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Research Article

Small Pre-Quasi Banach Operator Ideals of Type Orlicz-Cesáro Mean Sequence Spaces

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In this paper, we give the sufficient conditions on Orlicz-Cesáro mean sequence spaces ces_φ, where φ is an Orlicz function such that the class S_{ces_φ} of all bounded linear operators between arbitrary Banach spaces with its sequence of $s-$numbers which belong to ces_φ forms an operator ideal. The completeness and denseness of its ideal components are specified and S_{ces_φ} constructs a pre-quasi Banach operator ideal. Some inclusion relations between the pre-quasi operator ideals and the inclusion relations for their duals are explained. Moreover, we have presented the sufficient conditions on ces_φ such that the pre-quasi Banach operator ideal generated by approximation number is small. The above results coincide with that known for ces_p $(1 < p < \infty)$.

1. Introduction

Throughout the paper, by w, we mean the space of all real sequences, $\mathbb{R}$ the real numbers, and $\mathbb{N} = \{0, 1, 2, \ldots\}$ and $\mathfrak{L}(X, Y)$ the space of all bounded linear operators from a normed space X into a normed space Y. The operator ideals theory takes an importance in functional analysis, since it has numerous applications in fixed point theorem, geometry of Banach spaces, spectral theory, eigenvalue distributions theorem, etc. Some of the operator ideals in the class of normed spaces or Banach spaces in functional analysis are characterized by various scalar sequence spaces. For example the ideal of compact operators is defined by kolmogorov numbers and the space c_0 of convergent to zero sequences. Pietsch [1] inspected the operator ideals framed by the approximation numbers and the classical sequence space $\ell^p (0 < p < \infty)$. He proved that the ideals of Hilbert Schmidt operators and nuclear operators between Hilbert spaces are defined by ℓ^2 and ℓ^1, respectively, and the sequence of approximation numbers. In [2], Faried and Bakery examined the operator ideals developed by generalized Cesáro, Orlicz sequence spaces ℓ_M, and the approximation numbers. In [3], Faried and Bakery studied the operator ideals constructed by $s-$ numbers, generalized Cesáro and Orlicz sequence spaces ℓ_M and show that the operator ideal formed by the previous sequence spaces and approximation numbers is small under certain conditions. Also summation process and sequences spaces applications are closely related to Korovkin type approximation theorems and linear positive operators studied by Costarelli and Vinti [4] and Altomare [5]. The idea of this paper is to examine a generalized class S_{ces_φ} by using Orlicz-Cesáro mean sequence spaces ces_φ and the sequence of s-numbers, for which S_{ces_φ} constructs an operator ideal. The components of S_{ces_φ} as a pre-quasi Banach operator ideal containing finite dimensional operators as a dense subset and its completeness are proved. The inclusion relations between the pre-quasi operator ideals and the inclusion relations for their duals are determined. Finally, we show that the pre-quasi Banach operator ideal formed by the approximation numbers and ces_φ is small under certain conditions. These results coincide with that known for ces_p, $(1 < p < \infty)$ in [3]. Furthermore we give some examples which support our main results.

2. Definitions and Preliminaries

Definition 1 (see [6]). The sequence $(s_n(T))_{n=0}^{\infty}$, for all $T \in \mathfrak{L}(X,Y)$ is named an s-function and the number $s_n(T)$ is called the n^{th} s- number of T if the following are satisfied:

(a) monotonicity: $\|T\| = s_0(T) \geq s_1(T) \geq s_2(T) \geq \cdots \geq 0$ for all $T \in \mathfrak{L}(X,Y)$;

(b) additivity: $s_{m+n-1}(T_1 + T_2) \leq s_m(T_1) + s_n(T_2)$ for all $T_1, T_2 \in \mathfrak{L}(X,Y)$, $m, n \in \mathbb{N}$;

(c) property of ideal: $s_n(RTP) \leq \|R\| s_n(T) \|P\|$ for all $P \in \mathfrak{L}(X_0, X)$, $T \in \mathfrak{L}(X,Y)$, and $R \in \mathfrak{L}(Y, Y_0)$, where X_0 and Y_0 are normed spaces;

(d) $s_n(\beta T) = |\beta| s_n(T)$ for every $T \in \mathfrak{L}(X,Y)$, $\beta \in \mathbb{R}$;

(e) rank property: if $\operatorname{rank}(T) \leq n$ then $s_n(T) = 0$ for every $T \in \mathfrak{L}(X,Y)$;

(f) property of norming:

$$s_i\left(I_j\right) = \begin{cases} 1, & \text{if } i < j; \\ 0, & \text{if } i \geq j, \end{cases} \tag{1}$$

where I_j is the identity operator on $\mathbb{R}^j$.

There are a few instances of s-numbers; we notice the accompanying conditions:

(1) The n-th approximation number, denoted by $\alpha_n(T)$, is defined by $\alpha_n(T) = \inf\{\|T - B\| : B \in \mathfrak{L}(X,Y) \text{ and } \operatorname{rank}(B) \leq n\}$.

(2) The n-th Hilbert number, denoted by $h_n(T)$, is defined by

$$\begin{aligned} h_n(T) = \sup\{\alpha_n(ATB) : \|A : Y \longrightarrow \ell_2\| \\ \leq 1 \text{ and } \|B : \ell_2 \longrightarrow X\| \leq 1\}. \end{aligned} \tag{2}$$

(3) The n-th Weyl number, denoted by $x_n(T)$, is defined by

$$x_n(T) = \inf\{\alpha_n(TB) : \|B : \ell_2 \longrightarrow X\| \leq 1\}. \tag{3}$$

(4) The n-th Kolmogorov number, denoted by $d_n(T)$, is defined by

$$d_n(T) = \inf_{\dim Y \leq n} \sup_{\|x\| \leq 1} \inf_{y \in Y} \|Tx - y\|. \tag{4}$$

(5) The n-th Gel'fand number, denoted by $c_n(T)$, is defined by $c_n(T) = \alpha_n(J_Y T)$, where J_Y is a metric injection from the space Y to a higher space $l_\infty(\Psi)$ for an adequate index set Ψ. This number is independent of the choice of the higher space $l_\infty(\Psi)$.

(6) The n-th Chang number, denoted by $y_n(T)$, is defined by

$$y_n(T) = \inf\{\alpha_n(AT) : \|A : Y \longrightarrow \ell_2\| \leq 1\}. \tag{5}$$

Remark 2 (see [6]). Among all the s-number sequences characterized above, it is easy to check that the approximation number, $\alpha_n(T)$, is the largest and the Hilbert number, $h_n(T)$, is the smallest s-number sequence, i.e., $h_n(T) \leq s_n(T) \leq \alpha_n(T)$ for any bounded linear operator T. If T is defined on a Hilbert space and compact, then all the s-numbers correspond with the eigenvalues of $|T|$, where $|T| = (T^*T)^{1/2}$.

Theorem 3 ([6], p.115). *Let $T \in \mathfrak{L}(X,Y)$. Then*

$$\begin{aligned} h_n(T) \leq x_n(T) \leq c_n(T) \leq \alpha_n(T), \\ h_n(T) \leq y_n(T) \leq d_n(T) \leq \alpha_n(T). \end{aligned} \tag{6}$$

Theorem 4 ([6], p.90). *An s-number sequence is called injective if, for any metric injection $K \in \mathfrak{L}(Y, Y_0)$, $s_n(T) = s_n(KT)$ for all $T \in \mathfrak{L}(X,Y)$.*

Theorem 5 ([6], p.95). *An s-number sequence is called surjective if, for any metric surjection $P \in \mathfrak{L}(X_0, X)$, $s_n(T) = s_n(TP)$ for all $T \in \mathfrak{L}(X,Y)$.*

Theorem 6 ([6], pp.90-94). *The Weyl and Gel'fand numbers are injective.*

Theorem 7 ([6], pp.95). *The Chang and Kolmogorov numbers are surjective.*

Definition 8. A finite rank operator is a bounded linear operator whose dimension of the range space is finite.

Definition 9 ((dual s-numbers) [7]). For each s-number sequence $s = (s_n)$, a dual s-number function $s^d = (s_n^d)$ is defined by

$$s_n^d(T) = s_n\left(T'\right) \quad \text{for all } T \in \mathfrak{L}(X,Y), \tag{7}$$

where T' is the dual of T.

Definition 10 ([8], p.152)). An s-number sequence is called symmetric if $s_n(T) \geq s_n(T')$ for all $T \in \mathfrak{L}(X,Y)$. If $s_n(T) = s_n(T')$, then the s-number sequence is said to be completely symmetric.

Presently we express some known results of dual of an s-number sequence.

Theorem 11 ([8], p.152). *The approximation numbers are symmetric, i.e., $\alpha_n(T') \leq \alpha_n(T)$ for $T \in \mathfrak{L}(X,Y)$.*

Remark 12 (see [9]). $\alpha_n(T) = \alpha_n(T')$ for every compact operator T.

Theorem 13 ([8], p.153). *Let $T \in \mathfrak{L}(X,Y)$. Then*

$$\begin{aligned} c_n\left(T'\right) \leq d_n(T), \\ c_n(T) = d_n\left(T'\right). \end{aligned} \tag{8}$$

In addition, if T is a compact operator then $d_n(T) = c_n(T')$.

Theorem 14 ([6], p.96). *Let $T \in \mathfrak{L}(X, Y)$. Then*

$$y_n\left(T'\right) \leq x_n(T), \quad x_n(T) = y_n\left(T'\right), \tag{9}$$

i.e., Chang numbers and Weyl numbers are dual to each other.

Theorem 15 ([8], p.153). *The Hilbert numbers are completely symmetric, i.e., $h_n(T') = h_n(T)$ for all $T \in \mathfrak{L}(X, Y)$.*

Definition 16 (see [10, 11]). The operator ideal $\mathbb{U} := \{\mathbb{U}(X, Y)$; *X and Y are Banach Spaces*$\}$ is a subclass of linear bounded operators such that its components $\mathbb{U}(X, Y)$ which are subsets of $\mathfrak{L}(X, Y)$ fulfill the accompanying conditions:

(i) $I_A \in \mathbb{U}$ where A indicates one dimensional Banach space, where $\mathbb{U} \subset \mathfrak{L}$.

(ii) For $T_1, T_2 \in \mathbb{U}(X, Y)$, then $\beta_1 T_1 + \beta_2 T_2 \in \mathbb{U}(X, Y)$ for any scalars β_1, β_2.

(iii) If $T \in \mathfrak{L}(X_0, X)$, $R \in \mathbb{U}(X, Y)$, and $P \in \mathfrak{L}(Y, Y_0)$, then $PRT \in \mathbb{U}(X_0, Y_0)$.

Definition 17 (see [12, 13]). An Orlicz function is a function $\varphi : [0, \infty) \longrightarrow [0, \infty)$, which is nondecreasing, convex, and continuous with $\varphi(0) = 0$ and $\varphi(x) > 0$ for $x > 0$ and $\lim_{x \longrightarrow \infty} \varphi(x) = \infty$.

Definition 18. An Orlicz function φ is said to satisfy Δ_2-condition for every values of $x \geq 0$, if there is $a > 0$, such that $\varphi(2x) \leq a\varphi(x)$. The Δ_2-condition is corresponding to $\varphi(mx) \leq am\varphi(x)$ for every values of $m > 1$ and x.

Lindenstrauss and Tzafriri [14] utilized the idea of an Olicz function to define Orlicz sequence space:

$$\ell_\varphi = \{x \in \omega : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\} \quad \text{where } \rho(x) = \sum_{k=0}^{\infty} \varphi(|x_k|), \tag{10}$$

$(\ell_\varphi, \|.\|)$ is a Banach space with the Luxemburg norm:

$$\|x\|_{\ell_\varphi} = \inf\left\{\lambda > 0 : \rho\left(\lambda^{-1} x\right) \leq 1\right\}. \tag{11}$$

Every Orlicz sequence space contains a subspace that is isomorphic to ℓ^p, for some $1 \leq p < \infty$ or c_0 ([15], Theorem 4.a.9).

In the recent past lot of work has been done on sequence spaces defined by Orlicz functions by Altin et al. [16], Et et al. ([17, 18]), Tripathy et al. ([19–21]), and Mohiuddine et al. ([22–25]).

Given an Orlicz function φ, the Orlicz-Cesáro mean sequence spaces is defined by

$$ces_\varphi = \{u = (u_i) \in \omega : \rho(\beta u) < \infty \text{ for some } \beta > 0\}, \quad \rho(u) = \sum_{i=0}^{\infty} \phi\left(\frac{\sum_{j=0}^{i} |u_j|}{i+1}\right). \tag{12}$$

$(ces_\varphi, \|.\|)$ is a Banach space with the Luxemburg norm given by

$$\|u\|_{ces_\varphi} = \inf\left\{\beta > 0 : \rho\left(\beta^{-1} u\right) \leq 1\right\}. \tag{13}$$

It seems that Orlicz-Cesáro mean sequence spaces ces_φ appeared for the first time in 1988, when Lim and Yee found their dual spaces [26]. Recently Cui, Hudzik, Petrot, Suantai, and Szymaszkiewicz obtained important properties of spaces ces_φ [27]. In 2007 Maligranda, Petrot, and Suantai showed that ces_φ is not B-convex, if $\varphi \in \Delta_2$ and $ces_\varphi \neq 0$ [28]. The extreme points and strong X-points of ces_φ have been characterized by Foralewski, Hudzik, and Szymaszkiewicz in [29]. In the case when $\varphi(u) = u^p$, $1 \leq p < \infty$, the space ces_φ is just a Cesáro sequence space ces_p, with the norm given by

$$\|u\|_{ces_p} = \left[\sum_{i=0}^{\infty} \left(\frac{\sum_{j=0}^{i} |u_j|}{i+1}\right)^p\right]^{1/p}. \tag{14}$$

It is well known that $ces_1 = \{0\}$ [30].

Definition 19 (see [31]). The Matuszewska Orlicz lower index α_φ of an Orlicz function φ is defined as follows:

$$\alpha_\varphi = \sup\{p > 0 : \exists_{K>0}\ \forall_{0<\lambda, t \leq 1} \varphi(\lambda t) \leq K t^p \varphi(\lambda)\}. \tag{15}$$

Theorem 20 (see [31]). *For any Orlicz function φ, we have $\alpha_\varphi > 1$ if and only if $\ell_\varphi \subset ces_\varphi$. In particular, if $\alpha_\varphi > 1$ then $ces_\varphi \neq \{0\}$.*

Theorem 21 (see [31]). *Let φ_1 and φ_2 be Orlicz functions. If there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$ for all $t \in [0, t_0]$, then $ces_{\varphi_1} \subset ces_{\varphi_2}$.*

Theorem 22 (see [31]). *Let φ_1 and φ_2 be Orlicz functions and $\alpha_{\varphi_1} > 1$, then $ces_{\varphi_1} \subset ces_{\varphi_2}$ if and only if there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$ for all $t \in [0, t_0]$.*

Definition 23 (see [2]). A class of linear sequence spaces $\mathbb{E}$ is called a special space of sequences (sss) having three properties:

(1) $e_i \in \mathbb{E}$ for all $i \in \mathbb{N}$,

(2) if $x = (x_i) \in \omega$, $y = (y_i) \in \mathbb{E}$ and $|x_i| \leq |y_i|$ for every $i \in \mathbb{N}$, then $x \in \mathbb{E}$, "i.e., $\mathbb{E}$ is solid",

(3) if $(x_i)_{i=0}^{\infty} \in \mathbb{E}$, then $(x_{[i/2]})_{i=0}^{\infty} \in \mathbb{E}$, wherever $[i/2]$ means the integral part of $i/2$.

Definition 24 (see [2]). A subclass of the special space of sequences is called a premodular (sss) if there is a function $\varrho : \mathbb{E} \longrightarrow [0, \infty[$ fulfilling the accompanying conditions:

(i) $\varrho(x) \geq 0$ for each $x \in \mathbb{E}$ and $\varrho(x) = 0 \Longleftrightarrow x = \theta$, where θ is the zero element of $\mathbb{E}$,

(ii) there exists $L \geq 1$ such that $\varrho(\lambda x) \leq L|\lambda|\varrho(x)$ for all $x \in \mathbb{E}$, and for any scalar λ,

(iii) for some $K \geq 1$, we have $\varrho(x + y) \leq K(\varrho(x) + \varrho(y))$ for every $x, y \in \mathbb{E}$,

(iv) if $|x_i| \le |y_i|$ for all $i \in \mathbb{N}$, then $\varrho((x_i)) \le \varrho((y_i))$,

(v) for some $K_0 \ge 1$, we have

$$\varrho((x_i)) \le \varrho((x_{[i/2]})) \le K_0 \varrho((x_i)), \tag{16}$$

(vi) the set of all finite sequences is ϱ-dense in $\mathbb{E}$. This means for each $x = (x_i)_{i=o}^{\infty} \in \mathbb{E}$ and for each $\varepsilon > 0$ there exists $m \in \mathbb{N}$ such that $\varrho((x_i)_{i=m}^{\infty}) < \varepsilon$,

(vii) there exists a constant $\xi > 0$ such that $\varrho(\lambda, 0, 0, 0, \ldots) \ge \xi|\lambda|\varrho(1, 0, 0, 0, \ldots)$ for any $\lambda \in \mathbb{R}$.

We denote $(\mathbb{E}_{\varrho}, \varrho)$ for the linear space $\mathbb{E}$ equipped with the metrizable topology generated by ϱ.

Theorem 25 (see [32]). *If X, Y are infinite dimensional Banach spaces and λ_i is a monotonic decreasing sequence to zero, then there exists a bounded linear operator T such that*

$$\frac{1}{16}\lambda_{3i} \le \alpha_i(T) \le 8\lambda_{i+1}. \tag{17}$$

Notations 26 (see [3]).

$S_{\mathbb{E}} := \{S_{\mathbb{E}}(X, Y);\ X \text{ and } Y \text{ are Banach Spaces}\}$, where

$S_{\mathbb{E}}(X, Y) := \{T \in \mathfrak{L}(X, Y) : ((s_i(T))_{i=0}^{\infty} \in \mathbb{E}\}$. Also

$S_{\mathbb{E}}^{app} := \{S_{\mathbb{E}}^{app}(X, Y);\ X \text{ and } Y \text{ are Banach Spaces}\}$, where

$S_{\mathbb{E}}^{app}(X, Y) := \{T \in \mathfrak{L}(X, Y) : ((\alpha_i(T))_{i=0}^{\infty} \in \mathbb{E}\}$.

Theorem 27 (see [3]). *If $\mathbb{E}$ is a (sss), then $S_{\mathbb{E}}$ is an operator ideal.*

The concept of pre-quasi operator ideal which is more general than the usual classes of operator ideal.

Definition 28 (see [3]). A function $g : \Omega \longrightarrow [0, \infty)$ is said to be a pre-quasi norm on the ideal Ω fulfilling the accompanying conditions:

(1) for all $T \in \Omega(X, Y)$, $g(T) \ge 0$ and $g(T) = 0$ if and only if $T = 0$,

(2) there exists a constant $L \ge 1$ such that $g(\beta T) \le L|\beta|g(T)$, for all $T \in \Omega(X, Y)$ and $\beta \in \mathbb{R}$,

(3) there exists a constant $K \ge 1$ such that $g(T_1 + T_2) \le K[g(T_1) + g(T_2)]$, for all $T_1, T_2 \in \Omega(X, Y)$,

(4) there exists a constant $C \ge 1$ such that if $P \in \mathfrak{L}(X_0, X)$, $R \in \Omega(X, Y)$, and $T \in \mathfrak{L}(Y, Y_0)$, then $g(TRP) \le C\ \|T\| g(R)\ \|P\|$, where X_0 and Y_0 are normed spaces.

Theorem 29 (see [3]). *Every quasi norm on the ideal Ω is a pre-quasi norm on the ideal Ω.*

Here and after, we define $e_i = \{0, 0, \ldots, 1, 0, 0, \ldots\}$ where 1 appears at the i^{th} place for all $i \in \mathbb{N}$.

3. Main Results

We give here the conditions on Orlicz-Cesáro mean sequence spaces ces_{φ} such that the class $S_{ces_{\varphi}}$ of all bounded linear operators between arbitrary Banach spaces with its sequence of s−numbers which belong to ces_{φ} forms an operator ideal.

Theorem 30. *If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_{\varphi} > 1$, then $S_{ces_{\varphi}}$ is an operator ideal.*

Proof. (1-i) Let $x, y \in ces_{\varphi}$. Since φ is nondecreasing, convex, and satisfying Δ_2-condition, we get for some $k > 0$ that

$$\begin{aligned}&\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i + y_i|}{n+1}\right)\\ &\quad \le k\left[\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) + \sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|y_i|}{n+1}\right)\right] < \infty,\end{aligned} \tag{18}$$

then $x + y \in ces_{\varphi}$.

(1-ii) Let $\lambda \in \mathbb{R}$ and $x \in ces_{\varphi}$, and since φ is convex and satisfying Δ_2-condition, we get for some $k > 0$ that

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|\lambda x_i|}{n+1}\right) \le |\lambda|\, k \sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) < \infty, \tag{19}$$

then $\lambda x \in ces_{\varphi}$; from (1-i) and (1-ii) ces_{φ} is a linear space. Since $e_n \in \ell_{\varphi}$, for all $n \in \mathbb{N}$ and $\alpha_{\varphi} > 1$, then from Theorem 20, we get $e_n \in ces_{\varphi}$, for all $n \in \mathbb{N}$.

(2) Let $|x_n| \le |y_n|$ for all $n \in \mathbb{N}$ and $y \in ces_{\varphi}$; since φ is nondecreasing, then we have

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) \le \sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|y_i|}{n+1}\right) < \infty, \tag{20}$$

and we get $x \in ces_{\varphi}$.

(3) Let $(x_n) \in ces_{\varphi}$. Since φ is satisfying Δ_2-condition, we get for some $k > 0$ that

$$\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_{[i/2]}|}{n+1}\right) \le (k+1)\sum_{n=0}^{\infty}\varphi\left(\frac{\sum_{i=0}^{n}|x_i|}{n+1}\right) < \infty, \tag{21}$$

then $(x_{[n/2]}) \in ces_{\varphi}$. Then ces_{φ} is a (sss); hence by Theorem 27, $S_{ces_{\varphi}}$ is an operator ideal. □

Corollary 31. *S_{ces_q} is an operator ideal, if $1 < q < \infty$.*

We give the conditions on Orlicz-Cesáro mean sequence spaces ces_{φ} such that the ideal of the finite rank operators is dense in $S_{ces_{\varphi}}(X, Y)$.

Theorem 32. *$S_{ces_{\varphi}}(X, Y) = \overline{F(X, Y)}$, if φ is an Orlicz function satisfying Δ_2-condition and $\alpha_{\varphi} > 1$.*

Proof. Let us define $\varrho(u) = \sum_{i=0}^{\infty}\varphi(\sum_{j=0}^{i}|u_j|/(i+1))$ on ces_{φ}. First, we have to show that $\overline{F(X, Y)} \subseteq S_{ces_{\varphi}}(X, Y)$. Since $\alpha_{\varphi} > 1$, we have $e_i \in ces_{\varphi}$ for each $i \in \mathbb{N}$ and φ is an

Orlicz function satisfying Δ_2-condition, so for each finite operator $P \in F(X,Y)$, i.e., we obtain $(s_i(P))_{i=0}^{\infty}$ which contains only finitely many terms different from zero; hence $P \in S_{ces_\varphi}(X,Y)$. Currently we prove that $S_{ces_\varphi}(X,Y) \subseteq \overline{F(X,Y)}$; let $P \in S_{ces_\varphi}(X,Y)$; we have $(s_i(P))_{i=0}^{\infty} \in ces_\varphi$; and hence $\varrho(s_i(P))_{i=0}^{\infty} < \infty$. By taking $\varepsilon \in (0,1)$, hence there exists a $i_0 \in \mathbb{N} - \{0\}$ such that $\varrho((s_i(P))_{i=i_0}^{\infty}) < \varepsilon/9\delta C^2$ for some $c \geq 1$, where $\delta = \max\{1, \sum_{i=i_0}^{\infty} \varphi(1/(i+1))\}$. As $s_i(P)$ is decreasing for every $i \in \mathbb{N}$ and φ is nondecreasing, we have

$$\begin{aligned} i_0\varphi\left(s_{2i_0}(P)\right) &\leq \sum_{i=i_0+1}^{2i_0} \varphi\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) \\ &\leq \sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) < \frac{\varepsilon}{9\delta C^2}. \end{aligned} \tag{22}$$

Hence, there exists $B \in F_{2i_0}(X,Y)$ such that rank $B \leq 2i_0$ and

$$i_0\varphi(\|P-B\|) \leq \sum_{i=i_0+1}^{2i_0} \varphi\left(\frac{\sum_{j=0}^{i} \|P-B\|}{i+1}\right) < \frac{\varepsilon}{9\delta C^2}. \tag{23}$$

Since φ is right continuous at 0 and nondecreasing, then on considering this

$$\|P-B\| < \frac{\varepsilon}{6C^2 i_0 \delta}. \tag{24}$$

Let $k_1 > 0$, $k_2 > 0$ and $C = \max\{1, k_1, k_2\}$, since φ is Orlicz function and by using (22), (23), and (24), we have

$$\begin{aligned} d(P,B) &= \varrho\left(s_i(P-B)\right)_{i=0}^{\infty} = \sum_{i=0}^{3i_0-1} \varphi \\ &\cdot \left(\frac{\sum_{j=0}^{i} s_j(P-B)}{i+1}\right) + \sum_{i=3i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{i} s_j(P-B)}{i+1}\right) \\ &\leq \sum_{i=0}^{3i_0-1} \varphi\left(\frac{\sum_{j=0}^{i} \|P-B\|}{i+1}\right) + \sum_{i=i_0}^{\infty} \varphi \\ &\cdot \left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1}\right) \leq 3i_0\varphi(\|P-B\|) + \sum_{i=i_0}^{\infty} \varphi \\ &\cdot \left(\frac{\sum_{j=0}^{i+2i_0} s_j(P-B)}{i+1}\right) \leq 3i_0\varphi(\|P-B\|) \\ &+ \sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{2i_0-1} s_j(P-B) + \sum_{j=2i_0}^{i+2i_0} s_j(P-B)}{i+1}\right) \\ &\leq 3i_0\varphi(\|P-B\|) + k_1\left[\sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{2i_0-1} s_j(P-B)}{i+1}\right)\right. \\ &\left. + \sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=2i_0}^{i+2i_0} s_j(P-B)}{i+1}\right)\right] \leq 3i_0\varphi(\|P-B\|) \\ &+ k_1\left[\sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{2i_0-1} \|P-B\|}{i+1}\right)\right. \\ &\left. + \sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{i} s_{j+2i_0}(P-B)}{i+1}\right)\right] \leq 3i_0\varphi(\|P-B\|) \\ &+ 2i_0k_1k_2\|P-B\| \sum_{i=i_0}^{\infty} \varphi\left(\frac{1}{i+1}\right) + k_1\sum_{i=i_0}^{\infty} \varphi \\ &\cdot \left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) \leq 3i_0\varphi(\|P-B\|) + 2i_0C^2\|P \\ &- B\| \sum_{i=i_0}^{\infty} \varphi\left(\frac{1}{i+1}\right) + C\sum_{i=i_0}^{\infty} \varphi\left(\frac{\sum_{j=0}^{i} s_j(P)}{i+1}\right) < \varepsilon. \end{aligned} \tag{25}$$

□

Corollary 33. $S_{ces_p}(X,Y) = \overline{F(X,Y)}$, *if* $1 < p < \infty$.

We express the accompanying theorem without verification; these can be set up utilizing standard procedure.

Theorem 34. *The function* $g(P) = \sum_{i=0}^{\infty} \varphi(\sum_{j=0}^{i} |s_j(P)|/(i+1))$ *is a pre-quasi norm on* S_{ces_φ}, *if* φ *is an Orlicz function satisfying* Δ_2*-condition and* $\alpha_\varphi > 1$.

We give the sufficient conditions on Orlicz-Cesáro mean sequence spaces ces_φ such that the components of the pre-quasi operator ideal S_{ces_φ} are complete.

Theorem 35. *If* X *and* Y *are Banach spaces,* φ *is an Orlicz function satisfying* Δ_2*-condition and* $\alpha_\varphi > 1$*, then* $(S_{ces_\varphi}(X,Y), g)$ *is a pre-quasi Banach operator ideal.*

Proof. Since φ is an Orlicz function satisfying Δ_2-condition, then the function $g(P) = \varrho((s_n(P))_{n=0}^{\infty}) = \sum_{n=0}^{\infty} \varphi(\sum_{m=0}^{n} |s_m(P)|/(n+1))$ is a pre-quasi norm on S_{ces_φ}. Let (P_m) be a Cauchy sequence in $S_{ces_\varphi}(X,Y)$. Since $\mathfrak{L}(X,Y) \supseteq S_{ces_\varphi}(X,Y)$ and $\alpha_\varphi > 1$, we can find a constant $\xi > 0$ such that

$$\begin{aligned} g\left(P_i - P_j\right) &= \varrho\left(\left(s_n\left(P_i - P_j\right)\right)_{n=0}^{\infty}\right) \\ &\geq \varrho\left(s_0\left(P_i - P_j\right), 0, 0, 0, \ldots\right) \\ &= \varrho\left(\left\|P_i - P_j\right\|, 0, 0, 0, \ldots\right) \\ &\geq \xi\left\|P_i - P_j\right\| \varrho(1, 0, 0, 0, \ldots), \end{aligned} \tag{26}$$

then $(P_m)_{m\in\mathbb{N}}$ is also a Cauchy sequence in $\mathfrak{L}(X,Y)$. While the space $\mathfrak{L}(X,Y)$ is a Banach space, there exists $P \in \mathfrak{L}(X,Y)$

such that $\lim_{m\longrightarrow\infty} \|P_m - P\| = 0$, while $(s_n(P_m))_{n=0}^{\infty} \in ces_{\varphi}$ for every $m \in \mathbb{N}$. Since ϱ is continuous at θ and for some $K \geq 1$, we obtain

$$\begin{aligned} g(P) &= \varrho\left(\left(s_n(P)\right)_{n=0}^{\infty}\right) = \varrho\left(\left(s_n\left(P - P_m + P_m\right)\right)_{n=0}^{\infty}\right) \\ &\leq K\varrho\left(\left(s_{[n/2]}\left(P - P_m\right)\right)_{n=0}^{\infty}\right) \\ &\quad + K\varrho\left(\left(\alpha_{[n/2]}\left(P_m\right)_{n=0}^{\infty}\right)\right) \\ &\leq K\varrho\left(\left(\|P_m - P\|\right)_{n=0}^{\infty}\right) + K\varrho\left(\left(s_n\left(P_m\right)_{n=0}^{\infty}\right)\right) \\ &< \infty, \end{aligned} \tag{27}$$

we have $(s_n(P))_{n=0}^{\infty} \in ces_{\varphi}$, and then $P \in S_{ces_{\varphi}}(X, Y)$. □

Corollary 36. *If X and Y are Banach spaces and $1 < q < \infty$, then $(S_{ces_q}(X, Y), g)$ is quasi Banach operator ideal, where $g(P) = \varrho((s_n(P))_{n=0}^{\infty}) = [\sum_{n=0}^{\infty}(\sum_{m=0}^{n} |s_m(P)|/(n+1))^q]^{1/q}$.*

Theorem 37. *Let φ_1, φ_2 be Orlicz functions and $\alpha_{\varphi_1} > 1$. For any infinite dimensional Banach spaces X, Y and if there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$ for all $t \in [0, t_0]$, it is true that*

$$S^{app}_{ces_{\varphi_1}}(X, Y) \subsetneqq S^{app}_{ces_{\varphi_2}}(X, Y) \subsetneqq \mathfrak{L}(X, Y). \tag{28}$$

Proof. Let X and Y be infinite dimensional Banach spaces and there exist b, $t_0 > 0$ such that $\varphi_2(t_0) > 0$ and $\varphi_2(t) \leq \varphi_1(bt)$ for all $t \in [0, t_0]$; if $P \in S^{app}_{ces_{\varphi_1}}(X, Y)$, then $(\alpha_n(P)) \in ces_{\varphi_1}$. From Theorems 21, 22, and 25, we have $ces_{\varphi_1} \subset ces_{\varphi_2}$; hence $P \in S^{app}_{ces_{\varphi_2}}(X, Y)$. It is easy to see that $S^{app}_{ces_{\varphi_2}}(X, Y) \subset \mathfrak{L}(X, Y)$. □

Corollary 38. *For any infinite dimensional Banach spaces X, Y, and $1 < p < q < \infty$, then $S^{app}_{ces_p}(X, Y) \subsetneqq S^{app}_{ces_q}(X, Y) \subsetneqq \mathfrak{L}(X, Y)$.*

We now study some properties of the pre-quasi Banach operator ideal $S_{ces_{\varphi}}$.

Theorem 39. *The pre-quasi Banach operator ideal $(S_{ces_{\varphi}}, g)$ is injective, if the s-number sequence is injective.*

Proof. Let $T \in \mathfrak{L}(X, Y)$ and $P \in \mathfrak{L}(Y, Y_0)$ be any metric injection. Assume that $PT \in S_{ces_{\varphi}}(X, Y_0)$, then $\varrho(s_n(PT)) < \infty$. Since the s-number sequence is injective, we have $s_n(PT) = s_n(T)$, for all $T \in \mathfrak{L}(X, Y)$, $n \in \mathbb{N}$. So $\varrho(s_n(T)) = \varrho(s_n(PT)) < \infty$. Hence $T \in S_{ces_{\varphi}}(X, Y)$ and clearly $g(T) = g(PT)$ is verified. □

Remark 40. The pre-quasi Banach operator ideal $(S^{Weyl}_{ces_{\varphi}}, g)$ and the pre-quasi Banach operator ideal $(S^{Gel}_{ces_{\varphi}}, g)$ are injective pre-quasi Banach operator ideal.

Theorem 41. *The pre-quasi Banach operator ideal $(S_{ces_{\varphi}}, g)$ is surjective, if the s-number sequence is surjective.*

Proof. Let $T \in \mathfrak{L}(X, Y)$ and $P \in \mathfrak{L}(X_0, X)$ be any metric surjection. Suppose that $TP \in S_{ces_{\varphi}}(X_0, Y)$, then $\varrho(s_n(TP)) < \infty$. Since the s-number sequence is surjective, we have $s_n(TP) = s_n(T)$, for all $T \in \mathfrak{L}(X, Y)$, $n \in \mathbb{N}$. So $\varrho(s_n(T)) = \varrho(s_n(TP)) < \infty$. Hence $T \in S_{ces_{\varphi}}(X, Y)$ and clearly $g(T) = g(TP)$ is verified. □

Remark 42. The pre-quasi Banach operator ideal $(S^{Chang}_{ces_{\varphi}}, g)$ and the pre-quasi Banach operator ideal $(S^{Kol}_{ces_{\varphi}}, g)$ are surjective pre-quasi Banach operator ideal.

Likewise, we have the accompanying inclusion relations between the pre-quasi Banach operator ideals.

Theorem 43. (1) $S^{app}_{ces_{\varphi}} \subseteq S^{Kol}_{ces_{\varphi}} \subseteq S^{Chang}_{ces_{\varphi}} \subseteq S^{Hilb}_{ces_{\varphi}}$.

(2) $S^{app}_{ces_{\varphi}} \subseteq S^{Gel}_{ces_{\varphi}} \subseteq S^{Weyl}_{ces_{\varphi}} \subseteq S^{Hilb}_{ces_{\varphi}}$.

Proof. Since $h_n(T) \leq y_n(T) \leq d_n(T) \leq \alpha_n(T)$ and $h_n(T) \leq x_n(T) \leq c_n(T) \leq \alpha_n(T)$ for every $n \in \mathbb{N}$ and ϱ is nondecreasing, we obtain

$$\begin{aligned} &\varrho\left(h_n(T)\right) \leq \varrho\left(y_n(T)\right) \leq \varrho\left(d_n(T)\right) \leq \varrho\left(\alpha_n(T)\right), \\ &\varrho\left(h_n(T)\right) \leq \varrho\left(x_n(T)\right) \leq \varrho\left(c_n(T)\right) \leq \varrho\left(\alpha_n(T)\right). \end{aligned} \tag{29}$$

Hence the result is as follows. □

We presently express the dual of the pre-quasi operator ideal formed by different $s-$ number sequences.

Theorem 44. *The pre-quasi operator ideal $S^{Hilb}_{ces_{\varphi}}$ is completely symmetric and the pre-quasi operator ideal $S^{app}_{ces_{\varphi}}$ is symmetric.*

Proof. Since $h_n(T') = h_n(T)$ and $\alpha_n(T') \leq \alpha_n(T)$, for all $T \in \mathfrak{L}(X, Y)$, we have $S^{Hilb}_{ces_{\varphi}} = (S^{Hilb}_{ces_{\varphi}})'$ and $S^{app}_{ces_{\varphi}} \subseteq (S^{app}_{ces_{\varphi}})'$. □

In perspective on Theorem 13, we express the following result without proof.

Theorem 45. *The pre-quasi operator ideal $S^{Kol}_{ces_{\varphi}} \subseteq (S^{Gel}_{ces_{\varphi}})'$ and $S^{Gel}_{ces_{\varphi}} = (S^{Kol}_{ces_{\varphi}})'$. In addition if T is a compact operator from X to Y, then $S^{Kol}_{ces_{\varphi}} = (S^{Gel}_{ces_{\varphi}})'$.*

In perspective on Theorem 14, we express the following result without proof.

Theorem 46. *The pre-quasi operator ideal $S^{Chang}_{ces_{\varphi}} = (S^{Weyl}_{ces_{\varphi}})'$ and $S^{Weyl}_{ces_{\varphi}} = (S^{Chang}_{ces_{\varphi}})'$.*

Theorem 47. *If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_{\varphi} > 1$, then the pre-quasi Banach operator ideal $S^{app}_{ces_{\varphi}}$ is small.*

Proof. Since φ is an Orlicz function and $\alpha_{\varphi} > 1$, take $\beta = \sum_{i=0}^{\infty} \varphi(1/(i+1))$. Then $(S^{app}_{ces_{\varphi}}, g)$, where $g(T) =$

$\varrho((\alpha_n(T))_{n=0}^{\infty}) = (1/\beta)\sum_{n=0}^{\infty}\varphi(\sum_{m=0}^{n}\alpha_m(T)/(n+1))$ is a pre-quasi Banach operator ideal. Let X and Y be any two Banach spaces. Assume that $S^{app}_{ces_\varphi}(X,Y) = \mathfrak{L}(X,Y)$, then there exists a constant $C > 0$ such that $g(T) \le C\|T\|$ for all $T \in \mathfrak{L}(X,Y)$. Suppose that X and Y are infinite dimensional Banach spaces. Then by Dvoretzky's theorem [8] for $m \in \mathbb{N}$, we have quotient spaces X/M_m and subspaces N_m of Y which can be mapped onto ℓ_2^m by isomorphisms V_m and B_m such that $\|V_m\|\|V_m^{-1}\| \le 2$ and $\|B_m\|\|B_m^{-1}\| \le 2$. Consider I_m be the identity map on ℓ_2^m, P_m be the quotient map from X onto X/M_m, and Q_m be the natural embedding map from N_m into Y. Let v_n be the Bernstein numbers [7], then

$$\begin{aligned} 1 &= v_n(I_m) = v_n(B_mB_m^{-1}I_mV_mV_m^{-1}) \\ &\le \|B_m\|\, v_n(B_m^{-1}I_mV_m)\,\|V_m^{-1}\| \\ &= \|B_m\|\, v_n(Q_mB_m^{-1}I_mV_m)\,\|V_m^{-1}\| \\ &\le \|B_m\|\, d_n(Q_mB_m^{-1}I_mV_m)\,\|V_m^{-1}\| \\ &= \|B_m\|\, d_n(Q_mB_m^{-1}I_mV_mQ_m)\,\|V_m^{-1}\| \\ &\le \|B_m\|\, \alpha_n(Q_mB_m^{-1}I_mV_mQ_m)\,\|V_m^{-1}\|, \end{aligned} \tag{30}$$

for $1 \le i \le m$. Now since φ is nondecreasing and having Δ_2-condition, we have

$$\begin{aligned} &\sum_{j=0}^{i}(1) \le \sum_{j=0}^{i}\|B_m\|\,\alpha_j(Q_mB_m^{-1}I_mV_mP_m)\,\|V_m^{-1}\| \Longrightarrow \\ &\frac{1}{i+1}(i+1) \le \|B_m\|\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_mB_m^{-1}I_mV_mP_m)\right) \\ &\quad\cdot\|V_m^{-1}\| \Longrightarrow \\ &\varphi(1) \le L\left(\|B_m\|\,\|V_m^{-1}\|\right) \\ &\quad\cdot\varphi\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_mB_m^{-1}I_mV_mP_m)\right). \end{aligned} \tag{31}$$

Therefore

$$\begin{aligned} &\sum_{i=0}^{m}\varphi(1) \le L\|B_m\|\,\|V_m^{-1}\|\sum_{i=0}^{m}\varphi \\ &\quad\cdot\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_mB_m^{-1}I_mV_mP_m)\right) \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le L\|B_m\|\,\|V_m^{-1}\|\frac{1}{\beta}\sum_{i=0}^{m}\varphi \\ &\quad\cdot\left(\frac{1}{i+1}\sum_{j=0}^{i}\alpha_j(Q_mB_m^{-1}I_mV_mP_m)\right) \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le L\|B_m\|\,\|V_m^{-1}\|\,g(Q_mB_m^{-1}I_mV_mP_m) \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le LC\|B_m\|\,\|V_m^{-1}\|\,\|Q_mB_m^{-1}I_mV_mP_m\| \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le LC\|B_m\|\,\|V_m^{-1}\|\,\|Q_mB_m^{-1}\|\,\|I_m\|\,\|V_mP_m\| \\ &\quad = LC\|B_m\|\,\|V_m^{-1}\|\,\|B_m^{-1}\|\,\|I_m\|\,\|V_m\| \Longrightarrow \\ &\frac{\varphi(1)}{\beta}(m+1) \le 4LC, \end{aligned} \tag{32}$$

for some $L \ge 1$. Thus we arrive at a contradiction since m is arbitrary. Hence X and Y both cannot be infinite dimensional when $S^{app}_{ces_\varphi}(X,Y) = \mathfrak{L}(X,Y)$. □

Theorem 48. *If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then the pre-quasi Banach operator ideal $S^{Kol}_{ces_\varphi}$ is small.*

Corollary 49. *If $p \in (1,\infty)$, then the quasi Banach operator ideal $S^{app}_{ces_p}$ is small.*

Corollary 50. *If $p \in (1,\infty)$, then the quasi Banach operator ideal $S^{Kol}_{ces_p}$ is small.*

4. Examples

We give some examples which support our main results.

Example 1. Let φ be an Orlicz function; the subspace ces_φ^h of all order continuous elements of ces_φ is defined as [27]

$$ces_\varphi^h = \left\{x \in ces_\varphi : \forall_{k>0}\ \exists_{n_k\in\mathbb{N}} \sum_{n=n_k}^{\infty}\varphi\left(\frac{k}{n}\sum_{i=1}^{n}|x_i|\right) < \infty\right\}. \tag{33}$$

If φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then the following conditions are satisfied:

(1) $S_{ces_\varphi^h}$ is an operator ideal.

(2) $S_{ces_\varphi^h}(X,Y) = \overline{F(X,Y)}$.

(3) If X and Y are Banach spaces, then $(S_{ces_\varphi^h}(X,Y), g)$ is pre-quasi Banach operator ideal.

(4) The pre-quasi Banach operator ideal $S^{app}_{ces_\varphi^h}$ is small.

Proof. Since φ is an Orlicz function satisfying Δ_2-condition and $\alpha_\varphi > 1$, then from Theorem (5) in [31] we have $ces_\varphi^h = ces_\varphi$ which completes the proof. □

Example 2. Let φ be defined as

$$\varphi(t) = a_l t^l + a_{l-1} t^{l-1} + \ldots + a_1 t, \quad \text{where } a_i > 0 \text{ for all } 1 \le i \le l,\ l \in \mathbb{N},\ l > 1 \text{ and } t \ge 0. \tag{34}$$

It is clear that φ is an Orlicz function and $\alpha_\varphi = l > 1$. Also φ is satisfying Δ_2-condition since

$$\limsup_{t \longrightarrow 0^+} \frac{\varphi(2t)}{\varphi(t)} \le 2^l < \infty. \tag{35}$$

Then the following conditions are satisfied:

(1) S_{ces_φ} is an operator ideal.

(2) $S_{ces_\varphi}(X, Y) = \overline{F(X, Y)}$.

(3) If X and Y are Banach spaces, then $(S_{ces_\varphi}(X, Y), g)$ is pre-quasi Banach operator ideal.

(4) The pre-quasi Banach operator ideal $S^{app}_{ces_\varphi}$ is small.

In the following two examples we will explain the importance of the sufficient conditions.

Example 3. Let φ be defined as

$$\varphi(t) = \begin{cases} 0 & \text{if } t = 0, \\ \dfrac{-t}{\ln t} & \text{if } t \in \left(0, \dfrac{1}{e}\right], \\ \dfrac{3}{2} e t^2 - t + \dfrac{1}{2e} & \text{if } t \in \left(\dfrac{1}{e}, \infty\right). \end{cases} \tag{36}$$

It is clear that φ is an Orlicz function. Since $\sum_{n=1}^{\infty} \varphi(1/n) = \sum_{n=1}^{\infty} (1/n \ln n) = \infty$, hence $ces_\varphi = \{0\}$. The space S_{ces_φ} is not operator ideal since $I_K \notin S_{ces_\varphi}$. Also since φ is convex function and for $p > 1$, we have

$$\begin{aligned} \lim_{t \longrightarrow 0^+} \frac{\varphi(\lambda t)}{\varphi(\lambda) t^p} &= \lim_{t \longrightarrow 0^+} \frac{t^{1-p} \ln \lambda}{\ln \lambda t} \\ &= \lim_{t \longrightarrow 0^+} (1 - p) t^{1-p} \ln \lambda = \infty, \end{aligned} \tag{37}$$

for all $\lambda \in (0, 1]$, then $\alpha_\varphi = 1$. Although φ is satisfying Δ_2-condition since

$$\limsup_{t \longrightarrow 0^+} \frac{\varphi(2t)}{\varphi(t)} = \limsup_{t \longrightarrow 0^+} \frac{2 \ln t}{\ln 2t} \le 2 < \infty. \tag{38}$$

Example 4. Let $\varphi(u) = \int_0^u f(t) dt$, where $f(t)$ is defined as

$$f(t) = \begin{cases} 0 & \text{if } t = 0, \\ \dfrac{1}{n!} & \text{if } t \in \left[\dfrac{1}{(n+1)!}, \dfrac{1}{n!}\right) \text{ for } n = 1, 2, 3 \ldots, \\ t & \text{if } t \in [1, \infty). \end{cases} \tag{39}$$

It is clear that φ is an Orlicz function. Let $T \in S_{ces_\varphi}$ with $s_n(T) = 1/n!$ for all $n \in \mathbb{N}$. We have for $n > 2$ that

$$\begin{aligned} \varphi(s_n(2T)) &= \int_0^{2/n!} f(t)\, dt > \int_{1/n!}^{2/n!} f(t)\, dt \\ &> \int_{1/n!}^{1/(n-1)!} f(t)\, dt > \frac{1}{n!(n-1)!}, \\ n\varphi(s_n(T)) &= n \int_0^{1/n!} f(t)\, dt \\ &< n \sup_{0 \le t \le 1/n!} f(t) \int_0^{1/n!} 1\, dt < \frac{1}{n!(n-1)!}. \end{aligned} \tag{40}$$

Hence $2T \notin S_{ces_\varphi}$, so the space S_{ces_φ} is not operator ideal and $\varphi \notin \Delta_2$. Also since φ is convex function and for $p > 1$, we have

$$\lim_{t \longrightarrow 0^+} \frac{\varphi(\lambda t)}{\varphi(\lambda) t^p} = \lim_{t \longrightarrow 0^+} t^{-p} = \infty, \tag{41}$$

for all $\lambda \in (0, 1]$, then $\alpha_\varphi = 1$.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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Conflicts of Interest

The authors declare that have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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