tag{38}$$

So the mapping $\delta$ satisfies inequality (30) that holds for all $x \in A$.

Since $l_1, l_2 \in (0,1)$, inequality (30) shows that

$$\begin{aligned}&\lim_{i\longrightarrow\infty}\frac{\psi\left(2^i x, 2^i y, 2^i z, 0, 0\right)}{2^{4i}} = 0\\ &\& \lim_{i\longrightarrow\infty}\frac{\psi\left(0,0,0,2^i a, 2^i b\right)}{2^{8i}} = 0\\ &\left(\lim_{i\longrightarrow\infty} 2^{4i}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}, \frac{z}{2^i}, 0, 0\right) = 0\right.\\ &\quad \left.\& \lim_{i\longrightarrow\infty} 2^{8^i}\psi\left(0,0,0,\frac{a}{2^i},\frac{b}{2^i}\right) = 0, \textit{ respectively}\right)\end{aligned} \tag{39}$$

for all $x, y, z, a, b \in A$. Replacing $x, y$ by $2^i x, 2^i y$ and putting $z = 0$ in inequality (27), we have

$$\begin{aligned}&\left\|\frac{1}{2^{4i}}\mathbf{Q}_\mu f\left(2^i x, 2^i y\right)\right\| \le \frac{\psi\left(2^i x, 2^i y, 0, 0, 0\right)}{2^{4i}}\\ &\left(\left\|2^{4i}\mathbf{Q}_\mu f\left(\frac{x}{2^i}, \frac{y}{2^i}\right)\right\|\right.\\ &\quad \left.\le 2^{4i}\psi\left(\frac{x}{2^i}, \frac{y}{2^i}, 0, 0, 0\right), \textit{ respectively}\right)\end{aligned} \tag{40}$$

and taking the limit as $i$ tends to infinity, we get $\mathbf{Q}_\mu\delta(x,y) = 0$ for all $x, y \in A$ and all $\mu \in \mathbb{T}^1_{1/n_0}$. Also, by the same argument in the proof of Theorem 5, the mapping $\delta$ is quartic homogeneous. Next, replacing $a, b$ by $2^n a, 2^n b$ in inequality (28), we get

$$\begin{aligned}&\frac{1}{2^{8i}}\left\|\mathbf{D}f\left(2^i a, 2^i b\right)\right\| \le \frac{1}{2^{8i}}\psi\left(0,0,0,2^i a, 2^i b\right)\\ &\left(\left\|2^{8i}\mathbf{D}f\left(\frac{a}{2^i}, \frac{b}{2^i}\right)\right\|\right.\\ &\quad \left.\le 2^{8i}\psi\left(0,0,0,\frac{a}{2^i},\frac{b}{2^i}\right), \textit{ respectively}\right)\end{aligned} \tag{41}$$

for all $a, b \in A$. By (39), we have $\mathbf{D}\delta(a,b) = 0$ for all $a, b \in A$. Letting $x = y = 0$ and replacing $z$ by $2^n z$ in inequality (27), we have

$$\begin{aligned}&\left\|\frac{f\left(2^n z^*\right)}{2^{4n}} - \frac{f\left(2^n z\right)^*}{2^{4n}}\right\| \le \frac{1}{2^{4n}}\psi\left(0,0,2^n z, 0, 0\right)\\ &\left(\left\|2^{4n} f\left(\frac{z^*}{2^n}\right) - 2^{4n} f\left(\frac{z}{2^n}\right)^*\right\|\right.\\ &\quad \left.\le 2^{4n}\psi\left(0,0,\frac{z}{2^n},0,0\right), \textit{ respectively}\right)\end{aligned} \tag{42}$$

for all $z \in A$. Also by (39), we have $\delta(z^*) = \delta(z)^*$ for all $z \in A$. Therefore the mapping $\delta$ is a quartic $*$-derivation.

The rest of the proof is similar to the proof of Theorem 5. □

**Corollary 12.** *Let $\theta_i (i = 1, 2)$, $p, q$ be nonnegative reals with either $0 \le p < 4, q < 8$ or $p > 4, q > 8$ and let $f : A \longrightarrow B$ be a continuous mapping with $f(0) = 0$ such that*

$$\begin{aligned}&\left\|\mathbf{Q}_\mu f(x,y) + f(z^*) - f(z)^*\right\|\\ &\quad \le \theta_1\left(\|x\|^p + \|y\|^p + \|z\|^p\right)\\ &\left\|\mathbf{D}f(a,b)\right\| \le \theta_2\left(\|a\|^q + \|b\|^q\right)\end{aligned} \tag{43}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Then there exists a unique quartic $*$-derivation $\delta$ on $A$ satisfying*

$$\left\|f(x) - \delta(x)\right\| \le \frac{\theta_1\|x\|^p}{|16 - 2^p|} \tag{44}$$

*for all $x \in A$.*

*Proof.* Letting $\psi(x,y,z,a,b) := \theta_1(\|x\|^p + \|y\|^p + \|z\|^p) + \theta_2(\|a\|^q + \|b\|^q)$ and applying Theorem 11 with $l_1 = 2^{-|p-4|}$, $l_2 = 2^{-|q-8|}$, we obtain the desired results. □

**Corollary 13.** *Let $\theta_i (i = 1, 2)$, $p, q$ be nonnegative reals with either $0 \le p < 2, q < 4$ or $p > 2, q > 4$ and let $f : A \longrightarrow B$ be a continuous mapping with $f(0) = 0$ such that*

$$\begin{aligned}&\left\|\mathbf{Q}_\mu f(x,y) + f(z^*) - f(z)^*\right\|\\ &\quad \le \theta_1\left(\|x\|^p\|y\|^p + \|y\|^p\|z\|^p + \|z\|^p\|x\|^p\right)\\ &\left\|\mathbf{D}f(a,b)\right\| \le \theta_2\left(\|a\|^q\|b\|^q\right)\end{aligned} \tag{45}$$

*for all $\mu \in \mathbb{T}^1_{1/n_0}$ and $x, y, z, a, b \in A$. Then there exists a unique quartic $*$-derivation $\delta$ on $A$ satisfying*

$$\left\|f(x) - \delta(x)\right\| \le \frac{\theta_1\|x\|^{2p}}{2\left|16 - 2^{2p}\right|} \tag{46}$$

*for all $x \in A$.*

*Proof.* Letting $\psi(x,y,z,a,b) := \theta_1(\|x\|^p\|y\|^p + \|y\|^p\|z\|^p + \|z\|^p\|x\|^p) + \theta_2(\|a\|^q\|b\|^q)$ and applying Theorem 11 with $l_1 = 2^{-|2p-4|}$, $l_2 = 2^{-|2q-8|}$, we obtain the desired results. □

## Data Availability

No data were used to support this study.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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## References


[1] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, Chapter 6, Wiley, New York, NY, USA, 1940.

[2] D. H. Hyers, "On the stability of the linear functional equation," *Proceedings of the National Acadamy of Sciences of the United States of America*, vol. 27, pp. 222–224, 1941.

[3] T. Aoki, "On the stability of the linear transformation in Banach spaces," *Journal of the Mathematical Society of Japan*, vol. 2, pp. 64–66, 1950.

[4] Th. M. Rassias, "On the stability of the linear mapping in Banach spaces," *Proceedings of the American Mathematical Society*, vol. 72, no. 2, pp. 297–300, 1978.

[5] H. Kim and H. Shin, "Approximate cubic lie derivations on $\rho$-complete convex modular algebras," *Journal of Function Spaces*, vol. 2018, Article ID 3613178, 8 pages, 2018.

[6] S.-M. Jung and S. Min, "Stability of the wave equation with a source," *Journal of Function Spaces*, vol. 2018, Article ID 8274159, 4 pages, 2018.

[7] A. Khan, K. Shah, Y. Li, and T. S. Khan, "Ulam type stability for a coupled system of boundary value problems of nonlinear fractional differential equations," *Journal of Function Spaces*, vol. 2017, Article ID 3046013, 8 pages, 2017.

[8] S. Jung and J. Roh, "Approximation property of the stationary stokes equations with the periodic boundary condition," *Journal of Function Spaces*, vol. 2018, Article ID 5138414, 5 pages, 2018.

[9] S. Y. Jang and C. Park, "Approximate $*$-derivations and approximate quadratic $*$-derivations on C$*$-algebras," *Journal of Inequalities and Applications*, vol. 2011, no. 55, 2011.

[10] C. Park and A. Bodaghi, "On the stability of $*$-derivations on Banach $*$-algebras," *Advances in Difference Equations*, vol. 2012, no. 1, article no. 138, 2012.

[11] S. Y. Yang, A. Bodaghi, and K. A. M. Atan, "Approximate cubic $*$-derivations on banach $*$-algebras," *Abstract and Applied Analysis*, vol. 2012, Article ID 684179, 12 pages, 2012.

[12] H. Koh and D. Kang, "Approximate generalized cubic $*$-derivations," *Journal of Function Spaces*, vol. 2014, Article ID 757956, 6 pages, 2014.

[13] J. M. Rassias, "Solution of the Ulam stability problem for quartic mappings," *Glasnik Matematički*, vol. 34, no. 2, pp. 243–252, 1999.

[14] Y.-S. Lee and S.-Y. Chung, "Stability of quartic functional equations in the spaces of generalized functions," *Advances in Difference Equations*, vol. 2009, Article ID 838347, 16 pages, 2009.

[15] D. Kang, "On the stability of generalized quartic mappings in quasi-$\beta$-normed spaces," *Journal of Inequalities and Applications*, vol. 2010, Article ID 198098, 11 pages, 2010.

[16] A. Bodaghi, "Stability of a quartic functional equation," *The Scientific World Journal*, vol. 2014, Article ID 752146, 9 pages, 2014.

[17] S. H. Lee, S. M. Im, and I. S. Hwang, "Quartic functional equations," *Journal of Mathematical Analysis and Applications*, vol. 307, no. 2, pp. 387–394, 2005.

[18] D. Kang and H. Koh, "A fixed point approach to the stability of quartic lie $*$-derivations," *The Korean Journal of Mathematics*, vol. 24, no. 4, pp. 587–600, 2016.

[19] R. Saadati, Y. J. Cho, and J. Vahidi, "The stability of the quartic functional equation in various spaces," *Computers & Mathematics with Applications*, vol. 60, no. 7, pp. 1994–2002, 2010.

[20] D. Miheţ, R. Saadati, and S. M. Vaezpour, "The stability of the quartic functional equation in random normed spaces," *Acta Applicandae Mathematicae*, vol. 110, no. 2, pp. 797–803, 2010.

[21] J. A. Baker, "The stability of certain functional equations," *Proceedings of the American Mathematical Society*, vol. 112, no. 3, pp. 729–732, 1991.

[22] V. Radu, "The fixed point alternative and the stability of functional equations," *Fixed Point Theory and Applications*, vol. 4, no. 1, pp. 91–96, 2003.

[23] K. Ciepliński, "Applications of fixed point theorems to the Hyers-Ulam stability of functional equations—a survey," *Annals of Functional Analysis*, vol. 3, no. 1, pp. 151–164, 2012.

[24] J. B. Diaz and B. Margolis, "A fixed point theorem of the alternative, for contractions on a generalized complete metric space," *Bulletin of the American Mathematical Society*, vol. 74, pp. 305–309, 1968.

[25] T. Z. Xu and J. M. Rassias, "On the hyers-ulam stability of a general mixed additive and cubic functional equation in n-banach spaces," *Abstract and Applied Analysis*, vol. 2012, Article ID 926390, 23 pages, 2012.