Research Article

Gateaux Differentiability of Convex Functions and Weak Dentable Set in Nonseparable Banach Spaces

Shaoqiang Shang and Yunan Cui

1 Academy of Mathematical Sciences, Harbin Engineering University, Harbin 150080, China
2 Department of Mathematics, Harbin University of Science and Technology University, Harbin 150001, China

Correspondence should be addressed to Shaoqiang Shang; sqshang@163.com

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1. Introduction and Preliminaries

Let $(X, ||·||)$ denote a real Banach space. $B(X)$ and $S(X)$ denote the unit ball and unit sphere of $X$, respectively. Let $X^*$ denote the dual space of $X$. Let $N$, $R$, and $R^+$ denote the sets of natural number, reals, and nonnegative reals, respectively. Let $B(x, r)$ denote the closed ball centered at $x$ and of radius $r > 0$. Let $x_n \to x$ denote that \( \{x_n\}_{n=1}^{\infty} \) is weakly convergent to $x$.

Let $D$ be a nonempty open convex subset of $X$ and $f$ a continuous convex function on $D$. We called that $f$ is said to be Gateaux differentiable at the point $x$ in $D$ if the limit

\[
df(x)(y) = \lim_{t \to 0} \frac{f(x+ty) - f(x)}{t} \tag{*}
\]

exists for all $y \in X$. Moreover, if the difference quotient in $(*)$ converges to $df(x)(y)$ uniformly for $y$ in the unit ball, then $f$ is said to be Frechet differentiable at $x$.

Definition 1 (see [1]). $X$ is called a weak Asplund space [Asplund space] if, for every $f$ and $D$ as above, there exists a dense $G_δ$ subset $G$ of $D$ such that $f$ is Gateaux [Frechet] differentiable at each point of $G$.

It is well known that $l^1$ is weak Asplund space, but not Asplund space. Moreover, it is well known that $X$ is an Asplund space if and only if $X^*$ has the Radon-Nikodym property (see [1]). In 1933, Mazur proved that separable Banach spaces have the weak Asplund property (see [1]).

Definition 2 (see [1]). A Banach space is said to be a Gateaux differentiable space if every convex continuous function on it is Gateaux differentiable at the points of a dense set.

In 2006, Waren B. Moors and Sivajah Somasundaram proved that there exists a Gateaux differentiable space that is not a weak Asplund space (see [2]). In 1979, D.G. Larman and R.R.Phelps proved that if $X^*$ is a strictly convex space, then $X$ is a weak Asplund space (see [3]). In 1997, Cheng proved that if $f$ is a continuous convex function on a Banach space $X$, then every proper convex function $g$ on $X$ with $g \leq f$ is generically Frechet differentiable if and only if the image of the subdifferential map $\partial f$ has the Radon-Nikodym property (see [4]).
Definition 3 (see [1]). A point $x_0^* \in C^*$ is said to be weak exposed point of $C^*$ if there exists $x \in S(X)$ such that $x_0^*(x) > x^*(x)$ whenever $x^* \in C^* \setminus \{x_0^*\}$.

Definition 4 (see [1]). Suppose that $f$ is a convex function on $X$, then the set-valued mapping $\partial f(x) = \{x^* \in X^* : (x^*, y - x) \leq f(y) - f(x) \text{ for all } y \in X\}$ is said to be subdifferential mapping.

Remark 5. It is well known that if $f$ is a continuous convex function, then the set-valued mapping $\partial f$ is norm-weak upper semicontinuous (see [1]). Moreover, it is well known that $\partial f$ is a singleton at $x$ if and if $f$ is Gâteaux differentiable at $x$ (see [1]).

Let $C$ be a bounded subset of $X$. Let $C^* = \{x^* \in X^* : x^*(x) \leq 1, x \in C\}$ and $C^{**} = \{x^{**} \in X^{**} : x^{**}(x^*) \leq 1, x^* \in C^*\}$. Then it is easy to see that $C^{**}$ is a weak closed convex set. Define the sublinear functional

$$\sigma_C(x^*) = \sup \{x^*(x) : x \in C\}$$

where

$$\sigma_C(x^*) = \sup \{x^*(x^{**}) : x^{**} \in C^{**}\}. \quad (1)$$

Then $\sigma_C$ is a continuous sublinear functional. It is well known that $\partial \sigma_C(0) = C^*, \overline{\partial \sigma_C'}(C) = C^*$, and $C^* = \{x^* \in X^* : \sigma_C(x^*) \leq 1\}$. Let $p$ be a continuous Minkowski functional on $X$ and $C = \{x \in X : p(x) \leq 1\}$. Then $p(x) = \inf \{\lambda \geq 0 : \lambda^{-1} x \in C\}$ whenever $x \in X$. In this case, we called $p$ is generated by $C$. Let $C^* = \{x^* \in X^* : x^*(x) \leq 1, x \in C\}$. Then $p(x) = \sigma_C(x^*) = \sup \{x^*(x) : x^* \in C^*\}. \quad (2)$

Moreover, it is well known that

1. $\partial p(0) = C^*$;
2. $x^* \in \partial p(x) \iff x^* \in C^*$
3. and $x^*(x) = p(x)$

It is well known that if $C$ is a bounded subset of $X$, then

$$\sigma_{\overline{\partial p(C^*)}}(x^*) = \inf \{\lambda : \lambda^{-1} x^* \in C^*\}. \quad (4)$$

denotes epigraph of $f$. It is well known that $epif$ is closed if and only if $f$ is lower semicontinuous.

Lemma 6 (see [4]). Let $f$ be a continuous convex function on $X$ and $f(0) = -1$. Let $p : E \times R \to R$ be the Minkowski functional generated by $epif$. Then $x^* \in \overline{\partial p(x)}$ if and only if $(y^*, r) \in \partial p(x, f(x))$ with $y^* = [x^*(x) - f(x)]^{-1} x^*$ and $r = -[x^*(x) - f(x)]^{-1}$.

Definition 7. A set $D \subset X$ is said to be weak dentable set if for any weak neighborhood $U$ of origin, there exists $z \in D$ such that $z \notin \overline{\partial p(D)}(z + U)$.

Definition 8 (see [5]). A set $D \subset X$ is said to be dentable set if, for any $\varepsilon > 0$, there exists $z \in D$ such that $z \notin \overline{\partial p(D)}(z + \varepsilon)$.

Definition 9 (see [5]). A Banach space $X$ is said to have the Radon-Nikodym property (see [1]) if $(T, \Sigma, \mu)$ is a nonatomic measure space and $v$ is a vector measure on $\Sigma$ with values in $X$ which is absolutely continuous with respect to $\mu$ and has a bounded variation, then there exists $f \in L_1(X)$ such that, for any $A \in \Sigma$,

$$v(A) = \int_A f(t) \, dt. \quad (5)$$

It is well known that a Banach space $X$ has the Radon-Nikodym property if and only if every bounded subset of $X$ is dentable. By Definitions 8 and 9, it is easy to see that if $C$ is dentable, then $C$ is weak dentable. Moreover, there exists a weak dentable set such that it is not dentable. We will give two examples in Sections 3 and 4.

Proposition 10. The weak neighborhood $A = \{x \in X : \bigcap_{i=1}^k \{x_i^*, x\} \leq r\}$ is not a weak dentable set, where $\{x_i^*, x_2^*, \ldots, x_k^*\} \subset S(X^*)$ and $r \in (0, 1)$. Pick $x_0 \in S(X)$ and

$$x_0 \in \bigcap_{i=1}^k \{x \in X : (x_i^*, x) = 0\}. \quad (6)$$

Then, by the Hahn-Banach theorem, there exists $x_{0}^* \in S(X^*)$ such that $x_{0}^*(x_0) = 1$. Define a weak neighborhood

$$V = \{x \in X : (x_{0}^*, x) < \frac{1}{4}\} \quad (7)$$

of origin. Then, for any $x \in A$, we have

$$|(x_{0}^*, x + x_0 - x)| = |(x_{0}^*, x_0)| = 1$$

and

$$|(x_{0}^*, x - x_0 - x)| = |(x_{0}^*, -x_0)| = 1. \quad (8)$$

Therefore, by formula (7), we have

$$x + x_0 \notin x + V$$

and

$$x - x_0 \notin x + V. \quad (9)$$

Moreover, for any $i \in \{1, 2, \ldots, k\}$, we have

$$|(x_i^*, x + x_0)| = |(x_i^*, x)| \leq r$$

and

$$|(x_i^*, x - x_0)| = |(x_i^*, -x_0)| \leq r. \quad (10)$$

Therefore, by formula (6), we have

$$x + x_0 \in A \setminus (x + V)$$

and

$$x - x_0 \in A \setminus (x + V). \quad (11)$$

Hence we obtain that

$$x = \frac{1}{2} (x + x_0) + \frac{1}{2} (x - x_0) \in \overline{\partial p(A \setminus (x + V))} \quad (12)$$

$$c \overline{\partial p}(A \setminus (x + V)).$$
This implies that $A$ is not weak dentable, which finishes the proof.

\begin{definition}
A set $A \subset X^*$ is said to be $\varepsilon$-separable if there exists a sequence $\{x_{n}\}_{n=1}^{\infty} \subset X$ such that $\sup_{n \geq 1} x^*(x_n) > 0$ for any $x^* \in A \setminus \{0\}$.
\end{definition}

It is well known that if $X$ is a separable space, then every subset of $X^*$ is $\varepsilon$-separable.

\begin{proposition}
Suppose that $A \subset X^*$ is separable and $A \neq \{0\}$. Then $A$ is $\varepsilon$-separable.
\end{proposition}

\begin{proof}
Since $A$ is a separable subset of $X^*$, there exists a sequence $\{x_{n}\}_{n=1}^{\infty} \subset A$ such that $\{x_{n}\}_{n=1}^{\infty} = A$. Then we may assume without loss of generality that $x_{n}^* \neq 0$ for any $n \in \mathbb{N}$. Hence there exists a sequence $\{x_{n}\}_{n=1}^{\infty} \subset S(X)$ such that $(x_{n}^*, x_{n}) \geq \|x_{n}^*\|/2$.

Pick $y^* \in A \setminus \{0\}$. We will prove that $\sup_{n \geq 1} y^*(x_{n}) > 0$. In fact, suppose that $\sup_{n \geq 1} y^*(x_{n}) = 0$. Then $(x_{n}, y^*) \leq 0$ for all $n \in \mathbb{N}$. Since $\{x_{n}\}_{n=1}^{\infty} = A$, there exists a natural number $n_0$ such that $\|x_{n_0}^* - y^*\| < (1/8)\|y^*\|$. Then $\|y^*\| - \|x_{n_0}^*\| < (1/8)\|y^*\|$. Hence we obtain that $\|y^*\| < (8/7)\|x_{n_0}^*\|$. Therefore, by $(x_{n}, y^*) \leq 0$ for any $n \in \mathbb{N}$, we have

$$
(x_{n_0}, x_{n_0}^* - y^*) + (x_{n_0}, y^*) \\
\leq \frac{1}{8}\|y^*\| - \frac{8}{7}\|x_{n_0}^*\| = \frac{1}{7}\|x_{n_0}^*\|,
$$

which contradicts $\|x_{n_0}^*\| \geq \|x_{n_0}^*\|/2$ for every $n \in \mathbb{N}$. This implies that $\sup_{n \geq 1} y^*(x_{n}) > 0$. Hence $A$ is $\varepsilon$-separable, which finishes the proof.
\end{proof}

\begin{example}
Let $X = c_0$. Then $X^* = l^1$ is separable and $X^{**} = l^{\infty}$ is not a separable space. Let $Y$ be a Banach space and $Y$ be not a separable space. Define $C = B(X) \times \{0\} \subset X \times Y$. Then

$$
C^* = B(X^*) \times Y^* \subset X^* \times Y^*
$$

and $C^{**} = B(Y^{**}) \times \{0\} \subset X^{**} \times Y^{**}$.

Let $(x^{**}, 0) \in C^{**}\setminus \{(0, 0)\}$. Since $X^{**}$ is separable, there exists $\{x^{**}_{n}\}_{n=1}^{\infty} \subset S(Y^*)$ such that $\{x^{**}_{n}\}_{n=1}^{\infty} = S(Y^*)$. Then $\sup_{n \geq 1} x^{**}(x^{**}_{n}) > 0$. Hence

$$
\sup_{n \geq 1} (x^{**}, 0)(x^{**}_{n}, 0) = \sup_{n \geq 1} (x^{**}, x^{**}_{n}) > 0.
$$

This implies that $C^{**}$ is $\varepsilon$-separable and bounded. Moreover, it is easy to see that $C^*$ and $C^{**}$ are not separable and $C$ is not dentable.

The paper is organized as follows. In Section 1 some necessary definitions and notations are collected. In Section 2 we prove that if $C^{**}$ is an $\varepsilon$-separable bounded subset of $X^{**}$, then every convex function $g \leq \sigma_c$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$ if and only if every subset of $\partial \sigma_c(0) \cap X$ is weakly dentable. In Section 3 we prove that if $C$ is a closed convex set, then $\partial \sigma_c(x^*) = x$ if and only if $x$ is a weakly exposed point of $C$ exposed by $x^*$. Moreover, we also prove that $X$ is an Asplund space if and only if for every bounded closed convex set $C^*$ of $X^*$, there exists a dense subset $G$ of $X^*$ such that $\sigma_{C^*}$ is Gâteaux differentiable on $G$ and $d\sigma_{C^*}(G) \subset C^*$. We also prove that $X$ is an Asplund space if and only if for every $w^*$-lower semicontinuous convex function $f$, there exists a dense subset $G$ of $X^{**}$ such that $f$ is Gâteaux differentiable on $G$ and $df(G) \subset C^*$. In Section 4 we prove that there exists an exposed point such that it is not a weak exposed point in Orlicz function spaces. The topic of this paper is related to the topic of [5–12].

2. Gâteaux Differentiability, Weakly Dentable, Set, and $\varepsilon$-Separable Set

\begin{theorem}
Suppose that $C^{**}$ is a $\varepsilon$-separable bounded subset of $X^{**}$, and $C$ is a closed convex set.

\begin{enumerate}
\item Every $w^*$-lower semicontinuous convex function $g \leq \sigma_C$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$.
\item Every convex subset of $\partial \sigma_c(0) \cap X$ is a weakly dentable set.
\item Every weak* closed convex subset of $\partial \sigma_c(0)$ is the weak* convex hull of its weak* exposed points.
\end{enumerate}

\end{theorem}

In order to prove the theorem, we give some lemmas.

\begin{lemma}
Suppose that $C^{**}$ is a $\varepsilon$-separable bounded subset of $X^{**}$ and $C$ is a closed convex set;

\begin{enumerate}
\item $(f)\; f$ is a convex function and $df(X^*) \subset C^{**}$;
\item $(g)$ for any $D \subset C$ and weak neighborhood $U$ of origin, there exists a slice $S(x^{**}, D, \alpha)$ such that $S(x^{**}, D, \alpha) - S(x^{**}, D, \alpha) \subset U$.
\end{enumerate}

\end{lemma}

\begin{proof}
We claim that if $C^{**}$ is $\varepsilon$-separable, then the set $C^{**} - C^{**}$ is $\varepsilon$-separable. In fact, since $C^{**}$ is $\varepsilon$-separable, there exists a sequence $\{x^{**}_{N}\}_{n=1}^{\infty} \subset S(X^*)$ such that $\{x^{**}_{N}\}_{n=1}^{\infty} = S(X^*)$. Then $\sup_{n \geq 1} x^{**}(x^{**}_{N}) > 0$. Hence we obtain that $\sup_{n \geq 1} y^{**}(x^{**}_{N}) > 0$. Then, for any $x^{**} \in C^{**}$, we have

\begin{align}
\left(\frac{1}{2}x^{**} - \frac{1}{2}y^{**}, x^{**}\right) & \\
\leq & \left(\frac{1}{2}x^{**}, x^{**}\right)
\end{align}

This implies that $(x^{**} - y^{**})/2 \in C^{**}$. Hence we obtain that $C^{**} - C^{**}$ is $\varepsilon$-separable.

\end{proof}

This implies that $(x^{**} - y^{**})/2 \in C^{**}$. Hence we obtain that $C^{**} - C^{**}$ is $\varepsilon$-separable. Since $C^{**} - C^{**}$ is an $\varepsilon$-separable bounded subset of $X^{**}$, there exists a sequence $\{x^{**}_{N}\}_{n=1}^{\infty} \subset S(X^*)$ such that $\sup_{n \geq 1} x^{**}(x^{**}_{N}) > 0$. Hence, for every natural number $n \in \mathbb{N}$, we define a weak* neighborhood

$$W_n = \left\{x^{**} \in X^{**} : \sup_{i=1}^{n} \left\|\left(x^{**}_{i}, x^{**}\right)\right\| \leq \frac{1}{n}\right\}.$$
of origin in $X^{**}$. Moreover, for every natural number $n \in \mathbb{N}$, we define a weak neighborhood

$$U_n = \left\{ x \in X : \left| \frac{1}{n} \sum_{i=1}^{n} (x_i^*, x) \right| \leq 1 \right\}$$

(18)
of origin in $X$. Hence, if $x^* \in \bigcap_{n=1}^{\infty} W_n$ and $y^* \in C^{**} \setminus C^*$, then $(x^*, y^*) = 0$ for all $n \in \mathbb{N}$. Since $\partial f(W_n) \cap X = C^*$, we have $y^* = 0$. Hence, for each $n > 1$, let $G_n$ be the set of all $x^* \in X^*$ for which there exists a norm neighborhood $V_n$ of $x^*$ such that $\partial f(V_n) \cap X = C^*$. Let

$$x^* \in \bigcap_{n=1}^{\infty} G_n.$$ Pick $x^{**} \in \partial f(x^*)$ and $y^{**} \in \partial f(y^*)$. Then $x^{**} \in \partial f(V_n)$ and $y^{**} \in \partial f(V_n)$ for every $n \in \mathbb{N}$. Hence we obtain that

$$x^{**} - y^{**} \in \partial f(V_n) - \partial f(V_n) \subset W_n$$

(19)

for every $n \in \mathbb{N}$. This implies that $(x^{**} - y^{**}, x^*) = 0$ for every $x^* \in \bigcap_{n=1}^{\infty} G_n$. Since $\partial f(X^*) \subset C^*$, we have $x^{**} - y^{**} \in C^{**} \setminus C^*$. Therefore, by the previous proof, we have $x^{**} = y^{**}$. This implies that

$$\bigcap_{n=1}^{\infty} G_n = \{ x^* \in X^* : \partial f(x^*) \text{ is a singleton} \}.$$ (20)

Hence we obtain that $f$ is Gâteaux differentiable at each point of $G = \bigcap_{n=1}^{\infty} G_n$.

Since $X^*$ is a Baire space, we next will prove that, for any $n \in \mathbb{N}$, the set $G_n$ is open and dense in $X^*$. It is easy to see that $G_n$ is an open set. We next will prove that $G_n$ is dense in $X^*$. Let $x^* \in X^*$ and let $U$ be a neighborhood of $x^*$ in $X^*$. We claim that $\partial f(U) \cap X = \emptyset$. In fact, since $f$ is a $w^*$-lower semicontinuous function on $X^*$, we obtain that the set $epif$ is a weak* closed set of $X^*$. Moreover, we may assume without loss of generality that $f(0) = -1$. Let $C^* = epif$ and $\mu_{C^*}(x^*, r) = \inf \{ \lambda \in \mathbb{R}^+ : \lambda^{-1}(x^*, r) \in C^* \}$. Since $0, 0 \in \int \mathbb{C}^*$, we obtain that $\mu_{C^*}$ is continuous. Pick $(x^*, f(x^*)-1) \notin epif$. Since $epif$ is weak* closed, by the separation theorem, there exists $(x, r) \in X \times R$ such that

$$x^* (x) + rf(x^*) - r \geq \sup \{ z^* (x) + rh : (z^*, h) \in epif \}.$$ (21)

Hence we may assume without loss of generality that $sup(z^* (x) + rh : (z^*, h) \in epif) = 1$. This implies that the set

$$C = \{ (x, r) \in X \times R : x^* (x) + rh \leq 1, (x^*, h) \in epif \}$$

(22)
is a nonempty bounded closed convex subset of $X \times R$. Therefore, by the Bishop-Phelps Theorem, we obtain that

$$\{(x^*, r) \in X^* \times R : x^* (x) + rh \}$$

(23)
is a dense set of $X^* \times R$. Hence

$$\{(x^*, r) \in X^* \times R : \partial_{C^*} (x^*, r) \cap (X \times R) \neq \emptyset \}$$

(24)
is a dense set of $X^* \times R$. Therefore, by Lemma 6, it is easy to see that $\partial f(U) \cap X \neq \emptyset$. Therefore, by formulas $\partial f(U) \subset C^{**} \setminus C^*$ and $C^{**} \subset X = C$, we obtain that $\partial f(U) \cap X \subset C$. Pick $n \in \mathbb{N}$. Then, by hypothesis, there exist a slice

$$S(z^*, \partial f(U) \cap X, \alpha)$$

(25)

and $x_0 \in S(z^*, \partial f(U) \cap X, \alpha)$ such that $S(z^*, \partial f(U) \cap X, \alpha) \subset x_0 + U_n$. Moreover, if $x \in S(z^*, \partial f(U) \cap X, \alpha)$, then $x \in \partial f(x_0) \cap X$ for some point $x_1 \in U$ and $x_0 = x_1 + rz^*$ is in $U$ for sufficiently small $r > 0$. We claim that

$$\partial f(x_0)$$

$$\subset \{ x^{**} \in X^* : (z^*, x^{**}) \geq \sigma_{f(U) \cap X} (z^*) - \alpha \}.$$ (26)

Indeed, if $y^{**} \in \partial f(x_0)$, then we have

$$0 \leq (y^{**} - x_0, x_0 - x_1) = (y^{**} - x, z^*).$$ (27)

This implies that

$$y^{**} \in \{ x^{**} \in X^* : (z^*, x^{**}) \geq \sigma_{f(U) \cap X} (z^*) - \alpha \}.$$ (28)

Since the set $\{ x^{**} \in X^* : (z^*, x^{**}) > \sigma_{f(U) \cap X} (z^*) - \alpha \}$ is a weak* open set in $X^*$ and since $\partial f$ is norm-to-weak* upper semicontinuous, there exists $\delta > 0$ such that $B(x_0, \delta) \subset U$ and

$$\partial f(y^*)$$

$$\subset \{ x^{**} \in X^* : (z^*, x^{**}) > \sigma_{f(U) \cap X} (z^*) - \alpha \}.$$ (29)

for any point $y^* \in B(x_0, \delta)$. Moreover, since $\partial f(y^*) \subset \partial f(U)$, we obtain that

$$\partial f(y^*)$$

$$\subset \{ x^{**} \in \partial f(U) : (z^*, x^{**}) \geq \sigma_{f(U) \cap X} (z^*) - \alpha \}.$$ (30)

Pick

$$z_0^{**}$$

$$\in \{ x^{**} \in \partial f(U) : (z^*, x^{**}) > \sigma_{f(U) \cap X} (z^*) - \alpha \}.$$ (31)

Since $C^{**} = C^{**}$, there exists a net $\{ z_\beta \}_{\beta \in \Delta} \subset C$ such that $z_\beta \overset{w^*}{\to} z_0^{**}$. Therefore, by formula (31), we obtain that

$$\{ z_\beta \} \supset (z^*, z_0^{**}) \geq \sigma_{f(U) \cap X} (z^*) - \alpha.$$ (32)

Hence we may assume that $\{ z_\beta \} \supset (z^*, z_\beta) \geq \sigma_{f(U) \cap X} (z^*) - \alpha$. Moreover, by formula (31), there exists $z_0^{**} \in U$ such that

$$\{ z^{**} \in U : (z^*, z^{**}) \geq \sigma_{f(U)} (z^*) - \alpha \}$$

(33)

for any $z^* \in X^*$. Therefore, by $z_\beta \overset{w^*}{\to} z_0^{**}$, we obtain that

$$\{ z_\beta, z^* - z_0^{**} \leq f(z^*) - f(z_0) \}$$

(34)
for any $x^* \in X^*$. This implies that $z_\beta \in \partial f(U) \cap X$. Therefore, by formula (3),

$$z_\beta \in \left\{ x \in \partial f(U) \cap X : (z^*, x) \geq \sigma_{\partial f(U) \cap X}(z^*) - \alpha \right\}$$

$$= S(z^*, \partial f(U) \cap X, \alpha).$$

Therefore, by the previous proof, we obtain that

$$z_\beta \in S(z^*, \partial f(U) \cap X, \alpha) \subset x_0^* + U_n.$$

We claim that $U_n^w \subset W_n$ for all $n \in \mathbb{N}$. In fact, let $z^{**} \in U_n^w$. Then there exists a net $(z_n)_{n \in \mathbb{N}} \subset U_n$ such that $z_n \rightarrow z^{**}$. Hence, for any $i \in \{1, 2, \ldots, k\}$, we obtain that $x_i^*(z_n) \rightarrow x_i^*(z^{**})$. Since $|x_i^*(z_n)| \leq 1/n$, we have $|x_i^*(z^{**})| \leq 1/n$. This implies that $z^{**} \in W_n$. Hence $U_n^w \subset W_n$. Since $z_n \rightarrow z^{**}$, by formulas (31) and (35), we have

$$z_0^{**} \in (x_0^* + U_n)^w = x_0^* + U_n^w \subset x_0^* + W_n.$$

Since $z_0^{**}$ is arbitrary, we have $\partial f(y^*) \subset x_0^* + W_n$. It follows that $\partial f(B(x_0, \delta)) \subset x_0^* + W_n$. This implies that $x_0^* \in G_n \cap U$. Hence $G_n$ is a dense open subset, which finishes the proof.

Lemma 16. Suppose that $X$ is a Banach space and $C$ is a bounded convex subset of $X$. Then (1) $\Longleftrightarrow$ (2) is true, where

1. For any continuous convex function $f$ on $X^*$, if $\partial f(X^*) \subset C^w$, then $f$ has the Gâteaux differentiable points on $X^*$;

2. For any weak neighborhood $U$ of origin and $D \subset C$, there exist a slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$ such that $S(x^*, D, \alpha) \subset x + U$.

Proof. Suppose that there exist $D \subset C$ and a weak neighborhood $U$ of origin such that, for any weak slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$, we have $S(x^*, D, \alpha) \cap x + U$. Since

$$\sigma_D(x^*) = \sup \left\{ (x, x^*) : x \in D \right\}$$

$$= \sup \left\{ (x^{**}, x^*) : x \in co^{**}(D) \right\},$$

by formula (3) and convexity of $C$, we have

$$\partial \sigma_D(x^*) = \partial \sigma_D(0) \subset co^{**}(D) \subset C^w.$$  

Hence the sublinear functional $\sigma_D$ has the Gâteaux differentiable points on $X^*$. Since $C$ is a bounded subset of $X$, we obtain that $D$ is a bounded subset of $X$. Hence there exists $M > 0$ such that $\|x\| < M$ whenever $x \in D$. This implies that $\sigma_D(x^*) \leq M\|x^*\|$ for every $x \in D$. Hence $\sigma_D$ is a continuous sublinear functional. Moreover, since $U$ is a weak neighborhood of origin, there exist $\varepsilon > 0$ and $\{x_1^*, \ldots, x_k^*\} \subset X^*$ such that

$$\left\{ x \in X : \sum_{i=1}^k |(x_i^*, x)| < \varepsilon \right\} \subset U.$$  

We will show that the function $\sigma_D$ is nowhere Gâteaux differentiable. Indeed, given any $x^* \in X^*$, for each slices $S(x^*, D, \varepsilon/3n)$, there exists $x_n \in S(x^*, D, \varepsilon/3n)$ such that $S(x^*, D, \varepsilon/3) \not\subset x_n + U$. Hence there exist $y_n \in S(x^*, D, \varepsilon/3n)$ and $i \in \{1, \ldots, k\}$ such that $|(x_i^*, x_n - y_n)| \geq \varepsilon$. Otherwise, for any $y \in S(x^*, D, \varepsilon/3n)$ and $i \in \{1, \ldots, k\}$, we have $|(x_i^*, x_n - y_n)| < \varepsilon$. Hence we have

$$\sigma_D(x^*) < \frac{1}{3n}$$  

and

$$\sigma_D(x^*) < \frac{1}{3n}.$$  

This implies that

$$2\sigma_D(x^*) < (x^*, x_n) + \frac{1}{3n} (x^*, y_n) + \frac{1}{3n}$$

$$= (x^*, x_n + y_n) + \frac{2}{3n}.$$  

Therefore, by formula (42), we have

$$\frac{1}{n} \left[ \sigma_D \left( \frac{x^* + 1}{n} x^* \right) + \sigma_D \left( \frac{x^* - 1}{n} x^* \right) - 2\sigma_D(x^*) \right]$$

$$\geq \frac{1}{n} \left[ \left( x^* + \frac{1}{n} x^* \right) x_n + \left( x^* - \frac{1}{n} x^* \right) y_n \right] - (x^*, x_n + y_n) - \frac{2}{3n} \geq \frac{1}{n} \left[ \frac{1}{n} (x^*_i, x_n - y_n) \right]$$

$$- \frac{2}{3n} \geq \frac{1}{n} \left[ \frac{\varepsilon - 2}{3n} \right] = \frac{1}{3}.$$  

This implies that the sublinear functional $\sigma_D$ is nowhere Gâteaux differentiable, a contradiction, which finishes the proof.

Lemma 17. Let $C$ be a bounded subset of $X$. Then the following statements are equivalent.

1. For any weak neighborhood $U$ of origin, there exist a slice $S(x^*, \alpha, C)$ and $x_0 \in S(x^*, \alpha, C)$ such that $S(x^*, \alpha, C) \subset x_0 + U$.

2. For any weak neighborhood $U$ of origin, there exists a point $x_0 \in C \cap (x_0 + U)$.

3. For any weak neighborhood $U$ of origin, there exists a slice $S(x^*, \alpha, C)$ such that $x_1 - x_2 \in U$ for any $x_1, x_2 \in S(x^*, \alpha, C)$. 

Proof. (3)⇒(2). Let $\sigma_C(x^*) = \sup\{x^*(y) : y \in C\}$. Then, by condition (3), it is easy to see that, for any weak neighborhood $U$ of origin, there exists an open slice $S(x^*, \alpha, C) = \{x \in C : x^*(x) > \sigma_C(x^*) - \alpha\}$ such that $x_1 - x_2 \in U$ for any $x_1, x_2 \in S(x^*, \alpha, C)$ and $x_2 \in S(x^*, \alpha, C)$. Hence, for any weak neighborhood $U$ of origin, there exists $x_0 \in S(x^*, \alpha, C)$ such that $S(x^*, \alpha, C) \subset x_0 + U$, it follows that $x^*(x_0) > \sigma_C(x^*) - \alpha$. Therefore, by $x^*(x_0) > \sigma_C(x^*) - \alpha$, we obtain that $x_0 \notin \partial C \cap \{x \in C \cap (x_0 + U)\}$.

(2)⇒(1). For any weak neighborhood $U$ of origin, there exists $x_0 \in C$ such that $x_0 \notin \partial C \cap \{x \in C \cap (x_0 + U)\}$. Therefore, by the separation theorem, there exist $x^* \in X^*$ and $r > 0$ such that $x^*(x_0) - r > \sup \{x^*(x) : x \in \partial C \cap \{x \in C \cap (x_0 + U)\}\}$. (47)

Let $\alpha = \sigma_C(x^*) - x^*(x_0) + r$. Then $x^*(x_0) = \sigma_C(x^*) - \alpha + r > \sigma_C(x^*) - \alpha$. This implies that $x_0 \in S(x^*, C^*, \alpha)$. Hence, for any $y \in S(x^*, C, \alpha)$, we obtain that $x^*(y) > \sigma_C(x^*) - \alpha = \sigma_C(x^*) - \alpha + r = x^*(x_0) - r$. (48)

Therefore, by $x^*(y) > x^*(x_0) - r$ and formula (47), we obtain that $y \notin x_0 + U$.

(1)⇒(3). For any weak neighborhood $U$ of origin, there exists a weak neighborhood $V$ of origin such that $V \cap V \subset U$ and there exists a slice $S(x^*, C, \alpha)$ and $x_0 \in S(x^*, C, \alpha)$ such that $S(x^*, C, \alpha) \subset x_0 + V$. Hence, if $x_1, x_2 \in S(x^*, C, \alpha)$, then $x_1 \notin x_0 + V$ and $x_2 \in x_0 + V$. This implies that $x_1 - x_2 = (x_0 + V) - (x_0 + V) = V \cap V \subset U$. (49)

which finishes the proof. □

Lemma 18. Suppose that $C^*$ is a $c$-separable bounded subset of $X^*$. Then the following statements are equivalent.

(1) Every $w^*$-lower convex function $g \leq \sigma_C$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$.

(2) For any weak neighborhood $U$ of origin and $D \subset \partial \sigma_C(0) \cap X$, there exist a slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$ such that $S(x^*, D, \alpha) \subset x + U$.

Proof. (1)⇒(2). Let $p = \sigma_C$. Suppose that there exist a set $D \subset \partial \sigma_C(0) \cap X$ and a weak neighborhood $U$ of origin such that for any weak slice $S(x^*, D, \alpha)$ and $x \in S(x^*, D, \alpha)$, we obtain that $S(x^*, D, \alpha) \cap x + U$. Since $p = \sigma_C$, we obtain that $C^* = \{x^* \in X^* : p(x^*) \leq 1\}$. Then

\[ \partial p(0) = \{x^{**} \in X^{**} : x^{**} (x^*) \leq 1, x^* \in C^*\} \]

and $\overline{\partial p(0) \cap X^{**}} = \partial p(0)$. (50)

Hence

\[ \partial p(0) \cap X = \{x \in X : x^* (x) \leq 1, x^* \in C^*\} \]

and $p(x^*) = \sigma_{\partial p(0)}(x^*)$. (51)

Moreover, by formula

\[ \{x \in X : x^* (x) \leq 1, x^* \in C^*\} \]

we obtain that

\[ p(x^*) = \sigma_{\partial p(0)}(x^*) = \sigma_{\partial p(0) \cap X}(x^*) = \sup \{x^*(x) : x \in \partial p(0) \cap X\} \]

Since $D \subset \partial p(0) \cap X$, we obtain that

\[ \sigma_{\partial p(0) \cap X}(x^*) = \sup \{x^*(x) : x \in \partial p(0) \cap X\} \]

(54)

Therefore, by formula $\sigma_C = \sigma_{\partial p(0) \cap X} \geq \sigma_D$, we obtain that $\sigma_D$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$. However, by the proof of Lemma 16, we obtain that $\sigma_D$ is nowhere Gâteaux differentiable, a contradiction.

(2)⇒(1). Let $g$ be a $w^*$-lower semicontinuous convex function on $X^*$ and $g \leq p$. Then $g$ is a continuous function. We claim that $\partial g(x^*) \subset \partial p(0)$. In fact, suppose that there exists a point $x^{**} \in \partial g(x^*)$ such that $x^{**} \notin \partial p(0)$. Then, by the separation theorem, there exists a point $x^* \in X^*$ and a real number $r > 0$ such that $x^{**} (z^*) > r + \sup \{z^{**} (z^{**}) : z^{**} \in \partial p(0)\} \geq r + p(z^*)$. (55)

Since $x^{**} \in \partial g(x^*)$, we obtain that $g(y^{**}) - g(x^*) \geq x^{**} (y^{**} - x^*)$ for all $y^{**} \in X^*$. Let $y^{**} = k z^*$. Then $g(k z^*) - g(x^*) \geq x^{**} (k z^*) - x^{**} (x^*) > \frac{1}{2} k r + k p(z^*) - x^{**} (x^*)$. (56)

This implies that $g(k z^*) > p(k z^*)$ for all sufficiently large $k > 0$, which contradicts formula $g \leq p$. Therefore, by Lemma 16, we obtain that $g$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$, which finishes the proof. □
Theorem 14. By Lemmas 15–18, we obtain that (1) \( \Longleftrightarrow \) (2) is true. (1) \( \Longleftrightarrow \) (3). Let \( p = \sigma_C \) and \( D \) be a weak' closed convex subset of \( \partial \Omega(p) \). Then \( p \geq \sigma_D \). This implies that \( \sigma_D \) is Gâteaux differentiable at a dense \( G_\delta \) subset for any \( D' \subset D \). Therefore, by Theorem 2 of [3], we obtain that \( D \) is the weak' closed convex hull of its weak' exposed points.

(3) \( \Rightarrow \) (2). Let \( D \) be a closed convex subset of \( \partial \Omega(0) \cap X \) and \( V \subset X \) be a weak neighborhood of origin. Since \( C^{**} \) is bounded, by the definition of \( C^* \), we obtain that \( C \) is bounded and \( C^{**} \) is a weak' bounded closed subset of \( X^{**} \). Hence \( D \) is bounded and \( D^{**} \) is a weak' bounded closed subset of \( X^* \). Since \( V \) is a weak neighborhood of origin, we may assume that there exist \( \{x_1^*, x_2^*, \ldots, x_k^*\} \subset S(X^*) \) and \( \varepsilon > 0 \) such that

\[
V = \left\{ x \in X : \left| (x_i^*, x) \right| < \frac{1}{2} \varepsilon \right\}
\]

Let

\[
V_0 = \left\{ x^{**} \in X^{**} : \left| (x_i^{**}, x^{**}) \right| < \frac{1}{2} \varepsilon \right\}
\]

Then \( V_0 \) is a weak' neighborhood of origin in \( X^{**} \). Since \( D^{**} \) is a weak' bounded closed subset of \( X^{**} \) and \( D^{**} \subset \partial \Omega(0) \), there exists \( x_0^{**} \in D^{**} \) such that \( x_0^{**} \) is a weak' exposed point of \( D^{**} \). Hence there exists \( x_0^* \in S(X^*) \) such that

\[
z_0^{**}(x_0^*) = \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \right\}
\]

and

\[
y^{**}(x_0^*) < \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \right\}
\]

for each \( y^{**} \in D^{**} \setminus \{z_0^{**}\} \). We claim that for any weak' neighborhood \( U \) of origin, there exists \( k > 0 \) such that

\[
z_0^{**}(x_0^*) - 4k \geq \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \setminus \{z_0^{**} + U\} \right\}
\]

(60)

In fact, suppose that there exists a sequence \( \{x_n^*\}_{n=1}^{\infty} \subset D^{**} \setminus \{z_0^{**} + U\} \) such that \( x_n^*(x_0^*) \to \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \right\} = \infty. \) Then we may assume without loss of generality that \( x_n^* \neq x_m^* \) for any \( m \neq n \). Since \( D^{**} \) is a weak' bounded closed convex set of \( X^{**} \), we obtain that \( D^{**} \) is weak' compact. Then there exists a point \( x_0^* \in D^{**} \) such that \( x_0^* \) is a weak' accumulation point of \( \{x_n^*\}_{n=1}^{\infty} \).

\[
\Delta = \left\{ U_{x_0^*} : U_{x_0^*} \text{ is weak' neighborhood of } x_0^{**} \right\}
\]

(61)

Hence we define an order by the containing relations, i.e., \( U_{x_0^*} \supseteq V_{x_0^*} \) if and only if \( V_{x_0^*} \supseteq U_{x_0^*} \). This implies that \( \Delta \) is an order set. Hence

\[
\Omega = \left\{ U_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty} : U_{x_0^*} \text{ is weak' neighbourhood of } x_0^{**} \right\}
\]

(62)

is an order set whenever \( V_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty} \supseteq U_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty} \) if and only if \( U_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty} \supseteq V_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty} \). Therefore, by the Zermelo Lemma, we obtain that there exists a mapping \( f \) on \( \Omega \) such that \( f(U_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty}) \subset U_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty}. \) Put \( x_0^{**} = f(U_{x_0^*} \cap \left\{ x^{**} \right\}_{n=1}^{\infty}). \) Hence we define a net \( \{x_\alpha^{**}\}_{\alpha \in \Delta} \subset \left\{ x^{**} \right\}_{n=1}^{\infty}. \) Therefore, by the definition of net \( \{x_\alpha^{**}\}_{\alpha \in \Delta} \), we have

\[
x_0^{**}(x_\alpha^*) = \sup \left\{ x^{**}(x_\alpha^*) : x^{**} \in D^{**} \right\}
\]

and

\[
x_\alpha^* \to x_0^*\]

(63)

which contradicts \( \{x_\alpha^{**}\}_{\alpha \in \Delta} \subset \left\{ x^{**} \right\}_{n=1}^{\infty} \subset D^{**} \setminus \{z_0^{**} + U\}. \) Moreover, since \( V_0 \) is a weak' neighborhood of origin in \( X^{**} \), there exists weak' neighborhood \( W \) of origin, such that \( W + W \subset V_0. \) Since there exists \( k > 0 \) such that

\[
z_0^{**}(x_0^*) - 4k \geq \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \setminus \{z_0^{**} + W\} \right\}
\]

(64)

by formula \( \overline{\partial^{**}X^*} = D^{**} \subset \{z_0^{**} + W\} \), therefore, \( z_0^{**} + W \subset x + W \subset x + V_0. \) Moreover, it is easy to see that \( D(x + V) \subset D^{**} \setminus (x + V_0) \). Then

\[
x(x_0^*) - 3k \geq \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \setminus \{z_0^{**} + W\} \right\}
\]

(65)

\[
\geq \sup \left\{ x^{**}(x_0^*) : x^{**} \in D^{**} \setminus (x + V_0) \right\}
\]

(66)

\[
\sup \left\{ \left| x(x_0^*) \right| : \left| z \right| \in D \setminus (x + V) \right\}
\]

This implies that

\[
x(x_0^*) - 3k \geq \sup \left\{ \left| x(x_0^*) \right| : \left| z \right| \in D \setminus (x + V) \right\}
\]

= \sup \left\{ \left| x(x_0^*) \right| : \left| z \right| \in D \setminus (x + V) \right\}

\[
= \sup \left\{ z(x_0^*) : \left| z \right| \in \overline{\partial} D \setminus (x + V) \right\}
\]

Hence we obtain that \( x \not\in \overline{\partial} D \setminus (x + V) \). This implies that \( D \) is weak' dentable, which finishes the proof.

Theorem 19. Suppose that \( X^* \) is a Gâteaux differentiable space. Then every bounded subset of \( X \) is weak' dentable.

Proof. By the proof of Theorem 14, we obtain that every closed convex subset of \( X \) is weak' dentable. Let \( C \) be a bounded subset of \( X \). Suppose that \( C \) is not weak' dentable. Then, by Lemma 17, there exists a weak' neighborhood \( U \) of origin such that \( S(x^*, \alpha, C) = S(x^*, \alpha, C) \cap U \) for any \( x^* \in X^* \) and \( \alpha > 0 \). Since \( \sigma(x^*) = \sigma(C)(x^*) \) for any \( x^* \in X^* \), we have \( S(x^*, \alpha, C) \subset S(x^*, \alpha, \overline{\partial} C) \). This implies that

\[
S\left( x^*, \alpha, \overline{\partial} C \right) = S(x^*, \alpha, \overline{\partial} C) \supset S(x^*, \alpha, C) \not\subset U
\]

(68)

Therefore, by Lemma 17, we obtain that \( \overline{\partial} C \) is not weak' dentable, a contradiction, which finishes the proof.
Example 20. Let $X = c_0$. Then $X^* = l^1$. Since $l^1$ is separable, by Theorem 19, we obtain that every bounded subset of $c_0$ is weak deniable. Moreover, it is well known that $c_0$ has not the Radon-Nikodym property. Hence there exists a bounded subset $D$ of $c_0$ such that $D$ is not deniable.

Example 21. Let $X = c_0$. Then $X^* = l^1$ is separable and $X^{**} = l^\infty$ is not a separable space. Let $Y$ be a Banach space and $Y$ be not a separable space. Define $C = B(X) \times \{0\} \subset X \times Y$. Then
\[
C^* = B(X^*) \times Y^* \subset X^* \times Y^*
\]
and
\[
C^{**} = B(X^{**}) \times \{0\} \subset X^{**} \times Y^{**}.
\]
By Theorem 19, we obtain that $C$ is weak deniable. By Example 13, we obtain that $C^{**}$ is $\varepsilon$-separable and bounded.

3. Gateaux Differentiability and Weakly Exposed Point

Definition 22. A point $x_0 \in C$ is said to be weakly exposed point of $C$ if there exist $x^* \in S(X^*)$ and $\{x_n^*\}_{n=1}^\infty$ such that $x^*(x_n) \to \sigma_C(x^*)$; then $x_n^* \rightharpoonup x_0$.

Definition 23. A point $x_0 \in C$ is said to be exposed point of $C$ if there exists $x^* \in S(X^*)$ such that $x^*(x_0) > x^*(x)$ whenever $x \in C\setminus\{x_0\}$.

Definition 24. A point $x_0 \in C$ is said to be strongly exposed point of $C$ if there exist $x^* \in S(X^*)$ and $\{x_n^*\}_{n=1}^\infty$ such that $x^*(x_n) \to \sigma_C(x^*)$; then $x_n \to x_0$.

Definition 25. A point $x \in A$ is said to be an extreme point of $A$ if $2x = y + z$ and $y, z \in A$ imply $y = z$. The set of all extreme points of $A$ is denoted by $ExtA$. If $ExtB(X) = S(X)$, then $X$ is said to be a strictly convex space.

It is easy to see that if $x$ is a strongly exposed point of $C$, then $x$ is a weakly exposed point of $C$ and if $x$ is a weakly exposed point of $C$, then $x$ is a strongly exposed point of $C$. Moreover, weakly exposed point, exposed point, and strongly exposed point are different. We will give two examples in Sections 3 and 4. A Banach space $X$ is said to have the Krein-Milman property if every bounded closed convex subset of $X$ is the closed convex hull of its extreme points. It is well known that if $X$ has the Radon-Nikodym property, then $X$ has the Krein-Milman property. Moreover, we know that $X^*$ has the Krein-Milman property if and only if $X^*$ has the Radon-Nikodym property (see [12]).

Theorem 26. Suppose that $C$ is a bounded closed convex set. Then $d\sigma_C(x^*) = x$ if and only if $x$ is a weakly exposed point of $C$ exposed by $x^*$.

Proof. Necessity. Let $p(x^*) = \sigma_C(x^*)$ and $d\sigma_C(x^*) = x$. Then we have $x \in C^{**} = \partial p(0)$. Therefore, by Lemma 1 of [1], we obtain that $x^*$ exposes $\partial p(0)$ at $x$; i.e., $x^*(x) = \sigma_C(x^*)$. Let $\{x_n\}_{n=1}^\infty \subset C$ and $x^*(x_n) \to \sigma_C(x^*)$ as $n \to \infty$. We next will prove that $x_n \rightharpoonup x$ as $n \to \infty$. In fact, we may assume without loss of generality that $x_n \not= x_m$ for any $m \neq n$.

Since $C$ is a bounded closed convex set, we obtain that $C^{**}$ is a bounded set. Hence we obtain that $\partial p(0) = C^{**}$ is a bounded set. This implies that $\partial p(0)$ is a weak* compact set in $X^{**}$. Hence there exists $x^{**}_0 \in \partial p(0)$ such that $x^{**}_0$ is a weak* accumulation point of $\{x_n\}_{n=1}^\infty$. Hence there exists a net $\{x_n\}_{n \in \Lambda} \subset \{x_n\}_{n=1}^\infty$ such that
\[
x^{**}_0 = \sup \{x^{**}(x^*) : x^{**} \in \partial p(0)\}
\]
and $x_n \rightharpoonup x^{**}_0$.

This implies that $x^{**}_0 \in \partial p(x^*)$. Since $\partial p(x^*) = x$, we have $x^{**}_0 = x$. Moreover, by $x_n \rightharpoonup x^{**}_0$, we obtain that $x_n \rightharpoonup x$.

Suppose that $\{x_n\}_{n \in \Lambda}$ does not converge weakly to $x$. Then there exists a weak neighbourhood $V$ of $x$ and a subsequence $\{n_k\}$ of $\{n\}$ such that $x_{n_k} \not\to V$. Repeat the previous proof; there exists a net $\{x_{n_k}\}_{n \in \Lambda} \subset \{x_n\}_{n=1}^\infty$ such that $x_{n_k} \rightharpoonup x$, a contradiction. Hence we have $x_n \rightharpoonup x$ as $n \to \infty$. This implies that $x$ is a weakly exposed point of $C$.

Sufficiency. Suppose that there exists $x^{**} \in C^{**}$ such that $x^{**}(x^*) = x(x^*)$. Since $x$ is a weakly exposed point of $C$ and $C^{**} = C^{***}$, we have
\[
x^{**}(x^*) = x(x^*) = \sup \{y(x^*) : y \in C\} = \sup \{y^{**}(y^*) : y^* \in C^{**}\}.
\]

We next will prove that $x^{**} = x$. Suppose that $x^{**} \neq x$. Then there exists a weak* neighborhood $V$ of origin such that $(x^{**} + V) \cap (X + V) = \emptyset$. Let $U_n = \{y^{**} \in X^{**} : \left| y^{**}(x^*) - x^{**}(x^*) \right| < \frac{1}{n}\}$. (72)

Then, by $(x^{**} + V) \cap (X + V) = \emptyset$, we obtain that $(x^{**} + (V \cap U_n)) \cap (X + V) = \emptyset$. Moreover, by $x^{**} \in C^{**}$ and $C^{**} = C^{***}$, we obtain that there exists $x_0 \in x^{**} + (V \cap U_n)$ such that $x_0 \in C$. Hence we have $x_0(x_n) \to x(x^*) = \sup\{x(y) : y \in C\}$. Since $x$ is a weakly exposed point of $C$ and exposed by $x^*$, by formula
\[
\lim_{n \to \infty} x^*(x_n) = x^*(x) = \sup\{x^*(y) : y \in C\},
\]
we obtain that $x_n \rightharpoonup x$, which contradicts $(x^{**} + (V \cap U_n)) \cap (X + V) = \emptyset$. Hence $x^{**} = x$. This implies that $x \in C^{**}$ is a weak* exposed point of $C^{**}$ exposed by $x^*$. Therefore, by Lemma 1 of [3], we obtain that $d\sigma_C(x^*) = x$, which completes the proof.

Lemma 27 (see [1]). Suppose that $x^* \in S(X^*)$, $y^* \in S(X^*)$, and $\varepsilon > 0$. If $|y^*(x)| \leq 1$ whenever $x \in X$ satisfies
\[
x^*(x) = 0
\]
and $\|x\| \leq 2\varepsilon^{-1}$,

then either $\|x^* - y^*\| \leq \varepsilon$ or $\|x^* + y^*\| \leq \varepsilon$. 

Theorem 28. The following statements are equivalent:
(1) $X$ is an Asplund space.
(2) $X^∗$ has the Radon-Nikodym property.
(3) Every bounded closed convex subset of $X^∗$ is the closed convex hull of its weakly exposed points.
(4) For every bounded closed convex set $C^∗$ of $X^∗$, there exists $x^{**} ∈ X^{**}$ such that $σ_{C^∗}$ is Gâteaux differentiable at $x^{**}$ and $dσ_{C^∗}(x^{**}) ∈ C^∗$.

Moreover, by Theorem 26, it is easy to see that (3) holds.

Proof. It is well known that (1)⇐⇒(2) and (2)⇒(3) are true. Moreover, by Theorem 26, it is easy to see that (3)⇒(4) is true. (3)⇒(2).

(4)⇒(2). Let $C^∗$ be a bounded closed convex set of $X^∗$ and $J$ be the closed convex hull of the extreme points of $C^∗$. Suppose that $J ≠ C^∗$. Then, by the separation theorem and the Bishop-Phelps Theorem, there exist $x^{**}_0 ∈ X^{**}$ and $x^0 ∈ X$ such that
\[ \sup \{ x^{**}_0(x^∗) : x^∗ ∈ C^∗ \} = x^{**}_0(x^0) > \sup \{ x^{**}_0(x^∗) : x^∗ ∈ J \}. \]

Then
\[ D = \{ x ∈ X : x^{**}_0(x^∗) = \sup \{ x^{**}_0(y^∗) : y^∗ ∈ C^∗ \} \} \]

is a nonempty bounded closed convex set. By Theorem 26, there exists $x^1_0 ∈ D^∗$ such that $x^1_0$ is a weakly exposed point of $D^∗$. Then $x^1_0$ is an extreme point of $D^∗$. Let $2x^1_0 = y^1_1 + z^1_1$, $y^1_1 ∈ C^∗$, and $z^1_1 ∈ C^∗$. Then
\[ x^{**}_0(2x^1_0) = x^{**}_0(y^1_1 + z^1_1) = x^{**}_0(y^1_1) + x^{**}_0(z^1_1) = 2 \sup \{ x^{**}_0(y^∗) : y^∗ ∈ C^∗ \}. \]

This implies that
\[ x^{**}_0(y^1_1) = x^{**}_0(z^1_1) = \sup \{ x^{**}_0(y^∗) : y^∗ ∈ C^∗ \}. \]

Hence $y^1_1 ∈ D^*$ and $z^1_1 ∈ D^*$. Since $x^1_0$ is an extreme point of $D^*$, we have $y^1_1 = z^1_1$. This implies that $x^1_0$ is an extreme point of $C^∗$, which contradicts $J ≠ C^∗$. Hence every bounded closed convex subset of $X^*$ is the closed convex hull of its extreme points. This implies that $X^*$ has the Krein-Milman property. Hence $X^*$ has the Radon-Nikodym property.

It is easy to see that (5)⇒(4) is true. We next will prove that (4)⇒(5) is true. Let $C^*$ be a bounded closed convex subset of $X^*$ and $C^*$ be not a singleton. Then we may assume without loss of generality that $C^* ⊂ B(X^*)$ and $0 ∈ C^*$. Let $∥x^{**}∥ = 1$ and $ε ∈ (0,1/8)$. Then we define the bounded closed convex set $C^∗_1 = \overline{σ(C^∗ ∪ N)}$, where
\[ N = \{ x^∗ ∈ X^* : x^∗(x^{**}) = 0, ∥x^∗∥ ≤ 2ε^{-1} \}. \]
Theorem 31. Let $X^*$ be a weak Asplund space and continuous convex function $f$ be above bounded in weak* neighborhood $U$. Then $f$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$ and $d f (G) \subset X$.

In order to prove the theorem, we give a lemma.

Lemma 32. Suppose that graph of convex function $f$ has an interior point in $(X^* \times R, w^*)$. Then for any $x^* \in X^*$, we have $d f (x^*) \subset X$.

Proof. Since continuous convex function $f$ has an interior point in $(X^* \times R, w^*)$, we obtain that $w^* - \text{int}(\text{epi} f) \neq \emptyset$. Pick $x^* \in X^*$. Then it is easy to see that $(x^*, f(x^*)) \notin w^* - \text{int}(\text{epi} f)$. Therefore, by the separation theorem, there exists $(x, r) \in (X \times R) \setminus (0, 0)$ such that

$$x^* (x) + r f (x^*) \geq \sup \{ z^* (x) + r \xi : (z^*, \xi) \in w^* - \text{int}(\text{epi} f) \}.$$  

(83)

Hence, for any $y^* \in X^*$, we have $x^* (x) + r f (x^*) \geq y^* (x) + r f (y^*)$ and $x^* (x) + r f (x^*) \geq y^* (x) + r f (y^*) + 1$. Let $y^* = x^*$. Then $r f (x^*) \leq r (f (x^*) + 1)$. This implies that $r \leq 0$. Suppose that $r = 0$. Then, for any $y^* \in X^*$, we obtain that $x^* (x) \geq y^* (x)$. This implies that $x = 0$. Hence we have $(x, r) = (0, 0)$, a contradiction. This implies that $r < 0$. Hence we may assume without loss of generality that $r = -1$. This implies that $x^* (x) - f (x^*) \geq y^* (x) - f (y^*)$. Hence $x \in d f (x^*)$. This implies that $d f (x^*) \neq \emptyset$ for any $x^* \in X^*$.

Let $x^* \in d f (x^*)$. We next will prove that $x^* \in X$. Since $x^* \in d f (x^*)$, we obtain that $(x^*, y^* + x^*) \leq f (y^*) - f (x^*)$. This implies that $x^* (y^*) - f (y^*) \leq x^* (x^*) - f (x^*)$. Hence, for any $(y^*, t) \in w^* - \text{int}(\text{epi} f)$, we have

$$x^* (y^*) - t \leq x^* (y^*) - f (y^*) \leq x^* (x^*) - f (x^*).$$  

(84)

This implies that

$$x^* (x^*) - f (x^*) \geq \sup \{ x^* (y^*) - t : (y^*, t) \in w^* - \text{int}(\text{epi} f) \}.$$  

(85)

We next will prove that the functional $x^*$ is a continuous functional of $(X^*, w^*)$. Suppose that $x^*$ is not continuous at origin. Then there exist a net $(x^*_\alpha)_{\alpha \in \Delta} \subset X^*$ and $r > 0$ such that $x^*_\alpha \rightharpoonup 0$ and $|x^* (x^*_\alpha)| > r$. Pick $x^*_0 \in \{ x^*_\alpha \}_{\alpha \in \Delta}$. Then,

$$\frac{x^*_0}{x^* (x^*_\alpha)} - \frac{x^*_0}{x^* (x^*_\alpha)} \rightharpoonup \left( - \frac{x^*_0}{x^* (x^*_0)} \right).$$  

(86)

and

$$x^* (x^*_0) = x^* (x^*_\alpha) = 0.$$  

(87)

This implies that the hyperplane $\{ h^* \in X^* : x^* (h^*) = 0 \}$ is not a weak* closed set. Pick

$$z^* \in \overline{\{ h^* \in X^* : x^* (h^*) = 0 \}}^w \setminus \{ h^* \in X^* : x^* (h^*) = 0 \}.$$  

(88)

Then

$$\{ \lambda z^* : \lambda \in R \} \subset \overline{\{ h^* \in X^* : x^* (h^*) = 0 \}}^w.$$  

(89)

Hence we have

$$X^* = \{ \lambda z^* : \lambda \in R \} \cup \{ h^* \in X^* : x^* (h^*) = 0 \} \subset \overline{\{ h^* \in X^* : x^* (h^*) = 0 \}}^w.$$  

Then

$$X^* = \overline{\{ h^* \in X^* : x^* (h^*) = 0 \}}^w.$$  

(91)

Moreover, there exist a weak* open set $U$ of $X^*$ and an open interval $(a, b)$ such that $U \times (a, b) \subset w^* - \text{int}(\text{epi} f)$. Pick $h^*_0 \in \{ h^* \in X^* : x^* (h^*) = 0 \}$. Then

$$X^* = \overline{h^*_0 + X^* = h^*_0 + \{ h^* \in X^* : x^* (h^*) = 0 \}}^w.$$  

(92)

Therefore, by formula (92), there exists $x_0^* \in U$ such that $x^* (x_0^*) = x^* (x^*) - f (x^*) + b$. Pick $r \in (a, b)$. Then

$$(x_0^*, r) \in U \times (a, b) \subset w^* - \text{int}(\text{epi} f).$$  

Therefore, by formula (85), we have

$$x^* (x^*) - f (x^*) \geq x^* (x_0^*) - r \geq x^* (x^*) - f (x^*) + (b - r).$$  

(93)

This implies that $r \geq b$, a contradiction. Hence we obtain that $x^*$ is continuous at origin. This implies that $x^*$ is a continuous functional of $(X^*, w^*)$. Since $(X^*, w^*)^* = X$, we have $x^* \in X$, which finishes the proof.

Proof of Theorem 31. Let $U$ be a weak* neighbourhood and $f$ is above bounded on $U$. Then we may assume without loss of generality that $f (x^*) < 0$ whenever $x^* \in U$. Then $U \times (0, 1) \subset \text{epi} f$. Therefore, by Lemma 32, we have $d f (x^*) \subset X$ for any $x^* \in X^*$. Since $X^*$ is a weak Asplund space, we obtain that $f$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$. Hence convex function $f$ is Gâteaux differentiable at a dense $G_δ$ subset $G$ of $X^*$ and $d f (G) \subset X$, which finishes the proof.

Definition 33. A point $x \in S (X)$ is called a smooth point if it has an unique supporting functional $f_x$. If every $x \in S (X)$ is a smooth point, then $X$ is called a smooth space.

Definition 34. A Banach space $X$ is said to have the $H$-property if \( \{ x_\alpha \}_{\alpha = 1}^\infty \subset S (X), x \in S (X), \text{ and } x_n \rightharpoonup x \) as $n \to \infty$. 


Example 35. It is well known that there exists a Banach space X such that X is reflexive and strictly convex and does not have the H-property. Then it is easy to see that there exists x ∈ B(X) such that x is a weakly exposed point of B(X) and not a strongly exposed point of B(X).

4. Some Examples in Orlicz Function Spaces

Definition 36. \( M : R \rightarrow R \) is called a N-function if it has the following properties:

1. \( M \) is even, continuous, convex and \( M(0) = 0 \).
2. \( M(u) > 0 \) for all \( u \neq 0 \).
3. \( \lim_{u \rightarrow 0} M(u)/u = 0 \) and \( \lim_{u \rightarrow \infty} M(u)/u = \infty \).

Let \( M \) be a \( N \)-function and \( (G, \Sigma, \mu) \) be a finite nonatomic measure space. Let \( p(u) \) denote the right derivative of \( M(u) \) and \( q(v) \) be the generalized inverse function of \( p(u) \) by
\[
q(v) = \sup_{u \geq 0} \{ u \geq 0 : p(u) \leq v \}.
\]

Then the function \( N(v) \) defined by \( N(v) = \int_0^v q(s)ds \) for any \( v \in R \) is called the complementary function to \( M \) in the sense of Young. We define the modular of \( x \) by
\[
\rho_M(x) = \int_G M(x(t)) dt.
\]

Let us define the Orlicz function space \( L_M \) by
\[
L_M = \{ x(t) : \rho_M(\lambda x) < \infty \text{ for some } \lambda > 0 \},
\]
\[
E_M = \{ x(t) : \rho_M(\lambda x) < \infty \text{ for all } \lambda > 0 \}.
\]

It is well known that \( L_M \) and \( E_M \) are Banach spaces when it is equipped with the Luxemburg norm
\[
\|x\| = \inf \left\{ \lambda > 0 : \rho_M \left( \frac{x}{\lambda} \right) \leq 1 \right\}
\]
or equipped with the Orlicz norm
\[
\|x\| = \inf_{k>0} \left\{ 1 + \rho_M(kx) \right\}.
\]

\( L_M, E_M \) denote Orlicz spaces equipped with the Luxemburg norm. \( L^0_M, E^0_M \) denote Orlicz spaces equipped with the Orlicz norm.

Definition 37 (see [12]). We say that an N-function \( M \) satisfies condition \( \Delta_2 \) if there exist \( K > 2 \) and \( u_0 \geq 0 \) such that \( M(2u) \leq KM(u) \) whenever \( u \geq u_0 \).

It is well known that \( (E_M) = L^0_M \) and \( (E^0_M) = L_M \) (see [12]). Moreover, it is well known that \( E_M = L_M \) if and only if \( M \in \Delta_2 \).

Theorem 38 (see [12]). Orlicz space \( E_M \) is smooth if and only if \( p \) is continuous.

Theorem 39 (see [12]). Orlicz space \( L_M \) is strictly convex if and only if \( M \) is strictly convex and \( M \in \Delta_2 \).

Theorem 40 (see [12]). Orlicz space \( E_M(L^0_M) \) has the Radon-Nikodym property if and only if \( M \in \Delta_2 \).

Example 41. There exist a bounded set \( C \subset X \) and \( x \in C \) such that \( x \) is a exposed point of \( C \) and is not a weakly exposed point. Let \( X = L^1_N \), where \( M \) is continuous, \( N \) is strictly convex, \( N \in \Delta_2 \), and \( M \notin \Delta_2 \). Since \( N \) is strictly convex and \( N \in \Delta_2 \), we obtain that \( L^1_N \) is strictly convex. Since \( M \) is continuous and \( N \in \Delta_2 \), we obtain that \( L^1_N \) is smooth. Since \( M \) is continuous and \( N \in \Delta_2 \), we obtain that \( L^1_N \) is smooth. Pick \( r \in (0, 1/8) \) and \( u_1 \in L^1_N \) such that \( \|u_1\| < r \). Pick \( u_2 \in E_M \) such that \( \|u_2\| = 8 \). Then it is easy to see that \( u_1 + u_2 \in L^1_N \) such that
\[
\|u_1 + u_2\| > 8 - r
\]
and
\[
dist(u_1 + u_2, E_M) \leq \|u_1 + u_2 - u_2\| = \|u_1\| < r.
\]

Let
\[
u' = u_1 + u_2.
\]

Then, by Theorem 1.44 of [12], there exists \( a \in (0, 1) \) such that
\[
\theta(u') = \inf \left\{ \lambda > 0 : \rho_M \left( \frac{u'}{\lambda} \right) < \infty \right\}
\]
\[
dist(u', E_M) = \frac{\|u_1 + u_2\|}{\|u_1 + u_2 - u_2\|} < \frac{r}{8 - r}
\]
\[
= 8a < 1 - 4a < 1.
\]

Moreover, by the Bishop-Phelp theorem, there exist \( u \in S(L^1_N) \) and \( v \in S(L_N) \) such that \( (u,v) = \int_G (u(t), v(t)) dt = 1 \) and \( \|u - u'\| < a \). Since \( \text{dist}(u', E_M) = 8a \), we have \( \text{dist}(u, E_M) > 4a \). This implies that \( u \in L_M \setminus E_M \). Therefore, by formula (101), we have
\[
\theta(u) = \inf \left\{ \lambda > 0 : \rho_M \left( \frac{u}{\lambda} \right) < \infty \right\} = \text{dist}(u, E_M)
\]
\[
\leq 10a < 1.
\]

Since \( \rho \) is continuous, by formula (102) and Theorem 2.49 of [12], we obtain that \( u \) is a smooth point. Let
\[
E_n = \{ t \in G : |u(t)| \leq n \}
\]
and \( u_n = u|_{E_n} \).

Then, by holder inequality, we have
\[
\nu \left( u|_{E_n} - u(t), v(t) \right) dt \leq \|u|_{E_n} - u\|_{\nu|_{E_n}} \|v|_{E_n} \|_{E_n}
\]
\[
\leq \|v|_{E_n} \|_{E_n} \rightarrow 0.
\]

This implies that \( (u|_{E_n} - u, v) \rightarrow 1 \) as \( n \rightarrow \infty \). Moreover, by the Hahn-Banach theorem, there exists \( \phi \in (L^0_M)^* \) such that \( \phi(v) > 0 \) and \( \phi(E_M') = 0 \). Hence
\[
\phi(u|_{E_n} - u) = \phi(u|_{E_n}) - \phi(u) = \phi(u) < 0.
\]
Since \( u \) is a smooth point and \( (u, v) = 1 \), by formulas (104) and (105), we obtain that \( u \) is not a weakly exposed point of \( B(L_M^0) \). Since \( q \) is continuous, we obtain that \( M \) is strictly convex. Therefore, by Theorem 2.4 of [12], we obtain that \( L_M^0 \) is strictly convex. This implies that \( u \) is a exposed point of \( B(L_M^0) \).

**Example 42.** Let \( X = E_N^0 \) and \( M \in \Delta_2 \) and \( N \notin \Delta_2 \). Then \( X^* = L_M \). Since \( M \in \Delta_2 \), we obtain that \( L_M \) is separable. Therefore, by Theorem 19 and \( (E_N^0)^* = L_M \), we obtain that every bounded subset of \( E_N^0 \) is weak dentable. Moreover, by Theorem 40, we obtain that \( E_N \) has not the Radon-Nikodym property. Hence there exists a bounded subset \( D \) of \( E_N^0 \) such that \( D \) is not dentable.

**Data Availability**

The data used to support the findings of this study are available from the corresponding author upon request.

**Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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