Hindawi
Journal of Function Spaces
Volume 2019, Article ID 6750892, 7 pages
https://doi.org/10.1155/2019/6750892

Research Article

Solution of Hamilton-Jacobi-Bellman Equation in Optimal Reinsurance Strategy under Dynamic VaR Constraint

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Received 18 August 2018; Accepted 19 December 2018; Published 8 January 2019

Academic Editor: Yong H. Wu

This paper analyzes the optimal reinsurance strategy for insurers with a generalized mean-variance premium principle. The surplus process of the insurer is described by the diffusion model which is an approximation of the classical Cramér-Lunderberg model. We assume the dynamic VaR constraints for proportional reinsurance. We obtain the closed form expression of the optimal reinsurance strategy and corresponding survival probability under proportional reinsurance.

1. Introduction

In practice, reinsurance is an important way for an insurer to control its risk exposure. In the actuarial literature, the optimal reinsurance problem of minimising ruin probability or equivalently maximising survival probability has been studied extensively in the past two decades. As one type of typical reinsurance strategy, proportional reinsurance has received great attention from both the academics and practitioners. Among others, Choulli et al. (2003), Højgaard and Taksar [1, 2], Schmidli [3, 4], Taksar [5], and Zhang et al. [6] work on the proportional reinsurance.

In the existing literature, the expected value principle is commonly used as the reinsurance premium principle due to its simplicity and popularity in practice. For details, the readers are referred to Bäuerle [7], Bai and Zhang [8], and Liang and Bayraktar [9]. Generally speaking, expected value principle is commonly used in life insurance whose claim frequency and claim sizes are stable and smooth, while the variance premium principle is extensively used in property insurance; see Zhou and Yuen [10] and Sun et al. [11]. Similarly to Zhang et al. [6], in this paper, we focus on a generalized mean-variance premium principle, which includes the expected value principle and the variance principle as special cases.

More recently, the problem of optimal reinsurance design has been studied by using risk measures such as the Value-at-Risk (VaR), Conditional Value-at-Risk (CVaR), and conditional tail expectation (CTE) (to name a few, Cai and Tan [12], Cheung et al. [13], and Cai et al. [14, 15]). Latterly static risk measures have been extended to the dynamic version; see Yiu [16], Alexander and Baptista [17], Cuoco et al. [18], Chen et al. [19], and Zhang et al. [6], all of which investigate the optimal reinsurance problem under dynamic VaR constraint.

In this paper, we investigate an optimal proportional reinsurance problem under dynamic VaR constraint. Assume that an insurer aims to maximize the survival probability. With this assumption, we obtain the closed form expressions. The rest of the paper is organized as follows. In Section 2, we provide a general formulation of the optimal reinsurance problem. Then we investigate the insurance company's maximum survival probability under dynamic VaR constraints, and the corresponding optimal reinsurance strategy is given in proportional reinsurance settings in Section 3.

2. Formulation

Let $(\Omega, \mathcal{F}, P)$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Consider a Cramér-Lundberg model with the surplus process of an insurance company being given by

$$U(t) = u_0 + ct - \sum_{i=1}^{N(t)} Y_i = u_0 + ct - S(t), \tag{1}$$

where u_0 is the initial surplus, the claim arrival process $\{N(t)\}_{t\geq 0}$ is a Poisson process with constant intensity $\lambda > 0$, and the random variables Y_i, $i = 1, 2, \ldots$, are i.i.d claim sizes independent of $N(t)$. We let T_i denote the i-th claim occurrence time and $F(y)$ denote the claim size distribution with finite first and second moments m_1, m_2. The premium rate c is assumed to be calculated via the expected value principle; that is,

$$c = (1+\eta)\lambda m_1, \tag{2}$$

where $\eta > 0$ is the relative loading factor.

In this paper, the insurer can purchase proportional reinsurance to adjust the exposure to insurance risk. The proportional reinsurance level is associated with the risk exposure $q(t)$ at time t. We assume $q(t) \in [0,1]$ for all t, and it means the insurer purchases proportional reinsurance. In this case, for each claim, the insurer only pays its $q(t)Y$, while the reinsurer pays the rest $(1-q(t))Y$ for each claim.

For a chosen reinsurance policy $q(t)$, let $\{U^q(t),\ t \geq 0\}$ denote the associated surplus process; that is, $U^q(t)$ is the surplus of insurer at time t. This process then evolves as

$$dU^q(t) = (c - c^q)\,dt - q(t)\,d\sum_{i=1}^{N(t)} Y_i, \tag{3}$$

where c^q is the net reinsurance rate which the reinsurer receives from the insurer. We assume that the reinsurance premium is calculated by the following generalized mean-variance principle $(1+\theta)[\mathbb{E}(\cdot) + \zeta\mathbb{D}(\cdot)]$, where $\theta, \zeta \geq 0$, and $\mathbb{E}$ and $\mathbb{D}$ denote the expectation and variance, respectively. Thus we have

$$c^q t = (1+\theta)\lambda\left[(1-q(t))m_1 + \zeta(1-q(t))^2 m_2\right]t, \tag{4}$$

and the premium rate for the insurer is

$$\begin{aligned} c - c^q &= (\eta-\theta)\lambda m_1 + (1+\theta)\lambda q(t) m_1 \\ &\quad + \lambda(1+\theta)\zeta(1-q(t))^2 m_2. \end{aligned} \tag{5}$$

According to Grandell (1991), the surplus process after reinsurance can be approximated by the following diffusion process:

$$\begin{aligned} U^q(t) &= u + \int_0^t \left[(\eta-\theta)\lambda m_1 + \theta\lambda q(s) m_1\right. \\ &\quad \left. + \lambda(1+\theta)\zeta(1-q(s))^2 m_2\right] ds + \int_0^t q(s) \\ &\quad \cdot \sqrt{\lambda m_2}\,dB(s), \end{aligned} \tag{6}$$

where $\{B(t)\}_{t\geq 0}$ is a standard Brownian motion.

We define the ruin time

$$\tau^q = \inf\{t > 0, U^q(t) \leq 0\} \tag{7}$$

where the superscript q emphasises that the surplus process and the ruin time are controlled by an admissible policy q. Denote the survival probability given the initial surplus u by

$$v^q(u) = \mathbb{P}(\tau^q = \infty \mid U^q(0) = u), \tag{8}$$

and the maximum survival probability by

$$v(u) = \max_{q\in\Pi} v^q(u). \tag{9}$$

Our objective is to find the value function $v(u)$ and the optimal policy q^* such that

$$v(u) = v^{q^*}(u). \tag{10}$$

3. Maximizing Survival Probability

Under the proportional reinsurance, the insurer could transfer a fraction $1 - q(t)$ of the incoming claims to a reinsurer, where $q(t)$ is $\mathcal{F}_t$-measurable and satisfies $0 \leq q(t) \leq 1$ for all t. The diffusion approximation of insurance company's claim process becomes

$$\begin{aligned} dS^q(t) &= \lambda q(t) m_1 dt + q(t)\sqrt{\lambda m_2}\,dB(t), \\ S^q(0) &= 0, \end{aligned} \tag{11}$$

where $B(t)$ is a standard Brownian motion. The insurer's surplus process satisfies the stochastic differential equation

$$\begin{aligned} dU^q(t) &= \left[(\eta-\theta)\lambda m_1 + \theta\lambda q(s) m_1\right. \\ &\quad \left. + \lambda(1+\theta)\zeta(1-q(s))^2 m_2\right] dt + q(t) \\ &\quad \cdot \sqrt{\lambda m_2}\,dB(t), \\ U^q(0) &= u_0. \end{aligned} \tag{12}$$

Taking $h > 0$ is small enough, we assume that risk exposure does not change over the short time period $[t, t+h]$. This means that the risk exposure remains roughly constant in the given time period; that is, $\mathbb{E}[q(s)Y] = \mathbb{E}[q(t)Y]$, $\sqrt{\lambda\mathbb{E}[(q(s)Y)^2]} = \sqrt{\lambda\mathbb{E}[(q(t)Y)^2]}$, $s \in [t, t+h]$. This setting is reasonable because the insurer can only adjust its reinsurance business at discrete time; and the decision made is based on the holding at time t. Thus, we rewrite the claim dynamics as

$$\begin{aligned} S^q(t+h) - S^q(t) &= \int_t^{t+h} \lambda\mathbb{E}[q(s)Y]\,ds \\ &\quad + \int_t^{t+h} \sqrt{\lambda\mathbb{E}\left[(q(s)Y)^2\right]}\,dB(s) \\ &:= \lambda q(t) m_1 h \\ &\quad + q(t)\sqrt{\lambda m_2}\int_t^{t+h} dB(s). \end{aligned} \tag{13}$$

3.1. Dynamic VaR, CVaR, and Worst-Case CVaR. For a given confidence level $1-\alpha \in (0,1)$ and a given horizon $h>0$, the VaR at time t of a proportional reinsurance policy q, denoted by $VaR_t^{\alpha,h}$, is defined as

$$VaR_t^{\alpha,h} \triangleq \inf\left\{L : \mathbb{P}\left(S^q(t+h)-S^q(t) \geq L \mid \mathscr{F}_t\right) < \alpha\right\}. \tag{14}$$

The dynamic Conditional Value-at-Risk $CVaR_t^{\alpha,h}$ is given by

$$CVaR_t^{\alpha,h} \triangleq \mathbb{E}\left[S^q(t+h)-S^q(t) \mid S^q(t+h)-S^q(t) \geq VaR_t^{\alpha,h}\right]. \tag{15}$$

The dynamic worst-case CVaR is defined as

$$wcCVaR_t^{\alpha,h} \triangleq \sup_{p(\cdot)\in\mathscr{P}_1} \inf_{a\in\mathbb{R}} \left\{a + \frac{1}{\alpha}\mathbb{E}_p\left[\left(S^q(t+h)-S^q(t)-a\right)_+\right]\right\}, \tag{16}$$

where

$$\begin{aligned}\mathscr{P}_1 = \Big\{p(\cdot) : \mathbb{E}_p\left[S^q(t+h)-S^q(t)\right] &= \lambda h m_1 q(t), \mathbb{E}_p\left[\left(S^q(t+h)-S^q(t)\right)^2\right] \\ &= \lambda h m_2 q(t)^2 + \left(\lambda h m_1 q(t)\right)^2\Big\}\end{aligned} \tag{17}$$

Proposition 1 (Zhang et al. [6]).

$$\begin{aligned} VaR_t^{\alpha,h} &= \lambda h q(t) m_1 - \Phi^{-1}(\alpha) q(t)\sqrt{\lambda h m_2}, \\ CVaR_t^{\alpha,h} &= \lambda h q(t) m_1 + \frac{\phi\left(\Phi^{-1}\right)(\alpha)}{\alpha} q(t)\sqrt{\lambda h m_2}, \\ wcCVaR_t^{\alpha,h} &= \lambda h q(t) m_1 + \sqrt{\frac{1-\alpha}{\alpha}} q(t)\sqrt{\lambda h m_2}, \\ 0 \leq VaR_t^{\alpha,h} &\leq CVaR_t^{\alpha,h} \leq wcCVaR_t^{\alpha,h} \\ &< U^q(t), \end{aligned} \tag{18}$$

where $\phi(x)$ and $\Phi(x)$ denote the probability density function and the cumulative distribution function of a standard normal random variable, respectively. $\Phi^{-1}(x)$ is the inverse function of $\Phi(x)$.

3.2. HJB Equation. Using the dynamic programming technique, we obtain that the value function $v(u)$ satisfies the following Hamilton-Jacobi-Bellman (HJB) equation:

$$\max_{q\in[0,1]} \left\{\left[(\eta-\theta)\lambda m_1 + \theta\lambda q m_1 - \lambda(1+\theta)\zeta(1-q)^2 m_2\right] v'(u) + \frac{1}{2}\lambda q^2 m_2 v''(u)\right\} = 0, \tag{19}$$

$$\begin{aligned} \text{s.t.}\quad & q \in [0,1]; \\ & VaR_t^{\alpha,h} \leq ku; \\ & v(0) = 0, \quad v(+\infty) = 1, \end{aligned} \tag{20}$$

where k $(0<k<+\infty)$ is a constant.

Next we try to construct a solution of the HJB equation (19) with the boundary condition (20). Suppose that $\eta m_1 \leq \theta m_1 + (1+\theta)\zeta m_2$, $v(u)$ with $v'(u) > 0$, $v''(u) \neq 0$ satisfies (19) and (20).

Theorem 2. *(a) If $\theta \geq 2\eta$, the function*

$$\varphi(u) = \begin{cases} \varphi\left(\dfrac{A}{k}\right) - \varphi\left(\dfrac{A}{k}\right)\dfrac{\int_u^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}{\int_0^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}, & \text{if } 0 < u \leq \dfrac{A}{k}, \\ \varphi\left(\dfrac{A}{k}\right) + \left[1-\varphi\left(\dfrac{A}{k}\right)\right]\left[1-e^{-(2\eta m_1/m_2)(u-A/k)}\right], & \text{if } u \geq \dfrac{A}{k}, \end{cases} \tag{21}$$

is a smooth ($\mathscr{C}^2$) solution to the HJB equation, where

$$\begin{aligned} A &= \lambda m_1 h - \Phi^{-1}(\alpha)\sqrt{\lambda m_2 h}, \\ \varphi\left(\frac{A}{k}\right) &= \frac{2\eta m_1/m_2}{2\eta m_1/m_2 + (A/k)^{-\triangle_2} e^{-k\triangle_1/A} e^{A\triangle_3/k} / \int_0^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}, \end{aligned} \tag{22}$$

$\triangle_1 = 2A^2[(\eta-\theta)m_1 - (1+\theta)\zeta m_2]/k^2 m_2$, $\triangle_2 = 2A[\theta m_1 + 2(1+\theta)\zeta m_2]/km_2$, $\triangle_3 = 2(1+\theta)\zeta$. *The maximum of the left side of HJB equation is attained at*

$$q^*(u) = \begin{cases} \dfrac{ku}{A}, & \text{if } u < \dfrac{A}{k}, \\ 1, & \text{if } u \geq \dfrac{A}{k}. \end{cases} \tag{23}$$

(b) If $\theta < 2\eta$, the function

$$\varphi(u) = \begin{cases} \varphi(u_1) - \varphi(u_1)\dfrac{\int_u^{u_1} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}{\int_0^{u_1} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}, & \text{if } 0 < u \leq u_1, \\ \varphi(u_1) + [1-\varphi(u_1)]\left[1 - e^{-\triangle_4(u-u_1)}\right], & \text{if } u \geq u_1, \end{cases} \tag{24}$$

is a smooth ($\mathscr{C}^2$) solution to the HJB equation, where

$$\varphi(u_1) = \frac{\triangle_4}{\triangle_4 + u_1^{-\triangle_2} e^{-\triangle_1/u_1} e^{\triangle_3 u_1} / \int_0^{u_1} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}, \tag{25}$$

$\triangle_4 = ((1/2)\theta^2 m_1^2 + 2\zeta\eta(1+\theta)m_1 m_2)/m_2[\zeta(1+\theta)m_2 + (\theta-\eta)m_1]$, $u_1 = (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)\theta m_1))$. *The maximum of the left side of HJB equation is attained at*

$$q^*(u) = \begin{cases} \dfrac{ku}{A}, & \text{if } u < u_1, \\ \dfrac{(\theta-\eta)m_1 + (1+\theta)\zeta m_2}{(1+\theta)\zeta m_2 + (1/2)\theta m_1}, & \text{if } u \geq u_1. \end{cases} \tag{26}$$

Proof. We solve the HJB equation analytically. First we need to determine the optimal strategy $q^*(u)$. Differentiating the terms inside the maximum in (19) with respect to $q(u)$ and setting to 0 yield

$$q^0(u) = \frac{\theta m_1 + 2\zeta(1+\theta)m_2}{2\zeta(1+\theta)m_2 - m_2(\varphi''/\varphi')}. \tag{27}$$

The dynamic VaR constraint implies $q(u) \leq ku/A$, when A is defined by (27). Normally, we take $0 < \alpha < 1/2$; hence, A is always positive.

(1) For $u \geq A/k$, we have $ku/A \geq 1$. Then, from $q(u) \leq ku/A$ obtained from the dynamic VaR constraint and the requirement that the retained proportion of claims $q(u)$ is always within $[0,1]$, we have $q(u) \in [0,1]$.

(a) If $q^0(u) \geq 1$, we let $q^*(u) = 1$, and then the HJB equation becomes

$$\eta m_1 \varphi'(u) + \frac{1}{2} m_2 \varphi''(u) = 0, \tag{28}$$

which implies

$$\frac{\varphi''(u)}{\varphi'(u)} = -\frac{2\eta m_1}{m_2}. \tag{29}$$

Inserting it into (28), we obtain

$$q^0(u) = \frac{\theta m_1 + 2\zeta(1+\theta)m_2}{2\zeta(1+\theta)m_2 + 2\eta m_1}. \tag{30}$$

(i) If $\theta \geq 2\eta$, we have $q^0(u) \geq 1$; consequently $q^*(u) = 1$, and then the HJB equation becomes (30).

(ii) If $\theta < 2\eta$, we have $q^0(u) < 1$, where conflict exits.

(b) If $q^0(u) < 1$, we have $q^*(u) = q^0(u)$, and then the HJB equation becomes

$$\frac{\varphi''(u)}{\varphi'(u)} = -\frac{(1/2)\theta^2 m_1^2 + 2\zeta\eta(1+\theta)m_1 m_2}{m_2[\zeta(1+\theta)m_2 + (\theta-\eta)m_1]}, \tag{31}$$

which implies

$$q^0(u) = \frac{\zeta(1+\theta)m_2 + (\theta-\eta)m_1}{(1/2)\theta m_1 + \zeta(1+\theta)m_2}. \tag{32}$$

(i) If $\theta \geq 2\eta$, we have $q^0(u) \geq 1$, where conflict exits.

(ii) If $\theta < 2\eta$, we have $q^0(u) < 1$; consequently $q^*(u) = q^0(u) = (\zeta(1+\theta)m_2 + (\theta-\eta)m_1)/((1/2)\theta m_1 + \zeta(1+\theta)m_2)$, and then the HJB equation becomes (32).

(2) When $0 < u < A/k$, we have $ku/A < 1$; thus $0 \leq q(u) \leq ku/A$.

(a) If $q^0(u) \geq ku/A$, we have $q^*(u) = ku/A$, and then the HJB equation becomes

$$\left[(\eta-\theta)m_1 + \theta\frac{ku}{A}m_1 - \zeta(1+\theta)\left(1-\frac{ku}{A}\right)^2 m_2\right] \cdot \varphi'(u) + \frac{1}{2}m_2\left(\frac{ku}{A}\right)^2 \varphi''(u) = 0. \tag{33}$$

We have

$$\frac{\varphi''(u)}{\varphi'(u)} = -\frac{(\eta-\theta)m_1 + \theta(ku/A)m_1 - \zeta(1+\theta)(1-ku/A)^2 m_2}{(1/2)m_2(ku/A)^2}, \tag{34}$$

which implies

$$q^0(u) = \frac{1}{2}\left(\frac{ku}{A}\right)^2 \cdot \frac{1}{ku/A - ((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/(\theta m_1 + 2\zeta(1+\theta)m_2)}. \tag{35}$$

We have $\varphi''(u) < 0$ when $u > (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)(\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1+\theta)\zeta\eta m_1 m_2})))$, and we have $q^0(u) > 0$ when $u > (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/(2(1+\theta)\zeta m_2 + \theta m_1))$. We have $\varphi''(u) < 0$ and $q^0(u) > 0$ when $u > (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)(\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1+\theta)\zeta\eta m_1 m_2})))$.

For $A/k > u > (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)(\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1+\theta)\zeta\eta m_1 m_2})))$, we have the following.

(i) If $\theta \geq 2\eta$, we have $q^0(u) \geq ku/A$; consequently $q^*(u) = q^0(u) = ku/A$, and then the HJB equation becomes (34).

(ii) If $\theta < 2\eta$, we have $q^0(u) < ku/A$, where conflict exits.

For $u \leq (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)(\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1+\theta)\zeta\eta m_1 m_2})))$, we have $\varphi''(u) \geq 0$; therefore $\varphi(u)$ is convex for small u. Through the analysis of the HJB equation (19), for $0 \leq u \leq (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)(\theta m_1 + \sqrt{\theta^2 m_1^2 + 4(1+\theta)\zeta\eta m_1 m_2})))$, the maximum of the left side of the HJB is attained at $q^*(u) = ku/A$ and the HJB equation becomes (34).

(b) When $q^0(u) \leq ku/A$, it is reasonable to let $q^*(u) = q^0(u)$. Similar to (a), we have the following conclusions.

For $0 \leq u \leq (A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)\theta m_1))$, the optimal strategy is obtained at $q^*(u) = ku/A$.

For $(A/k)(((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)\theta m_1)) < u \leq A/k$, we have the following.

(i) If $\theta \geq 2\eta$, we have $q^0(u) \geq 1 > ku/A$, where conflict exits.

(ii) If $\theta < 2\eta$, we have $q^0(u) < ku/A$; consequently, $q^*(u) = q^0(u) = ((\theta-\eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)\theta m_1)$, and then the HJB equation becomes (32).

From the previous analysis, we have the following conclusions.

(i) If $\theta \geq 2\eta$, the maximum of the left side of HJB equation is attained at

$$q^*(u) = \begin{cases} \dfrac{ku}{A}, & \text{if } u < \dfrac{A}{k}, \\ 1, & \text{if } u \geq \dfrac{A}{k}. \end{cases} \tag{36}$$

(ii) If $\theta < 2\eta$, the maximum of the left side of HJB equation is attained at

$$q^*(u) = \begin{cases} \dfrac{ku}{A}, & \text{if } u < u_1, \\ \dfrac{(\theta-\eta)m_1 + (1+\theta)\zeta m_2}{(1+\theta)\zeta m_2 + (1/2)\theta m_1}, & \text{if } u \geq u_1. \end{cases} \tag{37}$$

In the following, we will solve the HJB equation in each situation.

For $\theta \geq 2\eta$ and $u < A/k$, the HJB equation is (34), which is equivalent to (35). Taking integral from u to A/k, we obtain

$$\varphi(u) = \varphi\left(\frac{A}{k}\right) - K\int_u^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw, \tag{38}$$

where $\triangle_1 = 2A^2[(\eta-\theta)m_1 - (1+\theta)\zeta m_2]/k^2 m_2$, $\triangle_2 = 2A[\theta m_1 + 2(1+\theta)\zeta m_2]/k m_2$, and $\triangle_3 = 2(1+\theta)\zeta$. Applying the boundary condition $\varphi(0) = 0$ we obtain

$$K = \frac{\varphi(A/k)}{\int_0^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}. \tag{39}$$

For $\theta \geq 2\eta$ and $u \geq A/k$, the corresponding HJB is (29), which is equivalent to (30). Taking integral form A/k to u, we obtain

$$\begin{aligned} \varphi(u) &= \varphi\left(\frac{A}{k}\right) \\ &\quad + \frac{m_2}{2\eta m_1}\varphi'\left(\frac{A}{k}\right)\left[1 - e^{-(2\eta m_1/m_2)(u-A/k)}\right]. \end{aligned} \tag{40}$$

Applying the boundary condition $\delta(+\infty) = 1$ we obtain

$$\begin{aligned} \varphi(u) &= \varphi\left(\frac{A}{k}\right) \\ &\quad + [1 - \varphi(A/k)]\left[1 - e^{-(2\eta m_1/m_2)(u-A/k)}\right]. \end{aligned} \tag{41}$$

Considering that $\varphi(x)$ is twice continuously differentiable, it should satisfy $\varphi'(A/k-) = \varphi'(A/k+)$; that is,

$$\begin{aligned} &\frac{\varphi(A/k)(A/k)^{-\triangle_2} e^{-k\triangle_1/A} e^{A\triangle_3/k}}{\int_0^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw} \\ &\quad = \frac{2\eta m_1}{m_2}\left[1 - \varphi\left(\frac{A}{k}\right)\right], \end{aligned} \tag{42}$$

which leads to

$$\varphi\left(\frac{A}{k}\right) = \frac{2\eta m_1/m_2}{2\eta m_1/m_2 + (A/k)^{-\triangle_2} e^{-k\triangle_1/A} e^{A\triangle_3/k} / \int_0^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}. \tag{43}$$

Thus, if $\theta \geq 2\eta$, we have the function

$$\varphi(u) = \begin{cases} \varphi\left(\dfrac{A}{k}\right) - \varphi\left(\dfrac{A}{k}\right)\dfrac{\int_u^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}{\int_0^{A/k} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}, & \text{if } 0 < u \leq \dfrac{A}{k}, \\ \varphi\left(\dfrac{A}{k}\right) + \left[1 - \varphi\left(\dfrac{A}{k}\right)\right]\left[1 - e^{-(2\eta m_1/m_2)(u-A/k)}\right], & \text{if } u \geq \dfrac{A}{k}. \end{cases} \tag{44}$$

If $\theta < 2\eta$ and $u < u_1$, the HJB equation is (34), and the HJB equation is (32) for $\theta < 2\eta$ and $u \geq u_1$. From the procedure that is similar to the previous analysis we can get the following function is a $\mathscr{C}^2$ solution to HJB; that is,

$$\varphi(u) = \begin{cases} \varphi(u_1) - \varphi(u_1) \dfrac{\int_u^{u_1} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}{\int_0^{u_1} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}, & \text{if } 0 < u \leq u_1, \\ \varphi(u_1) + [1 - \varphi(u_1)]\left[1 - e^{-\triangle_4 (u-u_1)}\right], & \text{if } u \geq u_1, \end{cases} \quad (45)$$

where

$$\triangle_4 = \frac{(1/2)\theta^2 m_1^2 + 2\zeta\eta(1+\theta) m_1 m_2}{m_2[\zeta(1+\theta) m_2 + (\theta - \eta) m_1]},$$

$$\varphi(u_1) = \frac{\triangle_4}{\triangle_4 + u_1^{-\triangle_2} e^{-\triangle_1/u_1} e^{\triangle_3 u_1} / \int_0^{u_1} w^{-\triangle_2} e^{-\triangle_1/w} e^{\triangle_3 w} dw}. \quad (46)$$

This ends the proof. □

When the value function is twice continuously differentiable, then it is the unique solution of the HJB equation (see, e.g., [20]), and we have the following result.

Proposition 3. *The value function $v(x)$ coincides with the smooth function $\varphi(x)$ defined in Theorem 2 and the optimal control, which represents the optimal proportional reinsurance strategy, is described by the $q^*(u)$ in Theorem 2, where $\{U^{q^*}(t),\ t \geq 0\}$ is the corresponding surplus process.*

Remark 4. When $\zeta = 0$, Theorem 2 coincides with theorem 3.1 in Zhang et al. [6].

Corollary 5. *When $\theta = 0$, the generalized mean-variance premium principle is mean-variance premium principle, and we have the following:*

$$\varphi(u) = \begin{cases} \varphi(u_1) - \varphi(u_1) \dfrac{\int_u^{u_1} w^{-4A\zeta/k} e^{-2A^2(\eta m_1 - \zeta m_2)/wk^2 m_2} e^{2\zeta w} dw}{\int_0^{u_1} w^{-4A\zeta/k} e^{-2A^2(\eta m_1 - \zeta m_2)/wk^2 m_2} e^{2\zeta w} dw}, & \text{if } 0 < u \leq u_1, \\ \varphi(u_1) + [1 - \varphi(u_1)]\left[1 - e^{-(2\zeta\eta m_1/\zeta m_2 - \eta m_1)(u-u_1)}\right], & \text{if } u \geq u_1, \end{cases} \quad (47)$$

is a smooth ($\mathscr{C}^2$) solution to the HJB equation, where

$$\varphi(u_1) = \frac{2\zeta\eta m_1/(\zeta m_2 - \eta m_1)}{2\zeta\eta m_1/(\zeta m_2 - \eta m_1) + u_1^{-4A\zeta/k} e^{-2A^2(\eta m_1 - \zeta m_2)/u_1 k^2 m_2} e^{2\zeta u_1} / \int_0^{u_1} w^{-4A\zeta/k} e^{-2A^2(\eta m_1 - \zeta m_2)/wk^2 m_2} e^{2\zeta w} dw}, \quad (48)$$

$u_1 = (A/k)(1 - \eta m_1/\zeta m_2)$. The maximum of the left side of HJB equation is attained at

$$q^*(u) = \begin{cases} \dfrac{ku}{A}, & \text{if } u < u_1, \\ 1 - \dfrac{\eta m_1}{\zeta m_2}, & \text{if } u \geq u_1. \end{cases} \quad (49)$$

Corollary 6. *When there is no dynamic VaR, CVaR, or wcCVaR constraints, that is, $k = \infty$, and the model becomes the unconstrained reinsurance problem, we have the following.*

(a) If $\theta \geq 2\eta$, the optimal reinsurance strategy is $q^* = 1$, and the optimal survival probability is

$$\varphi(u) = 1 - e^{-(2\eta m_1/m_2)u}. \quad (50)$$

(b) If $\theta < 2\eta$, the optimal reinsurance strategy is $q^* = ((\theta - \eta)m_1 + (1+\theta)\zeta m_2)/((1+\theta)\zeta m_2 + (1/2)\theta m_1)$, and the optimal survival probability is

$$\varphi(u) = 1 - e^{-(((1/2)\theta^2 m_1^2 + 2\zeta\eta(1+\theta)m_1 m_2)/m_2[\zeta(1+\theta)m_2 + (\theta-\eta)m_1])u}. \quad (51)$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript. All authors read and approved the final manuscript.

Acknowledgments

The research was supported by the National Natural Science Foundation of China (No. 11501319, No. 11571198), the Education Department of Shandong Province Science and Technology Plan Project (No. J15L105), the Natural Science Foundation of Shandong Province (No. ZR2015AL013), and the China Postdoctoral Science Foundation (No. 2015M582064).

References

[1] B. Højgaard and M. Taksar, "Optimal proportional reinsurance policies for diffusion models," *Scandinavian Actuarial Journal*, vol. 1998, no. 2, pp. 166–180, 1998.

[2] B. Højgaard and M. Taksar, "Optimal dynamic portfolio selection for a corporation with controllable risk and dividend distribution policy," *Quantitative Finance*, vol. 4, no. 3, pp. 315–327, 2004.

[3] H. Schmidli, "Optimal proportional reinsurance policies in a dynamic setting," *Scandinavian Actuarial Journal*, no. 1, pp. 55–68, 2001.

[4] H. Schmidli, "On minimizing the ruin probability by investment and reinsurance," *The Annals of Applied Probability*, vol. 12, no. 3, pp. 890–907, 2002.

[5] M. I. Taksar, "Optimal risk and dividend distribution control models for an insurance company," *Mathematical Methods of Operations Research*, vol. 51, no. 1, pp. 1–42, 2000.

[6] N. Zhang, Z. Jin, S. Li, and P. Chen, "Optimal reinsurance under dynamic VaR constraint," *Insurance: Mathematics & Economics*, vol. 71, pp. 232–243, 2016.

[7] N. Bäuerle, "Benchmark and mean-variance problems for insurers," *Mathematical Methods of Operations Research*, vol. 62, no. 1, pp. 159–165, 2005.

[8] L. Bai and H. Zhang, "Dynamic mean-variance problem with constrained risk control for the insurers," *Mathematical Methods of Operations Research*, vol. 68, no. 1, pp. 181–205, 2008.

[9] Z. Liang and E. Bayraktar, "Optimal proportional reinsurance and investment in a stock market with Ornstein-Uhlenbeck process," *Insurance: Mathematics and Economics*, vol. 55, no. 2, pp. 156–166, 2014.

[10] M. Zhou and K. C. Yuen, "Optimal reinsurance and dividend for a diffusion model with capital injection: Variance premium principle," *Economic Modelling*, vol. 29, no. 2, pp. 198–207, 2012.

[11] Z. Sun, X. Zheng, and X. Zhang, "Robust optimal investment and reinsurance of an insurer under variance premium principle and default risk," *Journal of Mathematical Analysis and Applications*, vol. 446, no. 2, pp. 1666–1686, 2017.

[12] J. Cai and K. S. Tan, "Optimal retention for a stop-loss reinsurance under the VaR and CTE risk measures," *ASTIN Bulletin*, vol. 37, no. 1, pp. 93–112, 2007.

[13] K. C. Cheung, K. C. Sung, S. C. Yam, and S. P. Yung, "Optimal reinsurance under general law-invariant risk measures," *Scandinavian Actuarial Journal*, no. 1, pp. 72–91, 2014.

[14] J. Cai, K. S. Tan, C. Weng, and Y. Zhang, "Optimal reinsurance under VaR and CTE risk measures," *Insurance: Mathematics & Economics*, vol. 43, no. 1, pp. 185–196, 2008.

[15] J. Cai, C. Lemieux, and F. Liu, "Optimal reinsurance with regulatory initial capital and default risk," *Insurance: Mathematics & Economics*, vol. 57, no. 1, pp. 185–196, 2014.

[16] K. F. Yiu, "Optimal portfolios under a value-at-risk constraint," *Journal of Economic Dynamics & Control*, vol. 28, no. 7, pp. 1317–1334, 2004.

[17] G. J. Alexander and A. M. Baptista, "A comparison of VaR and CVaR constraints on portfolio selection with the mean-variance model," *Management Science*, vol. 50, no. 9, pp. 1261–1273, 2004.

[18] D. Cuoco, H. He, and S. Isaenko, "Optimal dynamic trading strategies with risk limits," *Operations Research*, vol. 56, no. 2, pp. 358–368, 2008.

[19] S. Chen, Z. Li, and K. Li, "Optimal investment-reinsurance policy for an insurance company with VaR constraint," *Insurance: Mathematics & Economics*, vol. 47, no. 2, pp. 13–24, 2010.

[20] W. Fleming and H. Soner, *Controlled Markov Processes and Viscosity Solutions*, Springer, New York, NY, USA, 2nd edition, 2006.