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Research Article

Approximation Properties of Durrmeyer Type of (p,q)-Bleimann, Butzer, and Hahn Operators

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In this study, we introduce a Durrmeyer type of Bleimann, Butzer, and Hahn operators (BBH) on (p,q)-integers. We derive the some approximation properties for these operators. We also give some graphs and numerical examples to illustrate the convergence properties of these operators to certain functions.

1. Introduction

In the literature, especially q-calculus in approximation theory is an important area and there are some applications of q-calculus in this theory (see [1–4]). Therefore, the researchers in the area of approximation theory put a particular importance to q-calculus. After that, the results in approximation theory related to q-calculus have been extended to the results in (p,q)-calculus. The researchers have presented the (p,q)-extensions of linear positive operators (see [5, 6]) and they have also begun to use these operators for applying in many different areas of science (see [7–9]).

We first give some notations related to (p,q)-calculus as in [10–13]. For $0<q<p\le 1$ and each natural number n, the (p,q)-integer of n denoted by $[n]_{p,q}$, (p,q)-factorial denoted by $[n]_{p,q}!$ was defined by

$$[n]_{p,q}=p^{n-1}+p^{n-2}q+p^{n-3}q^2+\cdots+pq^{n-2}+q^{n-1}=\begin{cases}\dfrac{p^n-q^n}{p-q}, & p\neq q\neq 1;\\ np^{n-1}, & p=q\neq 1;\\ [n]_q, & p=1;\\ n, & p=q=1,\end{cases}\tag{1}$$

and

$$[n]_{p,q}!=\begin{cases}[n]_{p,q}[n-1]_{p,q}\cdots[1]_{p,q}, & n\ge 1\\ 1, & n=0,\end{cases}\tag{2}$$

respectively. Also, the binomial representation was given by

$$\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{p,q}p^{\binom{n-k}{2}}q^{\binom{k}{2}}x^k=(1\oplus x)_{p,q}^n,\tag{3}$$

where

$$\begin{bmatrix}n\\k\end{bmatrix}_{p,q}=\frac{[n]_{p,q}!}{[n-k]_{p,q}!\,[k]_{p,q}!}\tag{4}$$

and

$$(1\oplus x)_{p,q}^n=(1+x)(p+qx)\cdots\left(p^{n-1}+q^{n-1}x\right).\tag{5}$$

In [1], q-beta function was defined by

$$B_q(r,s)=K(A,r)\int_0^{\infty/A}\frac{t^{r-1}}{(1+t)_q^{r+s}}d_qt,\tag{6}$$

where

$$K(A,r)=\frac{A^r}{1+A}\left(1+\frac{1}{A}\right)_q^r(1+A)_q^{1-r},\quad A>0.\tag{7}$$

We have $K(A,r) = q^{r(r-1)/2}$ for the special case of $r \in \mathbb{N}$ and $K(A,0) = 1$. Through formula (6), in [14], the authors define the (p,q)-beta function $B_{p,q}(k,n)$ which is a generalization of $B_q(k,n)$ as follows:

$$B_{p,q}(k,n) = p^{\binom{n}{2}} q^{\binom{k}{2}} \int_0^{\infty/A} \frac{t^{k-1}}{(1 \oplus t)_{p,q}^{n+k}} d_{p,q}t, \quad A > 0,\ n,k \in \mathbb{N} \setminus \{0\}. \tag{8}$$

Now we note that the definitions of (p,q)-integral are as follows:

$$\begin{aligned} \int_0^{\infty} f(t)\, d_{p,q}t &= (q-p) \sum_{k=-\infty}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}\right); \quad \left|\frac{p}{q}\right| < 1, \\ \int_0^{\infty} f(t)\, d_{p,q}t &= (p-q) \sum_{k=-\infty}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}\right); \quad \left|\frac{q}{p}\right| < 1 \end{aligned} \tag{9}$$

for an arbitrary function f in [15]. The BBH operators have been studied in [16–24] in the approximation theory. For this reason, we define a new (p,q)-Durrmeyer generalization of the BBH operators and obtain the approximation theorems involving these operators.

2. Durrmeyer Type of (p,q)-BBH Operators

There are many papers mentioned about Durrmeyer type positive linear operators, such as [25–28] and so on. The Durrmeyer type of (p,q)-BBH operators can be introduced via equations (1), (2), (3), (4), (5), and (8) as follows:

$$\begin{aligned} D_{n,p,q}(f,x) = {} & [n+1]_{p,q} \sum_{k=1}^{n} b_{n,k}(p,q,x) \\ & \cdot \int_0^{\infty/A} \gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right) f(t)\, d_{p,q}t \\ & + f(0)\, p^{\binom{n}{2}} \left((1 \oplus x)_{p,q}^{n}\right)^{-1}, \end{aligned} \tag{10}$$

where

$$b_{n,k}(p,q,x) = \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} \frac{x^k}{(1 \oplus x)_{p,q}^{n}} \tag{11}$$

and

$$\begin{aligned} & \gamma_{n,k-1}(p,q,x) \\ & \quad = \frac{p^{(n-k)^2+n-k} q^{k^2-k+1}}{(p^n + q^n x)(p^{n+1} + q^{n+1} x)} b_{n,k-1}(p,q,x). \end{aligned} \tag{12}$$

Let us give the following lemma to evaluate the values of the test functions.

Lemma 1. *If $e_0(t) = 1$, $e_1(t) = t/(1+t)$, and $e_m(t) = t^m/(1+t)(p^{n+3}+q^{n+3}t)\cdots(p^{n+1+m}+q^{n+1+m}t)$ for integer m with $m \geq 2$, then, we get the following result:*

$$\int_0^{\infty/A} \gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right) e_m(t)\, d_{p,q}t = \begin{cases} \dfrac{p^{\binom{n-k}{2}} q^{\binom{k}{2}}}{[n+1]_{p,q}}, & m = 0 \\ \dfrac{(q/p)^{k-1} p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-(2m+1)k-m^2+m+2)/2} [k+m-1]_{p,q}! [n]_{p,q}!}{[k-1]_{p,q}! [n+m+1]_{p,q}!}, & m \geq 1. \end{cases} \tag{13}$$

Proof. Firstly, let us give the proof of $m = 0$. Writing

$$\begin{aligned} & \gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right) \\ & \quad = \frac{p^{(n-k)^2+n-k} q^{k^2-k+1}}{(p^n + q^n (q/p)t)(p^{n+1} + q^{n+1}(q/p)t)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ & \quad \cdot \frac{((q/p)t)^{k-1}}{(1 \oplus (q/p)t)_{p,q}^{n}} \end{aligned} \tag{14}$$

instead of $\gamma_{n,k-1}(p,q,(q/p)t)$, we have

$$\begin{aligned} & \int_0^{\infty/A} \gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right) d_{p,q}t \\ & \quad = p^{(n-k)^2+n-k} q^{k^2-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ & \quad \cdot \int_0^{\infty/A} \frac{1}{(p^n + q^n (q/p)t)(p^{n+1} + q^{n+1}(q/p)t)} \\ & \quad \cdot \frac{((q/p)t)^{k-1}}{(1 \oplus (q/p)t)_{p,q}^{n}} d_{p,q}t. \end{aligned} \tag{15}$$

Substitute $u = (q/p)t$, and use $d_{p,q}u = (q/p)d_{p,q}t$ and

$$\left(p^n + q^n u\right)\left(p^{n+1} + q^{n+1}u\right)(1 \oplus u)_{p,q}^{n} = (1 \oplus u)_{p,q}^{n+2} \tag{16}$$

in the following integral; we get the result

$$\begin{aligned}\int_0^{\infty/A} \gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) d_{p,q}t &= p^{(n-k)^2+n-k+1} q^{k^2-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &\cdot \int_0^{\infty/((p/q)A)} \frac{u^{k-1}}{(1 \oplus u)_{p,q}^{n+2}} d_{p,q}u.\end{aligned} \tag{17}$$

And then by (8) and the equation

$$B_{p,q}(k, n+2-k) = \frac{[k-1]_{p,q}!\,[n-k+1]_{p,q}!}{[n+1]_{p,q}!}, \tag{18}$$

respectively, we get the proof of $m = 0$

$$\begin{aligned}\int_0^{\infty/A} \gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) d_{p,q}t &= p^{(n-k)^2+n-k+1} q^{k^2-k} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \frac{B_{p,q}(k, n+2-k)}{p^{\binom{n+2-k}{2}} q^{\binom{k}{2}}} \\ &= \frac{p^{\binom{n-k}{2}} q^{\binom{k}{2}}}{[n+1]_{p,q}}.\end{aligned} \tag{19}$$

Secondly, let us give the proof of $m = 1$. Writing

$$\begin{aligned}\gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) &= \frac{p^{(n-k)^2+n-k} q^{k^2-k+1}}{\left(p^n + q^n (q/p)t\right)\left(p^{n+1} + q^{n+1}(q/p)t\right)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &\cdot \frac{\left((q/p)t\right)^{k-1}}{(1 \oplus (q/p)t)_{p,q}^{n}}\end{aligned} \tag{20}$$

instead of $\gamma_{n,k-1}(p, q, (q/p)t)$, we have

$$\begin{aligned}\int_0^{\infty/A} \gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) \frac{t}{1+t} d_{p,q}t &= p^{(n-k)^2+n-k} q^{k^2-k+1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &\cdot \int_0^{\infty/A} \frac{1}{\left(p^n + q^n (q/p)t\right)\left(p^{n+1} + q^{n+1}(q/p)t\right)} \frac{\left((q/p)t\right)^{k-1}}{(1 \oplus (q/p)t)_{p,q}^{n}} \frac{t}{1+t} d_{p,q}t.\end{aligned} \tag{21}$$

Now using

$$\begin{aligned}&(1+t)\left(1 \oplus \frac{q}{p}t\right)_{p,q}^{n}\left(p^n + q^n \frac{q}{p}t\right)\left(p^{n+1} + q^{n+1}\frac{q}{p}t\right) \\ &= \frac{(1 \oplus t)_{p,q}^{n+3}}{p^{n+2}},\end{aligned} \tag{22}$$

we get

$$\begin{aligned}\int_0^{\infty/A} \gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) \frac{t}{1+t} d_{p,q}t &= p^{(n-k)^2+n-k} q^{k^2-k+1} p^{n+2} \left(\frac{q}{p}\right)^{k-1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &\cdot \int_0^{\infty/A} \frac{t^k}{(1 \oplus t)_{p,q}^{n+3}} d_{p,q}t.\end{aligned} \tag{23}$$

And then by (8) and the equation

$$B_{p,q}(k+1, n+2-k) = \frac{[k]_{p,q}!\,[n-k+1]_{p,q}!}{[n+2]_{p,q}!}, \tag{24}$$

respectively, we get the proof of $m = 1$:

$$\begin{aligned}\int_0^{\infty/A} \gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) \frac{t}{1+t} d_{p,q}t &= p^{(n-k)^2+n-k} q^{k^2-k+1} p^{n+2} \left(\frac{q}{p}\right)^{k-1} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &\cdot \frac{B_{p,q}(k+1, n+2-k)}{p^{\binom{n+2-k}{2}} q^{\binom{k+1}{2}}} \\ &= \frac{(q/p)^{k-1} p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-3k+2)/2} [k]_{p,q}!\,[n]_{p,q}!}{[k-1]_{p,q}!\,[n+2]_{p,q}!}.\end{aligned} \tag{25}$$

Lastly, let us give the proof of $m > 1$. Writing

$$\begin{aligned}\gamma_{n,k-1}\left(p, q, \frac{q}{p}t\right) &= \frac{p^{(n-k)^2+n-k} q^{k^2-k+1}}{\left(p^n + q^n (q/p)t\right)\left(p^{n+1} + q^{n+1}(q/p)t\right)} \begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q} \\ &\cdot \frac{\left((q/p)t\right)^{k-1}}{(1 \oplus (q/p)t)_{p,q}^{n}}\end{aligned} \tag{26}$$

instead of $\gamma_{n,k-1}(p,q,(q/p)t)$, we have

$$\int_0^{\infty/A} \gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right)\frac{t^m}{(1+t)\left(p^{n+3}+q^{n+3}t\right)\cdots\left(p^{n+1+m}+q^{n+1+m}t\right)}d_{p,q}t = p^{(n-k)^2+n-k}q^{k^2-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}$$
$$\times\int_0^{\infty/A}\frac{1}{\left(p^n+q^n(q/p)t\right)\left(p^{n+1}+q^{n+1}(q/p)t\right)}\frac{((q/p)t)^{k-1}}{(1\oplus(q/p)t)_{p,q}^n}\frac{t^m}{(1+t)\left(p^{n+3}+q^{n+3}t\right)\cdots\left(p^{n+1+m}+q^{n+1+m}t\right)}d_{p,q}t. \tag{27}$$

Now using

$$(1+t)\left(1\oplus\frac{q}{p}t\right)_{p,q}^n\left(p^n+q^n\frac{q}{p}t\right)\left(p^{n+1}+q^{n+1}\frac{q}{p}t\right)$$
$$\cdot\left(p^{n+3}+q^{n+3}t\right)\cdots\left(p^{n+1+m}+q^{n+1+m}t\right) \tag{28}$$
$$=\frac{(1\oplus t)_{p,q}^{n+m+2}}{p^{n+2}},$$

in the following integral, we get the result

$$\int_0^{\infty/A}\gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right)$$
$$\cdot\frac{t^m}{(1+t)\left(p^{n+3}+q^{n+3}t\right)\cdots\left(p^{n+1+m}+q^{n+1+m}t\right)}d_{p,q}t$$
$$=p^{(n-k)^2+n-k}q^{k^2-k+1}p^{n+2}\left(\frac{q}{p}\right)^{k-1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}$$
$$\cdot\int_0^{\infty/A}\frac{t^{k+m-1}}{(1\oplus t)_{p,q}^{n+m+2}}d_{p,q}t \tag{29}$$

And then by (8) and the equation

$$B_{p,q}(k+m,n+2-k)$$
$$=\frac{[k+m-1]_{p,q}!\,[n-k+1]_{p,q}!}{[n+m+1]_{p,q}!}, \tag{30}$$

respectively, we get the proof of $m>1$:

$$\int_0^{\infty/A}\gamma_{n,k-1}\left(p,q,\frac{q}{p}t\right)\frac{t^m}{(1+t)\left(p^{n+3}+q^{n+3}t\right)\cdots\left(p^{n+1+m}+q^{n+1+m}t\right)}d_{p,q}t$$
$$=p^{(n-k)^2+n-k}q^{k^2-k+1}p^{n+2}\left(\frac{q}{p}\right)^{k-1}\begin{bmatrix} n \\ k-1 \end{bmatrix}_{p,q}\frac{B_{p,q}(k+m,n+2-k)}{p^{\binom{n+2-k}{2}}q^{\binom{k+m}{2}}} \tag{31}$$
$$=\frac{(q/p)^{k-1}p^{n+1}p^{\binom{n-k}{2}}q^{(k^2-(2m+1)k-m^2+m+2)/2}[k+m-1]_{p,q}!\,[n]_{p,q}!}{[k-1]_{p,q}!\,[n+m+1]_{p,q}!}.$$

□

Lemma 2. *The following results for the Durrmeyer type of (p,q)-generalization of the BBH operators are verified:*

(i) $D_{n,p,q}(e_0,x)=1$,

(ii) $D_{n,p,q}(e_1,x)=(p^2[n]_{p,q}/[n+2]_{p,q})(x/(1+x))$,

(iii) $D_{n,p,q}(e_2,x)=(p^2[n-1]_{p,q}[n]_{p,q}/[n+2]_{p,q}[n+3]_{p,q})(x^2/(1+x)(p^{n-1}+q^{n-1}x))+(p^{n+1}q^{-2}[2]_{p,q}[n]_{p,q}/[n+2]_{p,q}[n+3]_{p,q})(x/(p^{n-1}+q^{n-1}x))$.

Proof. (i) Firstly, let us give the proof of (i). Using (10), (3), and Lemma 1, we get the following result:

$$D_{n,p,q}(e_0,x)=\sum_{k=1}^n p^{\binom{n-k}{2}}q^{\binom{k}{2}}b_{n,k}(p,q,x)$$
$$+p^{\binom{n}{2}}\left((1\oplus x)_{p,q}^n\right)^{-1} \tag{32}$$
$$=\sum_{k=0}^n\begin{bmatrix} n \\ k \end{bmatrix}_{p,q}p^{\binom{n-k}{2}}q^{\binom{k}{2}}\frac{x^k}{(1\oplus x)_{p,q}^n}=1.$$

(ii) Secondly, let us prove (ii). From (10) and Lemma 1, we have

$$\begin{aligned}
D_{n,p,q}(e_1,x) &= \sum_{k=1}^{n} \frac{[k]_{p,q}}{[n+2]_{p,q}} \left(\frac{q}{p}\right)^{k-1} \\
&\quad \cdot p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-3k+2)/2} b_{n,k}(p,q,x) \\
&= \sum_{k=1}^{n} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_{p,q} \\
&\quad \cdot \frac{(q/p)^{k-1} p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-3k+2)/2} [n]_{p,q}}{[n+2]_{p,q}} \frac{x^k}{(1\oplus x)_{p,q}^{n}} \\
&= \frac{[n]_{p,q}}{[n+2]_{p,q}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \\
&\quad \cdot \frac{p^{n+1} p^{\binom{n-1-k}{2}} q^{((k+1)^2-3(k+1)+2)/2} (q/p)^k x^{k+1}}{(1+x)(p+qx)\cdots(p^{n-1}+q^{n-1}x)} \\
&= \frac{[n]_{p,q}}{[n+2]_{p,q}} \frac{x}{1+x} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \\
&\quad \cdot \frac{p^{n+1} p^{\binom{n-1-k}{2}} q^{((k+1)^2-3(k+1)+2)/2} ((q/p)x)^k}{(p+qx)\cdots(p^{n-1}+q^{n-1}x)} \\
&= \frac{p^2 [n]_{p,q}}{[n+2]_{p,q}} \frac{x}{1+x} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix} \\
&\quad \cdot \frac{p^{\binom{n-1-k}{2}} q^{\binom{k}{2}} ((q/p)x)^k}{(1\oplus (q/p)x)_{p,q}^{n-1}}.
\end{aligned} \tag{33}$$

Thus from (3), we have the desired relation in (ii).

(iii) Lastly, let us prove (iii). We use (10) and Lemma 1 to get

$$\begin{aligned}
D_{n,p,q}(e_2,x) &= \sum_{k=1}^{n} b_{n,k}(p,q,x) \\
&\quad \cdot \frac{(q/p)^{k-1} p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-5k)/2} [k]_{p,q} [k+1]_{p,q}}{[n+2]_{p,q} [n+3]_{p,q}}.
\end{aligned} \tag{34}$$

Using the relation $[k+1]_{p,q} = q^2[k-1]_{p,q} + p^{k-1}[2]_{p,q}$ in the above equation, we reach

$$\begin{aligned}
D_{n,p,q}(e_2,x) &= q^2 \sum_{k=2}^{n} \frac{[n]_{p,q}!}{[k-2]_{p,q}!\,[n-k]_{p,q}!} \frac{(q/p)^{k-1} p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-5k)/2}}{[n+2]_{p,q}[n+3]_{p,q}} \frac{x^k}{(1\oplus x)_{p,q}^{n}} \\
&\quad + [2]_{p,q} \sum_{k=1}^{n} \frac{[n]_{p,q}!}{[k-1]_{p,q}!\,[n-k]_{p,q}!} \frac{p^{k-1}(q/p)^{k-1} p^{n+1} p^{\binom{n-k}{2}} q^{(k^2-5k)/2}}{[n+2]_{p,q}[n+3]_{p,q}} \frac{x^k}{(1\oplus x)_{p,q}^{n}} \\
&= \frac{q^2 [n-1]_{p,q}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} \frac{(q/p)^{k+1} p^{n+1} p^{\binom{n-2-k}{2}} q^{((k+2)^2-5(k+2))/2} x^{k+2}}{(1+x)\cdots(p^{n-1}+q^{n-1}x)} \\
&\quad + \frac{[2]_{p,q}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \frac{p^k (q/p)^k p^{n+1} p^{\binom{n-1-k}{2}} q^{((k+1)^2-5(k+1))/2} x^{k+1}}{(1+x)\cdots(p^{n-1}+q^{n-1}x)}.
\end{aligned} \tag{35}$$

Upon necessary arrangments that have been done, we have

$$\begin{aligned}
D_{n,p,q}(e_2,x) &= \frac{q^2 [n-1]_{p,q}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \\
&\quad \cdot \frac{x^2}{(1+x)(p^{n-1}+q^{n-1}x)} \times \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} \\
&\quad \cdot \frac{(q/p)^{k+1} p^{n+1} p^{\binom{n-2-k}{2}} q^{((k+2)^2-5(k+2))/2} x^k}{(p+qx)\cdots(p^{n-2}+q^{n-2}x)} \\
&\quad + \frac{[2]_{p,q}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \frac{x}{p^{n-1}+q^{n-1}x} \\
&\quad \times \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \\
&\quad \cdot \frac{p^k (q/p)^k p^{n+1} p^{\binom{n-1-k}{2}} q^{((k+1)^2-5(k+1))/2} x^k}{(1+x)(p+qx)\cdots(p^{n-2}+q^{n-2}x)}.
\end{aligned} \tag{36}$$

Finally, using (3), we derive the desired relation:

$$\begin{aligned}
&D_{n,p,q}(e_2,x) \\
&\quad = \frac{p^2 [n-1]_{p,q}[n]_{p,q}}{[n+2]_{p,q}[n+3]_{p,q}} \frac{x^2}{(1+x)(p^{n-1}+q^{n-1}x)}
\end{aligned}$$

$$\times \sum_{k=0}^{n-2} \begin{bmatrix} n-2 \\ k \end{bmatrix}_{p,q} \frac{p^{\binom{n-2-k}{2}} q^{\binom{k}{2}} ((q/p)x)^k}{(1 \oplus (q/p)x)_{p,q}^{n-2}}$$
$$+ \frac{p^{n+1} q^{-2} [2]_{p,q} [n]_{p,q}}{[n+2]_{p,q} [n+3]_{p,q}} \frac{x}{p^{n-1} + q^{n-1} x}$$
$$\times \sum_{k=0}^{n-1} \begin{bmatrix} n-1 \\ k \end{bmatrix}_{p,q} \frac{p^{\binom{n-1-k}{2}} q^{\binom{k}{2}} x^k}{(1 \oplus x)_{p,q}^{n-1}}. \tag{37}$$

which completes the proof. □

Remark 3. The following result is valid when $0 < q_n < p_n \leq 1$, $p_n \longrightarrow 1$, $q_n \longrightarrow 1$, $p_n^n \longrightarrow 1$, and $q_n^n \longrightarrow 1$ as $n \longrightarrow \infty$:

$$\begin{aligned} & D_{n,p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2, x\right) \\ &= D_{n,p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2, x\right) - \frac{2x}{1+x} D_{n,p_n,q_n}(e_1, x) \\ &\quad + \left(\frac{x}{1+x}\right)^2 D_{n,p_n,q_n}(e_0, x) \\ &\leq D_{n,p_n,q_n}(e_2, x) - \frac{2x}{1+x} D_{n,p_n,q_n}(e_1, x) \\ &\quad + \left(\frac{x}{1+x}\right)^2 D_{n,p_n,q_n}(e_0, x) \\ &= \frac{p_n^2 [n-1]_{p_n,q_n} [n]_{p_n,q_n}}{[n+2]_{p_n,q_n} [n+3]_{p_n,q_n}} \frac{x^2}{(1+x)(p_n^{n-1} + q_n^{n-1} x)} \\ &\quad - 2\frac{p_n^2 [n]_{p_n,q_n}}{[n+2]_{p_n,q_n}} \left(\frac{x}{1+x}\right)^2 + \left(\frac{x}{1+x}\right)^2 \\ &\quad + \frac{p_n^{n+1} q_n^{-2} [2]_{p_n,q_n} [n]_{p_n,q_n}}{[n+2]_{p_n,q_n} [n+3]_{p_n,q_n}} \frac{x}{p_n^{n-1} + q_n^{n-1} x} \longrightarrow 0. \end{aligned} \tag{38}$$

3. Genuine Type of (p,q)-Durrmeyer BBH Operators

Now, we define the genuine type of Durrmeyer BBH operators denoted by $D_{n,p,q}^*$ for $0 < q < p \leq 1$ as follows:

$$D_{n,p,q}^*(f, x) := D_{n,p,q}(f, \sigma_n(p, q, x)), \tag{39}$$

where

$$\sigma_n(p, q, x) := \frac{x}{p^2 [n]_{p,q} / [n+2]_{p,q} + \left(p^2 [n]_{p,q} / [n+2]_{p,q} - 1\right) x} \tag{40}$$

for $0 \leq x < \alpha_n/(1-\alpha_n)$, $\alpha_n = p^2 [n]_{p,q}/[n+2]_{p,q}$.

We first give the following lemma for the test functions.

Lemma 4. *For $x \in [0, \alpha_n/(1-\alpha_n))$, $0 < q < p \leq 1$, and $e_j(t) = (t/(1+t))^j$, where $j = 0, 1, 2$, the operators $D_{n,p,q}^*(e_j, x)$ satisfy the following properties:*

(i) $D_{n,p,q}^*(1, x) = 1$,

(ii) $D_{n,p,q}^*(t/(1+t), x) = x/(1+x)$,

(iii) $D_{n,p,q}^*((t/(1+t))^2, x) \leq ([n-1]_{p,q}/q^{n+1}[n]_{p,q})(x/(1+x))^2 + ([2]_{p,q}/q^{n+3}[n]_{p,q})(x/(1+x))$.

Lemma 5. *The operators $D_{n,p,q}^*$ satisfy the following inequality for $x \in [0, \alpha_n/(1-\alpha_n))$:*

$$\begin{aligned} 0 &\leq D_{n,p,q}^*\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2, x\right) \\ &\leq \left(\frac{[n-1]_{p,q}}{q^{n+1}[n]_{p,q}} - 1\right)\left(\frac{x}{1+x}\right)^2 + \frac{[2]_{p,q}}{q^{n+3}[n]_{p,q}} \frac{x}{1+x} \\ &\leq \left(\frac{[n-1]_{p,q}}{q^{n+1}[n]_{p,q}} + \frac{[2]_{p,q}}{q^{n+3}[n]_{p,q}} - 1\right) \\ &\quad \cdot \left(\left(\frac{x}{1+x}\right)^2 + \frac{x}{1+x}\right) \\ &\leq 2\left(\frac{[n-1]_{p,q}}{q^{n+1}[n]_{p,q}} + \frac{[2]_{p,q}}{q^{n+3}[n]_{p,q}} - 1\right). \end{aligned} \tag{41}$$

4. Approximation Properties of the Operators $D_{n,p,q}^*$

In this section, we will consider the space of all bounded and continuous functions on the interval $[0, \infty)$ denoted by $C_B[0, \infty)$ and use the supremum norm $\|f\|_{C_B} = \sup|f(t)|$ for $f \in C_B[0, \infty)$. Let H_ω be the space of all-real valued functions defined on $[0, \infty)$ satisfying the following condition:

$$|f(t) - f(x)| < \omega\left(f, \left|\frac{t}{1+t} - \frac{x}{1+x}\right|\right), \tag{42}$$

where any $t, x \in [0, \infty)$ and ω is the modulus of continuity in [22] satisfying the following conditions:

(i) ω is a nonnegative increasing function on $[0, \infty)$,

(ii) $\omega(f, \delta_1 + \delta_2) \leq \omega(f, \delta_1) + \omega(f, \delta_2)$,

(iii) $\lim_{\delta \longrightarrow 0} \omega(f, \delta) = 0$,

(iv) $\omega(f, \lambda\delta) \leq (1+\lambda)\omega(f, \delta)$.

Readers might see [29] for details. Now, we remember the Korovkin theorem for the linear positive operators acting from H_ω into $C_B[0, \infty)$ in [22].

Theorem 6 (see [22]). *If the linear positive operators L_n acting from H_ω into $C_B[0, \infty)$ satisfy the conditions,*

$$\left\| L_n\left(\left(\frac{t}{1+t}\right)^\gamma, x\right) - \left(\frac{x}{1+x}\right)^\gamma \right\|_{C_B} \longrightarrow 0, \quad n \longrightarrow \infty, \tag{43}$$

for $\gamma = 0, 1, 2$, then we get

$$\|L_n(f,x) - f\|_{C_B} \longrightarrow 0, \quad n \longrightarrow \infty \tag{44}$$

for any function $f \in H_\omega$.

Theorem 7. *Assume that (p_n) and (q_n) are two sequences with $0 < q_n < p_n \le 1$, $p_n \longrightarrow 1$, $q_n \longrightarrow 1$, $p_n^n \longrightarrow 1$, and $q_n^n \longrightarrow 1$ as $n \longrightarrow \infty$. Then the assertion*

$$\lim_{n\longrightarrow\infty} \sup_{x\in[0,\alpha_n/(1-\alpha_n))} \left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| = 0 \tag{45}$$

holds for any function $f \in H_\omega$.

Proof. Considering the operators,

$$\widetilde{D}_{n,p_n,q_n}(f,x) = \begin{cases} D^*_{n,p_n,q_n}(f,x); & \text{if } 0 \le x < \dfrac{\alpha_n}{1-\alpha_n} \\ f(x); & \text{if } x \ge \dfrac{\alpha_n}{1-\alpha_n}, \end{cases} \tag{46}$$

we have

$$\left\|\widetilde{D}_{n,p_n,q_n} f - f\right\|_{C_B} = \sup_{x\in[0,\alpha_n/(1-\alpha_n))} \left|D^*_{n,p_n,q_n}(f,x) - f(x)\right|. \tag{47}$$

It is clear that $\|\widetilde{D}_{n,p_n,q_n}(1,x) - 1\|_{C_B} = 0$ and $\|\widetilde{D}_{n,p_n,q_n}(t/(1+t),x) - x/(1+x)\|_{C_B} = 0$. Using Lemma 5, we have the following inequality:

$$\begin{aligned} &\left\|\widetilde{D}_{n,p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2, x\right) - \left(\frac{x}{1+x}\right)^2\right\|_{C_B} \\ &= \sup_{x\in[0,\alpha_n/(1-\alpha_n))} \left|D^*_{n,p_n,q_n}\left(\left(\frac{t}{1+t}\right)^2, x\right) - \left(\frac{x}{1+x}\right)^2\right| \\ &= \sup_{x\in[0,\alpha_n/(1-\alpha_n))} \left|D^*_{n,p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2, x\right)\right| \\ &\le 2\left(\frac{[n-1]_{p_n,q_n}}{q_n^{n+1}[n]_{p_n,q_n}} + \frac{[2]_{p_n,q_n}}{q_n^{n+3}[n]_{p_n,q_n}} - 1\right). \end{aligned} \tag{48}$$

Thus by (47), we get

$$\lim_{n\longrightarrow\infty} \left\|\widetilde{D}_{n,p_n,q_n}\left(\left(\frac{t}{1+t}\right)^\gamma, x\right) - \left(\frac{x}{1+x}\right)^\gamma\right\|_{C_B} = 0, \quad \gamma = 0, 1, 2. \tag{49}$$

Applying Theorem 6 for the operators $\widetilde{D}_{n,p_n,q_n}$, we have

$$\lim_{n\longrightarrow\infty} \left\|\widetilde{D}_{n,p_n,q_n}(f,x) - f(x)\right\|_{C_B} = 0 \tag{50}$$

for any function $f \in H_\omega$. Hence using (47), we obtain

$$\lim_{n\longrightarrow\infty} \sup_{x\in[0,\alpha_n/(1-\alpha_n))} \left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| = 0. \tag{51}$$

Thus, we finish the proof of Theorem 7. □

Theorem 8. *Assume that (p_n) and (q_n) are two sequences with $0 < q_n < p_n \le 1$, $p_n \longrightarrow 1$, $q_n \longrightarrow 1$, $p_n^n \longrightarrow 1$, and $q_n^n \longrightarrow 1$ as $n \longrightarrow \infty$. For the operators D^*_{n,p_n,q_n}, we get the following estimation property:*

$$\left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| \le 2\omega(f, \delta(n,p_n,q_n)), \tag{52}$$

where

$$\delta(n,p_n,q_n) := \sqrt{2\left(\frac{[n-1]_{p_n,q_n}}{q_n^{n+1}[n]_{p_n,q_n}} + \frac{[2]_{p_n,q_n}}{q_n^{n+3}[n]_{p_n,q_n}} - 1\right)}, \tag{53}$$

for each $x \in [0, \alpha_n/(1-\alpha_n))$ and $f \in H_\omega$.

Proof. Using $D^*_{n,p_n,q_n}(1,x) = 1$, we can write

$$\left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| \le D^*_{n,p_n,q_n}(|f(t) - f(x)|, x). \tag{54}$$

From the properties of the modulus of continuity, we have

$$\begin{aligned} |f(t) - f(x)| &< \omega\left(f, \left|\frac{t}{1+t} - \frac{x}{1+x}\right|\right) \\ &\le \left(1 + \frac{|t/(1+t) - x/(1+x)|}{\delta}\right)\omega(f,\delta). \end{aligned} \tag{55}$$

From (54) and (55), we get

$$\begin{aligned} &\left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| \\ &\le \omega(f,\delta)\left(1 + \frac{1}{\delta} D^*_{n,p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|, x\right)\right). \end{aligned} \tag{56}$$

Applying Cauchy-Schwarz inequality, we have

$$\begin{aligned} &\left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| \le \omega(f,\delta) \\ &\cdot\left(1 + \frac{1}{\delta}\left(D^*_{n,p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2, x\right)\right)^{1/2}\right). \end{aligned} \tag{57}$$

Choosing $\delta = \delta(n,p_n,q_n)$ as in (53) we obtain the result of Theorem 8. □

From [30] we give $\widetilde{W}_{\alpha,E}$ as

$$\widetilde{W}_{\alpha,E} = \left\{f : \sup(1+x)^\alpha f_\alpha(x,y) \le M\frac{1}{(1+y)^\alpha}, \ x \ge 0, \ y \in E\right\}, \tag{58}$$

the definition of general Lipschitz-type maximal functions on $E \subset [0,\infty)$, where f is a bounded and continuous function on $[0,\infty)$, M is a positive constant, $0 < \alpha \le 1$,

$$f_\alpha(x,y) = \frac{|f(x) - f(y)|}{|x-y|^\alpha}. \tag{59}$$

The distance of x and E is defined by

$$d(x,E) = \inf\{|x-y| : y \in E\}. \tag{60}$$

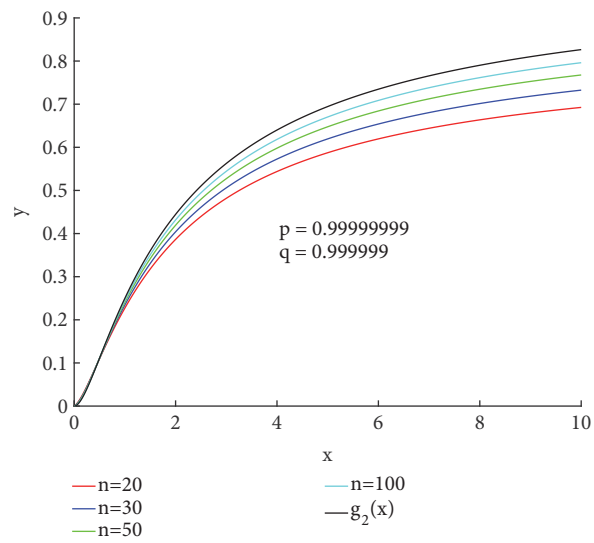

FIGURE 1: The figures of $g_2(x)$ and $D_{n,p,q}(g_2;x)$ with $p = 0.99999999$, $q = 0.999999$ and different values of n.

Theorem 9. *Assume that (p_n) and (q_n) are two sequences satisfying the conditions $0 < q_n < p_n \leq 1$, $p_n \longrightarrow 1$, $q_n \longrightarrow 1$, $p_n^n \longrightarrow 1$, and $q_n^n \longrightarrow 1$ as $n \longrightarrow \infty$. Then for each $f \in \widetilde{W}_{\alpha,E}$, we get the following inequality:*

$$\begin{aligned}&\left|D^*_{n,p_n,q_n}(f,x) - f(x)\right| \\ &\quad \leq M\left(\left(\widetilde{\delta}(n,p_n,q_n)\right)^{\alpha/2} + 2\left(d(x,E)\right)^{\alpha}\right),\end{aligned} \tag{61}$$

for each $x \in [0,\alpha_n/(1-\alpha_n))$ and $\widetilde{\delta}(n,p_n,q_n) = 2([n-1]_{p_n,q_n}/q_n^{n+1}[n]_{p_n,q_n} + [2]_{p_n,q_n}/q_n^{n+3}[n]_{p_n,q_n} - 1)$.

Proof. The closure of the set E is denoted by $\overline{E}$. Then we have $x_0 \in \overline{E}$ such that $|x - x_0| = d(x,x_0)$ for $x \in [0,\alpha_n/(1-\alpha_n))$. Thus we have

$$\left|f(t) - f(x)\right| \leq \left|f(t) - f(x_0)\right| + \left|f(x_0) - f(x)\right|. \tag{62}$$

Using $f \in \widetilde{W}_{\alpha,E}$ and linear and positive operator D^*_{n,p_n,q_n}, we get

$$\begin{aligned}&\left|D^*_{n,p_n,q_n}(f,x) - f\right| \\ &\quad \leq D^*_{n,p_n,q_n}\left(\left|f(t) - f(x_0)\right|,x\right) + \left|f(x_0) - f(x)\right| \\ &\quad \leq MD^*_{n,p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha},x\right) \\ &\qquad + M\frac{\left|x-x_0\right|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}.\end{aligned} \tag{63}$$

Using the well-known inequality $(a+b)^{\alpha} \leq a^{\alpha} + b^{\alpha}$ for $a \geq 0, b \geq 0$, for $0 < \alpha \leq 1$ and $t \in [0,\infty)$, we have

$$\begin{aligned}\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha} &\leq \left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha} \\ &\quad + \left|\frac{x}{1+x} - \frac{x_0}{1+x_0}\right|^{\alpha}.\end{aligned} \tag{64}$$

Thus we can write

$$\begin{aligned}&D^*_{n,p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha},x\right) \\ &\quad \leq D^*_{n,p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x}{1+x}\right|^{\alpha},x\right) \\ &\qquad + \frac{\left|x-x_0\right|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}.\end{aligned} \tag{65}$$

Because $D^*_{n,p_n,q_n}(1,x) = 1$, using Hölder inequality, we get

$$\begin{aligned}&D^*_{n,p_n,q_n}\left(\left|\frac{t}{1+t} - \frac{x_0}{1+x_0}\right|^{\alpha},x\right) \\ &\quad \leq \left(D^*_{n,p_n,q_n}\left(\left(\frac{t}{1+t} - \frac{x}{1+x}\right)^2,x\right)\right)^{\alpha/2} \\ &\qquad + \frac{\left|x-x_0\right|^{\alpha}}{(1+x)^{\alpha}(1+x_0)^{\alpha}}.\end{aligned} \tag{66}$$

Using (63), the proof of Theorem 9 is completed. □

5. Numerical Results

In this section, we will analyze the theoretical results presented in the previous sections by numerical examples.

Example 1. Let $g_2(x) = x^2/(1+x)(p^{n+3} + q^{n+3}x)$, the graphs of $D_{n,p,q}(g_2;x)$ with $p = 0.99999999$, $q = 0.999999$, and $n = 20, 30, 50, 100$ are shown in Figure 1. In Figure 2, fix $n = 100$ and let $p_1 = 0.9999$, $q_1 = 0.999$; the curves of $g_2(x)$ (the black curve, denoted by $g_2(x,p_1,q_1)$ in Figure 2) and $D_{n,p_1,q_1}(g_2;x)$ (the red curve) are shown; let $p_2 = 0.9999$, $q_2 = 0.9999$; the curves of $g_2(x)$ (the purple curve, denoted by $g_2(x,p_2,q_2)$ in Figure 2) and $D_{n,p_2,q_2}(g_2;x)$ (the brown curve) are shown; let $p_3 = 0.999999$, $q_3 = 0.99999$; the

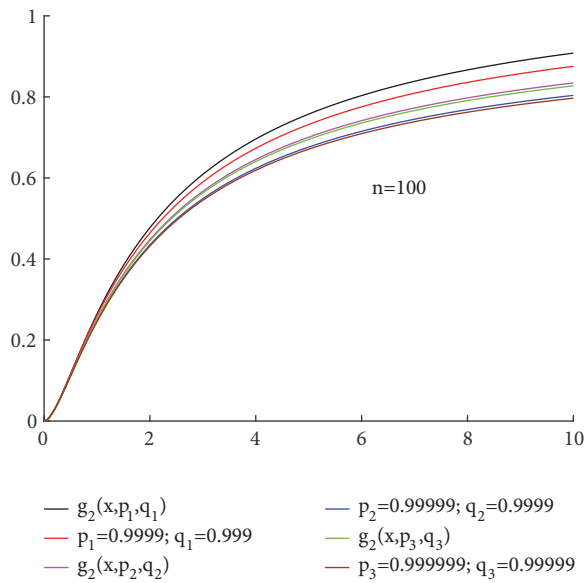

FIGURE 2: The figures of $g_2(x)$ and $D_{n,p_i,q_i}(g_2;x)$ with different values of p_i and q_i, i=1,2,3.

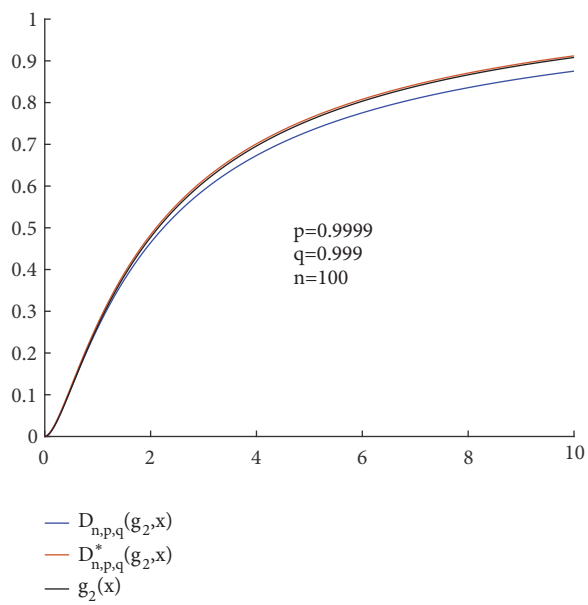

FIGURE 3: The figures of $g_2(x)$, $D_{n,p,q}(g_2;x)$, and $D^*_{n,p,q}(g_2;x)$ with $p = 0.9999$, $q = 0.999$ and $n = 100$.

curves of $g_2(x)$ (the green curve, denoted by $g_2(x, p_3, q_3)$ in Figure 2) and $D_{n,p_3,q_3}(g_2;x)$ (the brown curve) are shown. In Figure 3, the graphs of $D_{n,p,q}(g_2;x)$ and $D^*_{n,p,q}(g_2;x)$ with $p = 0.9999$, $q = 0.999$, and $n = 100$ are shown. Tables 1 and 2 are the absolute error bound of $D_{n,p,q}(g_2;x)$ to $g_2(x)$ and $D^*_{n,p,q}(g_2;x)$ to $g_2(x)$ with different values of p and q. One can see from Tables 1 and 2, the latter operators $D^*_{n,p,q}(g_2;x)$ are better than the former ones $D_{n,p,q}(g_2;x)$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Table 1: The absolute error bound of $D_{n,p,q}(g_2;x)$ to g_2 with $p = 1 - 10^{-m}$ and $q = 1 - 10^{-m+1}$.

m	$\left\|g_2(x) - D_{n,p,q}(g_2;x)\right\|_\infty$				
	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
5	0.0306017	0.01567566	0.00647738	0.00337185	0.00181546
6	0.03038806	0.01546181	0.00626191	0.00315343	0.00159127
7	0.03036675	0.01544054	0.00624065	0.00313215	0.00156992
8	0.03036463	0.01543841	0.00623853	0.00313002	0.0015678
9	0.03036441	0.0154382	0.00623832	0.00312981	0.00156759
10	0.03036439	0.01543818	0.0062383	0.00312979	0.00156757

Table 2: The absolute error bound of $D^*_{n,p,q}(g_2;x)$ to g_2 with $p = 1 - 10^{-m}$ and $q = 1 - 10^{-m+1}$.

m	$\left\|g_2(x) - D^*_{n,p,q}(g_2;x)\right\|_\infty$				
	$n = 100$	$n = 200$	$n = 500$	$n = 1000$	$n = 2000$
5	0.00493872	0.00258649	0.00113532	0.00065426	0.00043724
6	0.00481934	0.00246397	0.00100536	0.00051145	0.00026317
7	0.00480761	0.00245211	0.00099337	0.00049933	0.00025083
8	0.00480644	0.00245093	0.00099281	0.00049814	0.00024963
9	0.00480632	0.00245081	0.00099206	0.00049802	0.00024951
10	0.00480631	0.0024508	0.00099205	0.00049801	0.0002495

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