Research Article

Solution of Hamilton-Jacobi-Bellman Equation in Optimal Reinsurance Strategy under Dynamic VaR Constraint

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This paper analyzes the optimal reinsurance strategy for insurers with a generalized mean-variance premium principle. The surplus process of the insurer is described by the diffusion model which is an approximation of the classical Cramér-Lundberg model. We assume the dynamic VaR constraints for proportional reinsurance. We obtain the closed form expression of the optimal reinsurance strategy and corresponding survival probability under proportional reinsurance.

1. Introduction

In practice, reinsurance is an important way for an insurer to control its risk exposure. In the actuarial literature, the optimal reinsurance problem of minimising ruin probability or equivalently maximising survival probability has been studied extensively in the past two decades. As one type of typical reinsurance strategy, proportional reinsurance has received great attention from both the academics and practitioners. Among others, Choulli et al. (2003), Højgaard and Taksar [1,2], Schmidli [3,4], Taksar [5], and Zhang et al. [6] work on the proportional reinsurance.

In the existing literature, the expected value principle is commonly used as the reinsurance premium principle due to its simplicity and popularity in practice. For details, the readers are referred to Bäuerle [7], Bai and Zhang [8], and Liang and Bayraktar [9]. Generally speaking, expected value principle is commonly used in life insurance whose claim frequency and claim sizes are stable and smooth, while the variance premium principle is extensively used in property insurance; see Zhou and Yuen [10] and Sun et al. [11]. Similarly to Zhang et al. [6], in this paper, we focus on a generalized mean-variance premium principle, which includes the expected value principle and the variance principle as special cases.

More recently, the problem of optimal reinsurance design has been studied by using risk measures such as the Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), and conditional tail expectation (CTE) (to name a few, Cai and Tan [12], Cheung et al. [13], and Cai et al. [14, 15]). Latterly static risk measures have been extended to the dynamic version; see Yiu [16], Alexander and Baptista [17], Cuoco et al. [18], Chen et al. [19], and Zhang et al. [6], all of which investigate the optimal reinsurance problem under dynamic VaR constraint.

In this paper, we investigate an optimal proportional reinsurance problem under dynamic VaR constraint. Assume that an insurer aims to maximize the survival probability. With this assumption, we obtain the closed form expressions. The rest of the paper is organized as follows. In Section 2, we provide a general formulation of the optimal reinsurance problem. Then we investigate the insurance company's maximum survival probability under dynamic VaR constraints, and the corresponding optimal reinsurance strategy is given in proportional reinsurance settings in Section 3.
2. Formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$. Consider a Cramér-Lundberg model with the surplus process of an insurance company being given by

$$ U(t) = u_0 + ct - \sum_{i=1}^{N(t)} Y_i = u_0 + ct - S(t), $$

where $u_0$ is the initial surplus, the claim arrival process $\{N(t)\}_{t \geq 0}$ is a Poisson process with constant intensity $\lambda > 0$, and the random variables $Y_i$, $i = 1, 2, \ldots$, are i.i.d claim sizes independent of $N(t)$. We let $T_i$ denote the $i$-th claim occurrence time and $F(y)$ denote the claim size distribution with finite first and second moments $m_1, m_2$. The premium rate $c$ is assumed to be calculated via the expected value principle; that is,

$$ c = (1 + \eta) \lambda m_1, $$

where $\eta > 0$ is the relative loading factor.

In this paper, the insurer can purchase proportional reinsurance to adjust the exposure to insurance risk. The proportional reinsurance level is associated with the risk exposure $q(t)$ at time $t$. We assume $q(t) \in [0, 1]$ for all $t$, and it means the insurer purchases proportional reinsurance.

We assume the reinsurance premium is calculated by the following generalized mean-variance principle; that is,

$$ c - \mathcal{c}^q = (\eta - \theta) \lambda m_1 + (1 + \theta) \lambda q(t) m_1 + \lambda (1 + \theta) \xi (1 - q(t))^2 m_2, $$

where $\mathcal{c}^q$ is the net reinsurance rate which the reinsurer receives from the insurer. We assume that the reinsurance premium is calculated by the following generalized mean-variance principle $(1 + \theta)[E(\xi) + \xi D(\xi)], where $\theta, \xi \geq 0,$ and $E$ and $D$ denote the expectation and variance, respectively.

$$ c - \mathcal{c}^q = (\eta - \theta) \lambda m_1 + (1 + \theta) \lambda q(t) m_1 + \lambda (1 + \theta) \xi (1 - q(t))^2 m_2. $$

According to Grandell (1991), the surplus process after reinsurance can be approximated by the following diffusion process:

$$ \mathcal{U}^q(t) = u + \int_0^t \left[ (\eta - \theta) \lambda m_1 + \theta \lambda q(s) m_1 + \lambda (1 + \theta) \xi (1 - q(s))^2 m_2 \right] ds + \int_0^t q(s) \cdot \sqrt{\lambda m_2} dB(s), $$

where $\{B(t)\}_{t \geq 0}$ is a standard Brownian motion.

We define the ruin time

$$ \tau^q = \inf \{ t > 0, U^q(t) \leq 0 \} $$

where the superscript $q$ emphasises that the surplus process and the ruin time are controlled by an admissible policy $q$.

Denote the survival probability given the initial surplus $u$ by

$$ v^q(u) = \mathbb{P} (\tau^q = \infty \mid U^q(0) = u), $$

and the maximum survival probability by

$$ v(u) = \max_{q \in \Pi} v^q(u). $$

Our objective is to find the value function $v(u)$ and the optimal policy $q^*$ such that

$$ v(u) = v^{q^*}(u). $$

3. Maximizing Survival Probability

Under the proportional reinsurance, the insurer could transfer a fraction $1 - q(t)$ of the incoming claims to a reinsurer, where $q(t)$ is $\mathcal{F}_t$-measurable and satisfies $0 \leq q(t) \leq 1$ for all $t$. The diffusion approximation of insurance company’s claim process becomes

$$ dS^q(t) = \lambda q(t) m_1 dt + q(t) \sqrt{\lambda m_2} dB(t), $$

$$ S^q(0) = 0, $$

where $B(t)$ is a standard Brownian motion. The insurer’s surplus process satisfies the stochastic differential equation

$$ dU^q(t) = [(\eta - \theta) \lambda m_1 + \theta \lambda q(s) m_1 + \lambda (1 + \theta) \xi (1 - q(s))^2 m_2] dt + q(t) \sqrt{\lambda m_2} dB(t), $$

$$ U^q(0) = u_0. $$

Taking $h > 0$ is small enough, we assume that risk exposure does not change over the short time period $[t, t + h]$. This means that the risk exposure remains roughly constant in the given time period; that is, $E[q(s)Y] = E[q(t)Y]$, $\sqrt{\lambda E[(q(s)Y)^2]} = \sqrt{\lambda E[(q(t)Y)^2]}$, $s \in [t, t + h]$. This setting is reasonable because the insurer can only adjust its reinsurance business at discrete time; and the decision made is based on the holding at time $t$. Thus, we rewrite the claim dynamics as

$$ S^q(t+h) - S^q(t) = \int_t^{t+h} \lambda E[q(s)Y] ds + \int_t^{t+h} \sqrt{\lambda E[(q(s)Y)^2]} dB(s) $$

$$ = \lambda q(t) m_1 h + q(t) \sqrt{\lambda m_2} \int_t^{t+h} dB(s). $$
3.1. Dynamic VaR, CVaR, and Worst-Case CVaR. For a given confidence level \(1 - \alpha \in (0, 1)\) and a given horizon \(h > 0\), the VaR at time \(t\) of a proportional reinsurance policy \(q\), denoted by \(\text{VaR}^{a,h}_{t}\), is defined as

\[
\text{VaR}^{a,h}_{t} = \inf \{ L : \mathbb{P}(S^{l}(t+h) - S^{l}(t) \geq L | \mathcal{F}_{t}) < \alpha \}. \tag{14}
\]

The dynamic Conditional Value-at-Risk \(\text{CVaR}^{a,h}_{t}\) is given by

\[
\text{CVaR}^{a,h}_{t} = \mathbb{E}[(S^{l}(t+h) - S^{l}(t) | S^{l}(t+h) - S^{l}(t) \geq \text{VaR}^{a,h}_{t})]. \tag{15}
\]

The dynamic worst-case CVaR is defined as

\[
\text{wcCVaR}^{a,h}_{t} = \sup_{P \in \mathcal{P}_{1}} \inf_{a \in \mathbb{R}} \left\{ a + \frac{1}{\alpha} \mathbb{E}_{P}\left[(S^{l}(t+h) - S^{l}(t) - a)_{+} \right] \right\}, \tag{16}
\]

where \(\mathcal{P}_{1} = \{ P(\cdot) : \mathbb{E}_{P}[S^{l}(t+h) - S^{l}(t)] = \lambda m_{1} q(t), \mathbb{E}_{P}[(S^{l}(t+h) - S^{l}(t))^{2}] = \lambda m_{2} q(t)^{2} + (\lambda m_{1} q(t))^{2} \}\).

Proposition 1 (Zhang et al. [6]).

\[
\text{VaR}^{a,h}_{t} = \lambda m_{1} q(t) \Phi^{-1}(\alpha) \sqrt{\lambda m_{2}},
\]

\[
\text{CVaR}^{a,h}_{t} = \lambda m_{1} q(t) + \frac{\phi(\Phi^{-1}(\alpha))}{\alpha} q(t) \sqrt{\lambda m_{2}},
\]

\[
\text{wcCVaR}^{a,h}_{t} = \lambda m_{1} q(t) + \sqrt{1 - \alpha} \frac{\phi(\Phi^{-1}(\alpha))}{\alpha} q(t) \sqrt{\lambda m_{2}},
\]

\[
0 \leq \text{VaR}^{a,h}_{t} \leq \text{CVaR}^{a,h}_{t} \leq \text{wcCVaR}^{a,h}_{t} < U_{q}(t), \tag{18}
\]

where \(\phi(x)\) and \(\Phi(x)\) denote the probability density function and the cumulative distribution function of a standard normal random variable, respectively. \(\Phi^{-1}(x)\) is the inverse function of \(\Phi(x)\).

3.2. HJB Equation. Using the dynamic programming technique, we obtain that the value function \(\psi(u)\) satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

\[
\max_{q \in [0,1]} \left\{ \left[ (\eta - \theta) \lambda m_{1} + \theta \lambda q m_{1} - \lambda (1+\theta) \xi (1-q)^{2} m_{2} \right] \psi' (u) + \frac{1}{2} \lambda q^{2} m_{2} \psi'' (u) \right\} = 0, \tag{19}
\]

s.t. \(q \in [0,1]; \text{VaR}^{a,h}_{t} \leq ku; \psi(0) = 0, \psi(+\infty) = 1\),

where \(k (0 < k < +\infty)\) is a constant.

Next we try to construct a solution of the HJB equation (19) with the boundary condition (20). Suppose that

\[
\begin{align*}
\varphi (A) &= \begin{cases} 
\varphi \left( \frac{A}{k} \right) \varphi \left( \frac{A}{k} \right) \int_{0}^{A/k} w e^{-\Delta_{1}w - \Delta_{2}w} dw, & \text{if } 0 < u \leq \frac{A}{k}, \\
\varphi \left( \frac{A}{k} \right) + \left[ 1 - \varphi \left( \frac{A}{k} \right) \right] \left[ 1 - e^{-2\eta m_{1}/m_{2}} (u - A/k) \right], & \text{if } u \geq \frac{A}{k},
\end{cases}
\end{align*} \tag{21}
\]

is a smooth \((C^{2})\) solution to the HJB equation, where

\[
A = \lambda m_{1} h - \Phi^{-1}(\alpha) \sqrt{\lambda m_{2}},
\]

\[
\varphi \left( \frac{A}{k} \right) = \frac{2\eta m_{1}/m_{2}}{2\eta m_{1}/m_{2} + (A/k)^{-\Delta_{2}w e^{-\Delta_{1}w} e^{\Delta_{1}w} dw}, \tag{22}
\]

and where

\[
\mathcal{P}_{1} = \left\{ p(\cdot) : \mathbb{E}_{p}[S^{l}(t+h) - S^{l}(t)] = \lambda h m_{1} q(t), \mathbb{E}_{p}[S^{l}(t+h) - S^{l}(t))^{2}] = \lambda h m_{2} q(t)^{2} + (\lambda h m_{1} q(t))^{2} \right\}. \tag{17}
\]
\[ \Delta_1 = 2A^2[(\eta - \theta)m_1 - (1 + \theta)\xi m_2]/k^2m_2, \]  
\[ \Delta_2 = 2A[\theta m_1 + 2(1 + \theta)\xi m_2]/km_2, \]  
\[ \Delta_3 = 2(1 + \theta)\xi. \]  
The maximum of the left side of HJB equation is attained at
\[ q^*(u) = \begin{cases} 
\frac{ku}{A}, & \text{if } u < \frac{A}{k} \\
1, & \text{if } u \geq \frac{A}{k}.
\end{cases} \]  
(23)

(b) If \( \theta < 2\eta \), the function
\[ \phi(u) = \begin{cases} 
\phi(u_1) - \phi(u_1) - \int_{u_1}^u \frac{\partial}{\partial v} \phi(v) e^{(v-u)\eta} dv, & \text{if } 0 < u \leq u_1, \\
\phi(u_1) + (1 - \phi(u_1)) [1 - e^{(v-u)\eta}] & \text{if } u \geq u_1,
\end{cases} \]  
(24)
is a smooth \((C^2)\) solution to the HJB equation, where
\[ \phi(u_1) = \frac{\Delta_4 + \frac{1}{2} e^{(u_1-u)\eta}}{\int_{u_1}^u \frac{\partial}{\partial v} \phi(v) e^{(v-u)\eta} dv}. \]  
(25)

\[ \Delta_4 = (1/2\theta^2 m_2^2 + 2\theta(1 + \theta)\xi m_2)/(2(1/2 + \theta)\xi m_2). \]  
(26)
Proof. We solve the HJB equation analytically. First we need to determine the optimal strategy \( q^*(u) \). Differentiating the terms inside the maximum in (19) with respect to \( q(u) \) and setting to 0 yield
\[ q^0(u) = \frac{\theta m_1 + 2\xi (1 + \theta)\xi m_2}{2\xi(1 + \theta)\xi m_2 - m_2^{1/2} (\phi''/\phi')} . \]  
(27)
The dynamic VaR constraint implies \( q(u) \leq ku/A \), when \( A \) is defined by (27). Normally, we take 0 < \( \alpha < 1/2 \); hence, \( A \) is always positive.

(1) For \( u \geq A/k \), we have \( ku/A \geq 1 \). Then, from \( q(u) \leq ku/A \), obtained from the dynamic VaR constraint and the requirement that the retained proportion of claims \( q(u) \) is always within [0, 1], we have \( q(u) \in [0, 1] \).

(a) If \( q^0(u) \geq 1 \), we let \( q^0(u) = 1 \), and then the HJB equation becomes
\[ \eta m_1 q'(u) + \frac{1}{2} m_2 q''(u) = 0, \]  
(28)
which implies
\[ \frac{q''(u)}{q'(u)} = \frac{2\eta m_1}{m_2}. \]  
(29)
Inserting it into (28), we obtain
\[ q^0(u) = \frac{\theta m_1 + 2\xi (1 + \theta)\xi m_2}{2\xi(1 + \theta)\xi m_2 - 2\eta m_1}. \]  
(30)

(i) If \( \theta \geq 2\eta \), we have \( q^0(u) \geq 1 \); consequently \( q^*(u) = 1 \), and then the HJB equation becomes (30).
(i) If $\theta \geq 2\eta$, we have $q^0(u) \geq ku/A$; consequently $q^*(u) = q^0(u) = ku/A$, and then the HJB equation becomes (34).

(ii) If $\theta < 2\eta$, we have $q^0(u) < ku/A$, where conflict exits.

For $u \leq (A/k)((\theta - \eta)m_1 + (1 + \theta)\zeta m_2)/((1 + \theta)\zeta m_2 + (1/2)\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1 + \theta)\zeta m_2 m_1}))$, we have $q''(u) \geq 0$; therefore $q(u)$ is convex for small $u$. Through the analysis of the HJB equation (19), for $0 \leq u \leq (A/k)((\theta - \eta)m_1 + (1 + \theta)\zeta m_2)/((1 + \theta)\zeta m_2 + (1/2)\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1 + \theta)\zeta m_2 m_1}))$, the maximum of the left side of the HJB is attained at $q^*(u) = ku/A$ and the HJB equation becomes (34).

(b) When $q^0(u) \leq ku/A$, it is reasonable to let $q^*(u) = q^0(u)$. Similar to (a), we have the following conclusions.

For $0 \leq u \leq (A/k)((\theta - \eta)m_1 + (1 + \theta)\zeta m_2)/((1 + \theta)\zeta m_2 + (1/2)\theta m_1)$, the optimal strategy is obtained at $q^*(u) = ku/A$. For $(A/k)((\theta - \eta)m_1 + (1 + \theta)\zeta m_2)/((1 + \theta)\zeta m_2 + (1/2)\theta m_1) < u \leq A/k$, we have the following.

(i) If $\theta \geq 2\eta$, we have $q^0(u) \geq 1 > ku/A$, where conflict exits.

(ii) If $\theta < 2\eta$, we have $q^0(u) < ku/A$; consequently, $q^*(u) = q^0(u) = ((\theta - \eta)m_1 + (1 + \theta)\zeta m_2)/((1 + \theta)\zeta m_2 + (1/2)\theta m_1)$, and then the HJB equation becomes (32).

From the previous analysis, we have the following conclusions.

(i) If $\theta \geq 2\eta$, the maximum of the left side of HJB equation is attained at

$$q^*(u) = \begin{cases} ku/A, & \text{if } u < A/k, \\ 1, & \text{if } u \geq A/k. \end{cases} \quad (36)$$

(ii) If $\theta < 2\eta$, the maximum of the left side of HJB equation is attained at

$$q^*(u) = \begin{cases} ku/A, & \text{if } u < u_1, \\ (\theta - \eta)m_1 + (1 + \theta)\zeta m_2, & \text{if } u \geq u_1, \end{cases} \quad (37)$$

In the following, we will solve the HJB equation in each situation.

For $\theta \geq 2\eta$ and $u < A/k$, the HJB equation is (34), which is equivalent to (35). Taking integral form $u$ to $A/k$, we obtain

$$\varphi(u) = \varphi\left(\frac{A}{k}\right) - K \int_u^{A/k} w^{-\Delta_1} e^{\Delta_1} e^{\Delta_2} dw, \quad (38)$$

where $\Delta_1 = 2A^2[(\eta - \theta)m_1 - (1 + \theta)\zeta m_2]/k^2m_2$, $\Delta_2 = 2A[\theta m_1 + 2(1 + \theta)\zeta m_1]/km_2$, and $\Delta_3 = (1 + \theta)\zeta$. Applying the boundary condition $\varphi(0) = 0$ we obtain

$$K = \frac{\varphi(A/k)}{\int_0^{A/k} w^{-\Delta_2} e^{-\Delta_1} e^{\Delta_2} dw}. \quad (39)$$

For $\theta \geq 2\eta$ and $u \geq A/k$, the corresponding HJB is (29), which is equivalent to (30). Taking integral form $A/k$ to $u$, we obtain

$$\varphi(u) = \varphi\left(\frac{A}{k}\right) + \frac{m_2}{2\theta m_1} \varphi\left(\frac{A}{k}\right) \left[1 - e^{-2\eta m_1/(u-A/k)}\right]. \quad (40)$$

Applying the boundary condition $\delta(+\infty) = 1$ we obtain

$$\varphi(u) = \varphi\left(\frac{A}{k}\right) + \left[1 - \varphi\left(\frac{A}{k}\right)\right] \left[1 - e^{-2\eta m_1/(u-A/k)}\right]. \quad (41)$$

Considering that $\varphi(x)$ is twice continuously differentiable, it should satisfy $\varphi'(A/k-) = \varphi'(A/k+)$; that is,

$$\varphi'(A/k-) e^{-\Delta_1} e^{\Delta_2} \int_0^{A/k} w^{-\Delta_1} e^{-\Delta_1} e^{\Delta_2} dw = 2\eta m_1/m_2 \left[1 - \varphi\left(\frac{A}{k}\right)\right], \quad (42)$$

which leads to

$$\varphi\left(\frac{A}{k}\right) = \frac{2\eta m_1/m_2}{2\eta m_1/m_2 + (A/k)^{-\Delta_1} e^{-\Delta_1} e^{\Delta_2} \int_0^{A/k} w^{-\Delta_1} e^{-\Delta_1} e^{\Delta_2} dw}. \quad (43)$$

Thus, if $\theta \geq 2\eta$, we have the function

$$\varphi(u) = \begin{cases} \varphi\left(\frac{A}{k}\right) - \varphi\left(\frac{A}{k}\right) \int_u^{A/k} w^{-\Delta_1} e^{-\Delta_1} e^{\Delta_2} dw, & \text{if } 0 < u \leq A/k, \\ \varphi\left(\frac{A}{k}\right) + \left[1 - \varphi\left(\frac{A}{k}\right)\right] \left[1 - e^{-2\eta m_1/(u-A/k)}\right], & \text{if } u \geq A/k. \end{cases} \quad (44)$$
If \( \theta < 2\eta \) and \( u < u_1 \), the HJB equation is (34), and the HJB equation is (32) for \( \theta \geq 2\eta \) and \( u \geq u_1 \). From the procedure that is similar to the previous analysis we can get the following function is a \( C^2 \) solution to HJB; that is,

\[
\varphi(u) = \begin{cases} 
\varphi(u_1) - \varphi(u_1) \int_{u_1}^{u} w^{-\Delta_i} e^{\Delta_i/k} e^{-2A(t)w} dw, & \text{if } 0 < u \leq u_1, \\
\varphi(u_1) + [1 - \varphi(u_1)] [1 - e^{-\Delta_i(u-u_1)}], & \text{if } u \geq u_1,
\end{cases}
\]

where

\[
\Delta_i = \frac{(1/2) \theta^2 m_1^2 + 2C \eta (1 + \theta) m_1 m_2}{m_2 [\eta (1 + \theta) m_2 + (\theta - \eta) m_1]},
\]

\[
\varphi(u_1)
\]

is a smooth \( (C^2) \) solution to the HJB equation, where

\[
\varphi(u_1) = \frac{2C \eta m_1 / (\eta m_1 - \eta m_2)}{2C \eta m_1 / (\eta m_1 - \eta m_2) + u_1 \int_{u_1}^{u} w^{-4A(t)w} e^{-2A(t)w} dw,}
\]

\[
\varphi(u_1) = 1 - \eta m_1 / \eta m_2, \quad \text{if } u \geq u_1.
\]

**Corollary 6.** When there is no dynamic VaR, CVar, or wcCVar constraints, that is, \( k = \infty \), and the model becomes the unconstrained reinsurance problem, we have the following.

(a) If \( \theta \geq 2\eta \), the optimal reinsurance strategy is \( q^* = 1 \), and the optimal survival probability is

\[
\varphi(u) = 1 - e^{-2\eta m_1 / m_2} u.
\]

(b) If \( \theta < 2\eta \), the optimal reinsurance strategy is \( q^* = ((\theta - \eta)m_1 + (1 + \theta)\eta m_2)/(1 + \theta)m_2 + (1/2)\theta m_1) \), and the optimal survival probability is

\[
\varphi(u) = 1 - e^{-((1/2)\theta^2 m_1^2 + 2C\theta m_2)m_1 / m_2 [1 + \theta] m_2 + (\theta - \eta) m_1]} u.
\]

**Data Availability**

No data were used to support this study.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

**Authors’ Contributions**

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.
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