Research Article

Bivariate Chlodowsky-Stancu Variant of (p, q) -Bernstein-Schurer Operators

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In this study, it is proposed to define bivariate Chlodowsky variant of (p, q) -Bernstein-Stancu-Schurer operators. Therefore, Korovkin-type approximation theorems and the error of approximation by using full modulus of continuity are presented. Beside this, we introduce a generalization of the bivariate Chlodowsky variant of (p, q) -Bernstein-Stancu-Schurer operators and investigate its approximation in more general weighted space. Moreover, the numerical results are discussed in order to validate the accuracy of the bivariate Chlodowsky variant of (p, q) -Bernstein-Schurer operators.

1. Introduction

In this area, many operators about q -integer and (p, q) -integer [1–25] were studied. It may not be possible to find the exact solution for the models developed for the engineering fields because of its mathematical intractability. Therefore, these operators can be used effectively to find an algorithm for approximating solutions [26].

In 2014, Vedi and Ozarslan introduced the Chlodowsky variant of q -Bernstein-Schurer-Stancu operators in [27] as

$$C_{n,p}^{(\alpha,\beta)}(f; q; x) = \sum_{k=0}^{n+p} f\left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n\right) \begin{bmatrix} n+p \\ k \end{bmatrix}_q \left(\frac{x}{b_n}\right)^k \cdot \prod_{s=0}^{n+p-k-1} \left(1 - q^s \frac{x}{b_n}\right), \quad (1)$$

where $\{b_n\}$ is increasing sequence of real numbers satisfying $\lim_{n \rightarrow \infty} b_n = \infty$, $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \leq \alpha \leq \beta$, $0 \leq x \leq b_n$ and $0 < q < 1$ and investigated its approximation properties.

In 2017, Vedi and Ozarslan defined the two dimensional Chlodowsky variant of q -Bernstein-Schurer-Stancu operators in [28] by

$$C_{n,m,p}^{(\alpha,\beta)}(f; q_n, q_m; x, y) := \sum_{k=0}^{n+p} \sum_{j=0}^{m+p} f\left(\frac{[k]_{q_n} + \alpha}{[n]_{q_n} + \beta} a_n, \frac{[j]_{q_m} + \alpha}{[m]_{q_m} + \beta} b_m\right) \cdot \Phi_{k,n,q_n}\left(\frac{x}{a_n}\right) \Phi_{j,m,q_m}\left(\frac{y}{b_m}\right) \quad (2)$$

where $\{a_n\}$ and $\{b_m\}$ are increasing sequences of real numbers satisfying $\lim_{n \rightarrow \infty} a_n = \lim_{m \rightarrow \infty} b_m = \infty$ and $n \in \mathbb{N}$, $p \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$, $0 \leq \alpha \leq \beta$, $\Phi_{k,n,q_n}(z) = \begin{bmatrix} n+p \\ k \end{bmatrix}_{q_n} z^k \prod_{s=0}^{n+p-k-1} (1 - q_n^s z)$ and investigated its approximation properties on the rectangular unbounded domain.

Moreover, Gemikonakli and Vedi-Dilek [29] introduced the Chlodowsky variant of Bernstein-Schurer operators based on (p, q) -integers as

$$\begin{aligned} \bar{C}_{n,s}(f; p, q; x) &:= \sum_{k=0}^{n+s} f \left(p^{n-k} \frac{[k]_{p,q}}{[n]_{p,q}} b_n \right) \\ &\cdot p^{k(k-1)/2 - (n+s-1)(n+s)/2} \begin{bmatrix} n+s \\ k \end{bmatrix}_{p,q} \left(\frac{x}{b_n} \right)^k \\ &\cdot \prod_{j=0}^{n+s-k-1} \left(p^j - q^j \frac{x}{b_n} \right), \end{aligned} \quad (3)$$

where, $s \in \mathbb{N}$, $0 \leq x \leq b_n$ and $0 < q < p \leq 1$.

Let us discuss some well-known basic definitions of (p, q) -calculus. For $0 < q \leq p < 1$, the (p, q) -numbers are given as [30]

$$[k]_{p,q} = \frac{p^k - q^k}{p - q}. \quad (4)$$

For each $k \in \mathbb{N}_0$ the (p, q) -factorial is represented by

$$[k]_{p,q}! = \begin{cases} [k] [k-1] \dots [1], & k = 1, 2, 3, \dots, \\ 1, & k = 0 \end{cases} \quad (5)$$

and (p, q) -binomial coefficients are defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[n-k]_{p,q}! [k]_{p,q}!}, \quad (6)$$

where $n \geq k \geq 0$.

Then, Chlodowsky variant of (p, q) -Bernstein-Stancu-Schurer operators was constructed in [31] as

$$\begin{aligned} C_{n,m}^{(\alpha,\beta)}(f; x, p, q) &:= \frac{1}{p^{(n+m)(n+m-1)/2}} \sum_{k=0}^{n+m} \begin{bmatrix} n+p \\ k \end{bmatrix}_{p,q} p^{k(k-1)/2} \left(\frac{x}{b_n} \right)^k \\ &\times \prod_{s=0}^{n+m-k-1} \left(p^s - q^s \frac{x}{b_n} \right) f \left(\frac{p^{n+m-k} [k]_{p,q} + \alpha}{[n]_{p,q} + \beta} b_n \right), \end{aligned} \quad (7)$$

where $n \in \mathbb{N}$, $\alpha, \beta \in \mathbb{N}_0$ with $\alpha/\beta \approx 1$, $0 \leq x \leq b_n$, $0 < q < p \leq 1$ and b_n has the same properties as the operator $C_{n,p}^{(\alpha,\beta)}(f; q; x)$.

Lemma 1. Let $C_{n,m}^{(\alpha,\beta)}(f; x, p, q)$ be defined in [31].

Then the first few moments of the operators are

$$\begin{aligned} (i) \quad C_{n,m}^{(\alpha,\beta)}(1; x, p, q) &= 1, \\ (ii) \quad C_{n,m}^{(\alpha,\beta)}(t; x, p, q) &= \frac{[n+m]_{p,q} x + \alpha b_n}{[n]_{p,q} + \beta}, \\ (iii) \quad C_{n,m}^{(\alpha,\beta)}(t^2; x, p, q) &= \frac{q [n+m]_{p,q} [n+m-1]_{p,q} x^2}{([n]_{p,q} + \beta)^2} \\ &\quad + \frac{[n+m]_{p,q} (2\alpha + p^{n+m-1}) b_n x + \alpha^2 b_n^2}{([n]_{p,q} + \beta)^2}. \end{aligned} \quad (8)$$

The bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p, q) -integers is defined and then the first few moments of the operator are provided in Section 2. Next, in Section 3, some Korovkin-type theorems are studied. And following this, the order of convergence of the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p, q) -integers by means of the first modulus of continuity is obtained in Section 4. Moreover, in Section 5, we study the generalization of the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p, q) -integers and seek its approximation properties in more general weighted space. Finally, in Section 6, numerical results for the operators constructed are provided in detail.

2. Construction of the Operators

Let $\mathbb{D}_{a_n, b_m}$ denote

$$\mathbb{D}_{a_n, b_m} = \{(x, y) : 0 \leq x \leq a_n, 0 \leq y \leq b_m\} \quad (9)$$

and $\{a_n\}$ and $\{b_m\}$ are increasing sequences of real numbers satisfying

$$\lim_{n \rightarrow \infty} a_n = \lim_{m \rightarrow \infty} b_m = \infty, \quad (10)$$

$$\lim_{n \rightarrow \infty} \frac{a_n}{[n]_{p_n, q_n}} = \lim_{m \rightarrow \infty} \frac{b_m}{[m]_{p_m, q_m}} = 0.$$

For $(x, y) \in \mathbb{D}_{a_n, b_m}$, we construct the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p, q) -integers as

$$\begin{aligned} \bar{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) &:= \frac{1}{p_n^{(n+s)(n+s-1)/2}} \\ &\cdot \frac{1}{p_m^{(m+s)(m+s-1)/2}} \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} f \left(\frac{p_n^{n+s-k} [k]_{p_n, q_n} + \alpha}{[n]_{p_n, q_n} + \beta} a_n, \right. \\ &\quad \left. \frac{p_m^{m+s-l} [l]_{p_m, q_m} + \alpha}{[m]_{p_m, q_m} + \beta} b_m \right) \times \Phi_{k_n, p_n, q_n} \left(\frac{x}{a_n} \right) \\ &\quad \cdot \Phi_{k_m, p_m, q_m} \left(\frac{y}{b_m} \right), \end{aligned} \quad (11)$$

where $\Phi_{k_n, p_n, q_n}(z) = \binom{n+s}{k}_{p_n, q_n} p_n^{l(l-1)/2} (z/a_n)^k \prod_{j=0}^{n+s-k-1} (p_n^j - q_n^j(z/a_n))$, $0 < q_n, q_m < p_n, p_m \leq 1$, $0 \leq \alpha \leq \beta$ and $n, m \in \mathbb{N}$, $s \in \mathbb{N}_0 := \{0\} \cup \mathbb{N}$.

Lemma 2. Let $\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)$ be given in (11). Then, the first few moments of the operators are

$$\begin{aligned}
 (i) \quad & \overline{C}_{n,m,s}^{(\alpha,\beta)}(1; p_n, q_n, p_m, q_m; x, y) = 1, \\
 (ii) \quad & \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1; p_n, q_n, p_m, q_m; x, y) = \frac{[n+s]_{p_n, q_n} x + \alpha a_n}{[n]_{p_n, q_n} + \beta}, \\
 (iii) \quad & \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_2; p_n, q_n, p_m, q_m; x, y) = \frac{[m+s]_{p_m, q_m} y + \alpha b_m}{[m]_{p_m, q_m} + \beta}, \\
 (iv) \quad & \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1^2 + t_2^2; p_n, q_n, p_m, q_m; x, y) = \frac{(q_n [n+s-1]_{p_n, q_n} [n+s]_{p_n, q_n} x^2 + [n+s]_{p_n, q_n} (2\alpha + p_n^{n+s-1}) a_n x + \alpha^2 a_n^2)}{([n]_{p_n, q_n} + \beta)^2} \\
 & + \frac{(q_m [m+s-1]_{p_m, q_m} [m+s]_{p_m, q_m} y^2 + [m+s]_{p_m, q_m} (2\alpha + p_m^{m+s-1}) b_m y + \alpha^2 b_m^2)}{([m]_{p_m, q_m} + \beta)^2}.
 \end{aligned} \tag{12}$$

3. Korovkin-Type Approximation Theorems

In this section, Korovkin-type approximation theorems are given for the bivariate Chlodowsky variant of Bernstein-Stancu-Schurer operators based on (p, q) -integers. For fixed $\gamma \geq 0$ consider the space C_{ρ^γ} , which consists of all continuous functions f satisfying the condition

$$\begin{aligned}
 |f(x, y)| & \leq M_f \rho^\gamma(x, y), \\
 (x, y) & \in [0, \infty) \times [0, \infty) := \mathbb{R}_+^2, \quad \rho(x, y) = 1 + x^2 + y^2.
 \end{aligned} \tag{13}$$

Obviously, C_{ρ^γ} is a linear normed space with the following norm:

$$\|f\|_{\rho^\gamma} = \sup_{0 \leq x, y < \infty} \frac{|f(x, y)|}{\rho^\gamma(x, y)}. \tag{14}$$

Theorem 3. Let the numbers A and B be any fixed positive real numbers.

Let $\mathbb{D}_{A,B} = \{(x, y) : 0 \leq x \leq A, 0 \leq y \leq B\}$ with $0 < q_n, q_m < p_n, p_m < 1$, $\lim_{n \rightarrow \infty} q_n = \lim_{m \rightarrow \infty} q_m = \lim_{n \rightarrow \infty} p_n = \lim_{m \rightarrow \infty} p_m = 1$ and $\{a_n\}$ and $\{b_m\}$ be increasing sequences of positive real numbers that satisfy the following properties:

$$\begin{aligned}
 \lim_{n \rightarrow \infty} a_n & = \lim_{m \rightarrow \infty} b_m = \infty, \\
 \lim_{n \rightarrow \infty} \frac{a_n}{[n]_{p_n, q_n}} & = \lim_{m \rightarrow \infty} \frac{b_m}{[m]_{p_m, q_m}} = 0.
 \end{aligned} \tag{15}$$

For all $f \in C(\mathbb{D}_{A,B})$, we obtain

$$\begin{aligned}
 \lim_{n, m \rightarrow \infty} \max_{(x, y) \in \mathbb{D}_{A,B}} \left| \overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) \right. \\
 \left. - f(x, y) \right| & = 0.
 \end{aligned} \tag{16}$$

Proof. Using Lemma 2, we have

$$\begin{aligned}
 & \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)}(1; p_n, q_n, p_m, q_m; \cdot, \cdot) - 1 \right\|_{C(\mathbb{D}_{A,B})} = 0 \\
 & \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1; p_n, q_n, p_m, q_m; \cdot, \cdot) - x \right\|_{C(\mathbb{D}_{A,B})} \\
 & \leq A \left| \frac{[n+s]_{p_n, q_n}}{[n]_{p_n, q_n} + \beta} - 1 \right| + \frac{\alpha a_n}{[n]_{p_n, q_n} + \beta} \\
 & \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_2; p_n, q_n, p_m, q_m; \cdot, \cdot) - y \right\|_{C(\mathbb{D}_{A,B})} \\
 & \leq B \left| \frac{[m+s]_{p_m, q_m}}{[m]_{p_m, q_m} + \beta} - 1 \right| + \frac{\alpha b_m}{[m]_{p_m, q_m} + \beta}.
 \end{aligned} \tag{17}$$

And again using Lemma 2 we get

$$\begin{aligned}
 & \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1^2 + t_2^2; p_n, q_n, p_m, q_m; \cdot, \cdot) - (x^2 + y^2) \\
 & = \frac{q_n [n+s-1]_{p_n, q_n} [n+s]_{p_n, q_n}}{([n]_{p_n, q_n} + \beta)^2} x^2 \\
 & + \frac{[n+s]_{p_n, q_n} (2\alpha + p_n^{n+s-1})}{([n]_{p_n, q_n} + \beta)^2} a_n x \\
 & + \frac{\alpha^2 a_n^2}{([n]_{p_n, q_n} + \beta)^2} \\
 & + \frac{q_m [m+s-1]_{p_m, q_m} [m+s]_{p_m, q_m}}{([m]_{p_m, q_m} + \beta)^2} y^2
 \end{aligned}$$

$$\begin{aligned}
& + \frac{[m+s]_{p_m, q_m} (2\alpha + p_m^{m+s-1})}{([m]_{p_m, q_m} + \beta)^2} b_m y \\
& + \frac{\alpha^2 b_m^2}{([m]_{p_m, q_m} + \beta)^2}.
\end{aligned} \tag{18}$$

Finally, from the above equality, we obtain

$$\begin{aligned}
& \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)} (t_1^2 + t_2^2; p_n, q_n, p_m, q_m; \cdot, \cdot) - (x^2 + y^2) \right\|_{C(\mathbb{D}_{A,B})} \\
& \leq \left| \frac{q_n [n+s-1]_{p_n, q_n} [n+s]_{p_n, q_n}}{([n]_{p_n, q_n} + \beta)^2} \right| A^2 \\
& + \frac{[n+s]_{p_n, q_n} (2\alpha + p_n^{n+s-1})}{([n]_{p_n, q_n} + \beta)^2} a_n A \\
& + \frac{\alpha^2 a_n^2}{([n]_{p_n, q_n} + \beta)^2} \\
& + \left| \frac{q_m [m+s-1]_{p_m, q_m} [m+s]_{p_m, q_m}}{([m]_{p_m, q_m} + \beta)^2} \right| B^2 \\
& + \frac{[m+s]_{p_m, q_m} (2\alpha + p_m^{m+s-1})}{([m]_{p_m, q_m} + \beta)^2} b_m B \\
& + \frac{\alpha^2 b_m^2}{([m]_{p_m, q_m} + \beta)^2}.
\end{aligned} \tag{19}$$

Therefore, from the hypothesis of the theorem, we get

$$\begin{aligned}
& \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)} (t_1; p_n, q_n, p_m, q_m; \cdot, \cdot) - x \right\|_{C(\mathbb{D}_{A,B})} \rightarrow 0 \\
& \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)} (t_2; p_n, q_n, p_m, q_m; \cdot, \cdot) - y \right\|_{C(\mathbb{D}_{A,B})} \rightarrow 0 \\
& \left\| \overline{C}_{n,m,s}^{(\alpha,\beta)} (t_1^2 + t_2^2; p_n, q_n, p_m, q_m; \cdot, \cdot) - (x^2 + y^2) \right\|_{C(\mathbb{D}_{A,B})} \\
& \rightarrow 0
\end{aligned} \tag{20}$$

when n and $m \rightarrow \infty$.

Hence, the proof is completed by the two dimensional Korovkin theorem. $\square$

Theorem 4 (see [13]). *There exists a sequence of positive operators $T_{n,m}$, acting from $C_\rho(\mathbb{R}_+^2)$ to $C_\rho(\mathbb{R}_+^2)$, satisfying the conditions*

$$\begin{aligned}
& \lim_{n,m \rightarrow \infty} \|T_{n,m}(1; \cdot, \cdot) - 1\|_\rho = 0 \\
& \lim_{n,m \rightarrow \infty} \|T_{n,m}(t_1; \cdot, \cdot) - x\|_\rho = 0 \\
& \lim_{n,m \rightarrow \infty} \|T_{n,m}(t_2; \cdot, \cdot) - y\|_\rho = 0
\end{aligned} \tag{21}$$

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}(t_1^2 + t_2^2; \cdot, \cdot) - (x^2 + y^2)\|_\rho = 0$$

and there exists a function $f^* \in C_\rho$ for which

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}f^* - f^*\|_\rho \geq \frac{1}{4}, \tag{22}$$

where $\rho = 1 + x^2 + y^2$.

Now, consider the following operator:

$$\begin{aligned}
& T_{n,m}(f; p_n, q_n, p_m, q_m; x, y) \\
& = \begin{cases} \overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y), & (x, y) \in \mathbb{D}_{a_n, b_n} \\ f(x, y), & \mathbb{R}_+^2 \setminus \mathbb{D}_{a_n, b_n} \end{cases}
\end{aligned} \tag{23}$$

Theorem 5. *Let $f \in C_\rho(\mathbb{R}_+^2)$ for any $\gamma > 0$*

$$\lim_{n,m \rightarrow \infty} \|T_{n,m}(f; p_n, q_n, p_m, q_m; \cdot, \cdot) - f(\cdot, \cdot)\|_{C_{\rho^{1+\gamma}}} = 0 \tag{24}$$

where $\{a_n\}$, $\{b_m\}$, $\{q_n\}$, and $\{q_m\}$ have the same conditions as in Theorem 3.

Proof. For all $\varepsilon > 0$, there exist sufficiently large positive real numbers A and B such that

$$(1 + x^2 + y^2)^{-\gamma} < \varepsilon \tag{25}$$

when $x > A$ and $y > B$.

Let n, m be sufficiently large so that $\mathbb{D}_{A,B} \subset \mathbb{D}_{a_n, b_m}$

$$\begin{aligned}
& \|T_{n,m}(f; p_n, q_n, p_m, q_m; \cdot, \cdot) - f(\cdot, \cdot)\|_{C_{\rho^{1+\gamma}}} \\
& \leq \sup_{(x,y) \in \mathbb{D}_{A,B}} \frac{|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y)|}{(1 + x^2 + y^2)^{1+\gamma}} \\
& + \sup_{(x,y) \in \mathbb{D}_{a_n, b_n} \setminus \mathbb{D}_{A,B}} \frac{|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y)|}{(1 + x^2 + y^2)^{1+\gamma}} \\
& = y'_{n,m} + y''_{n,m}.
\end{aligned} \tag{26}$$

By Theorem 3, $\lim_{n,m \rightarrow \infty} y'_{n,m} = 0$ and for the proof of the second term we have

$$y''_{n,m} \leq (1 + x^2 + y^2)^{-\gamma} \cdot \left(\frac{|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)|}{1 + x^2 + y^2} + \frac{|f(x, y)|}{1 + x^2 + y^2} \right). \quad (27)$$

Finally, since $f \in C_\rho(\mathbb{R}_+^2)$, the term $|f(x, y)|/(1 + x^2 + y^2)$ is bounded. Furthermore, because of the fact that

$$\begin{aligned} & |\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)| \\ & \leq |C_{n,m,s}^{(\alpha,\beta)}(1 + t_1^2 + t_2^2; q_n, q_m; x, y)|, \end{aligned} \quad (28)$$

in case Lemma 2 is used, the term $|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)|/(1 + x^2 + y^2)$ is bounded for sufficiently large n and m . Hence, we get by (25) that

$$y''_{n,m} \leq \varepsilon(1 + M) \quad (29)$$

Since $\varepsilon > 0$ is arbitrary, then $\lim_{n,m \rightarrow \infty} y''_{n,m} = 0$. This completes the proof. $\square$

Now, consider the subspace C_ρ^0 of C_ρ which is defined by

$$C_\rho^0 := \left\{ f \in C_\rho : \lim_{x,y \rightarrow 0} \frac{|f(x, y)|}{1 + x^2 + y^2} = 0 \right\}. \quad (30)$$

Theorem 6. Let the sequences $\{q_n\}$, $\{q_m\}$, $\{a_n\}$, and $\{b_m\}$ satisfy the same properties as in Theorem 3. Then for all $f \in C_\rho^0(\mathbb{R}_+^2)$, we obtain

$$\begin{aligned} & \lim_{n,m \rightarrow \infty} \left\| T_{n,m} \left(\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; \cdot, \cdot) \right) - f(\cdot) \right\|_{C_\rho} \\ & = 0. \end{aligned} \quad (31)$$

Proof. For all $f \in C_\rho^0(\mathbb{R}_+^2)$, observe that

$$\begin{aligned} & \lim_{x,y \rightarrow \infty} \frac{|f(x, y)|}{1 + x^2 + y^2} = 0, \\ & \lim_{n,m \rightarrow \infty} \frac{|f(\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}\right)a_n, \left(\frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}\right)b_m)|}{1 + \left(\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}\right)a_n\right)^2 + \left(\left(\frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}\right)b_m\right)^2} = 0. \end{aligned} \quad (32)$$

Therefore, for all $\varepsilon > 0$, we can find sufficiently large numbers A and B such that

$$|f(x, y)| < \varepsilon(1 + x^2 + y^2) \quad (33)$$

for $x > A$ and $y > B$ and there exist natural numbers n_0 and m_0 such that

$$\begin{aligned} & \left| f \left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} a_n, \frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta} b_m \right) \right| \\ & < \varepsilon \left(1 + \left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta} a_n \right)^2 + \left(\frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta} b_m \right)^2 \right) \end{aligned} \quad (34)$$

for all $n > n_0$ and $m > m_0$.

Hence, for large n and m , we have

$$\begin{aligned} & \left\| T_{n,m}(f; p_n, q_n, p_m, q_m; \cdot, \cdot) - f(\cdot) \right\|_{C_\rho} \\ & \leq \sup_{(x,y) \in \mathbb{D}_{A,B}} \frac{|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y)|}{1 + x^2 + y^2} \\ & + \sup_{(x,y) \in \mathbb{D}_{a_n,b_m} \setminus \mathbb{D}_{A,B}} \frac{|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y)|}{1 + x^2 + y^2} \\ & = z'_{n,m} + z''_{n,m}. \end{aligned} \quad (35)$$

By Theorem 3, it is sufficient to show that $z''_{n,m} \rightarrow 0$ as $n \rightarrow \infty$.

Using (33) and (34), we get

$$\begin{aligned} & z''_{n,m} \\ & \leq \varepsilon \\ & + \sup_{(x,y) \in \mathbb{D}_{a_n,b_m} \setminus \mathbb{D}_{A,B}} \frac{|\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y)|}{1 + x^2 + y^2} \\ & \leq \varepsilon + \varepsilon \sup_{(x,y) \in \mathbb{D}_{a_n,b_m} \setminus \mathbb{D}_{A,B}} t_{n,m}(q_n, q_m; x, y) \\ & = \varepsilon \left(1 + \sup_{(x,y) \in \mathbb{D}_{a_n,b_m} \setminus \mathbb{D}_{A,B}} t_{n,m}(q_n, q_m; x, y) \right) \end{aligned} \quad (36)$$

where

$$t_{n,m}(q_n, q_m; x, y) := \frac{\overline{C}_{n,m,s}^{(\alpha,\beta)}(1; p_n, q_n, p_m, q_m; x, y) + \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_1^2; p_n, q_n, p_m, q_m; x, y) + \overline{C}_{n,m,s}^{(\alpha,\beta)}(t_2^2; p_n, q_n, p_m, q_m; x, y)}{1 + x^2 + y^2}. \quad (37)$$

By Lemma 2, it is clear that there exist K independent of n and m such that

$$\sup_{(x,y) \in \mathbb{D}_{a_n,b_m} / \mathbb{D}_{A,B}} t_{n,m}(p_n, q_n, p_m, q_m; x, y) \leq K. \quad (38)$$

Therefore, for $n > n_0$ and $m > m_0$ we have

$$z_{n,m}'' < (1 + K) \varepsilon. \quad (39)$$

This completes the proof. $\square$

4. Order of Convergence

In this section, we compute the rate of convergence of the operators in terms of the full modulus of continuity and partial modulus of continuities.

Let $f \in \mathbb{D}_{A,B}$ and $x \geq 0$. Then the definition of the modulus of continuity of f is given by

$$\begin{aligned} \omega(f; \delta) &= \max_{\substack{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \leq \delta \\ x, y \in C(\mathbb{D}_{A,B})}} |f(x_1, y_1) - f(x_2, y_2)|. \end{aligned} \quad (40)$$

It is known that [32] for any $\delta > 0$ we know that

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)| \\ &\leq \omega(f, \delta) \left(\frac{\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}}{\delta} + 1 \right). \end{aligned} \quad (41)$$

Also, for any $\delta > 0$ we have

$$\begin{aligned} &|f(x_1, y_1) - f(x_2, y_2)| \\ &\leq \omega^{(1)}(f, \delta) \left(\frac{|x_1 - x_2|}{\delta} + 1 \right), \\ &|f(x_1, y_1) - f(x_2, y_2)| \\ &\leq \omega^{(2)}(f, \delta) \left(\frac{|y_1 - y_2|}{\delta} + 1 \right). \end{aligned} \quad (42)$$

Theorem 7. For any $f \in C(\mathbb{D}_{A,B})$, the following inequalities

$$\left| \overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y) \right| \quad (43)$$

$$\leq 2 \left[\omega^{(1)}(f; \delta_m) + \omega^{(2)}(f; \delta_n) \right]$$

$$\begin{aligned} &\left| \overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y) \right| \\ &\leq 2\omega\left(f; \sqrt{\delta_m^2 + \delta_n^2}\right) \end{aligned} \quad (44)$$

are satisfied, where

$$\begin{aligned} \delta_n^2 &:= \left| \frac{q_n [n + s - 1]_{p_n, q_n} [n + s]_{p_n, q_n}}{([n]_{p_n, q_n} + \beta)^2} \right| A^2 \\ &+ \frac{[n + s]_{p_n, q_n} (2\alpha + p_n^{n+s-1})}{([n]_{p_n, q_n} + \beta)^2} a_n A \end{aligned} \quad (45)$$

$$\begin{aligned} &+ \frac{\alpha^2 a_n^2}{([n]_{p_n, q_n} + \beta)^2}, \\ \delta_m^2 &:= \left| \frac{q_m [m + s - 1]_{p_m, q_m} [m + s]_{p_m, q_m}}{([m]_{p_m, q_m} + \beta)^2} \right| B^2 \\ &+ \frac{[m + s]_{p_m, q_m} (2\alpha + p_m^{m+s-1})}{([m]_{p_m, q_m} + \beta)^2} b_m B \\ &+ \frac{\alpha^2 b_m^2}{([m]_{p_m, q_m} + \beta)^2}. \end{aligned} \quad (46)$$

Proof. We directly have

$$\begin{aligned} &\overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y) \\ &= \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \left[f \left(\frac{p_n^{n+s-k} [k]_{p_n, q_n} + \alpha}{[n]_{p_n, q_n} + \beta} a_n, \right. \right. \\ &\quad \left. \left. \frac{p_m^{m+s-l} [l]_{p_m, q_m} + \alpha}{[m]_{p_m, q_m} + \beta} b_m \right) - f(x, y) \right] \\ &\times \Phi_{k_n, p_n, q_n} \left(\frac{x}{a_n} \right) \Phi_{l_m, p_m, q_m} \left(\frac{y}{b_m} \right) \\ &= \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \left[f \left(\frac{p_n^{n+s-k} [k]_{p_n, q_n} + \alpha}{[n]_{p_n, q_n} + \beta} a_n, \right. \right. \\ &\quad \left. \left. \frac{p_m^{m+s-l} [l]_{p_m, q_m} + \alpha}{[m]_{p_m, q_m} + \beta} b_m \right) \right] \end{aligned}$$

$$\begin{aligned}
& -f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n, y\right) \\
& + f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n, y\right) - f(x, y) \Big] \\
& \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right).
\end{aligned} \tag{47}$$

By linearity and positivity of the operators, we get

$$\begin{aligned}
& \left| \overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y) \right| \\
& \leq \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \left| f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n, \right. \right. \\
& \quad \left. \left. \frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m\right) - f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n, y\right) \right| \times \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \\
& \quad \cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
& + \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \left| f\left(\frac{[k]_{p_n,q_n}}{[n]_{p_n,q_n}}a_n, y\right) - f(x, y) \right| \\
& \quad \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
& \leq \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \omega^{(2)}\left(f; \left| \frac{p_m^{m+s-k}[k]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m - y \right| \right) \\
& \quad \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
& + \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \omega^{(1)}\left(f; \left| \frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n - x \right| \right) \\
& \quad \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) = \Omega_1(x, y) \\
& + \Omega_2(x, y).
\end{aligned} \tag{48}$$

Using Lemma 1 and Cauchy-Schwartz inequality, we have

$$\begin{aligned}
& \Omega_1(x, y) \\
& = \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \omega^{(2)}\left(f; \left| \frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m - y \right| \right) \\
& \quad \cdot \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
& = \sum_{l=0}^{m+s} \omega^{(2)}\left(f; \left| \frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m - y \right| \right) \\
& \quad \cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \leq \omega^{(2)}(f; \delta_m) \left\{ 1 \right. \\
& \quad \left. + \frac{1}{\delta_m} \left[\sum_{j=0}^{m+p} \left(\frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m - y \right)^2 \right. \right. \\
& \quad \left. \left. \cdot \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \right]^{1/2} \right\}.
\end{aligned} \tag{49}$$

Then, we get

$$\Omega_1(x, y) \leq 2\omega^{(2)}(f; \delta_m), \tag{50}$$

where we choose δ_m as in (46).

In the same way, we obtain

$$\Omega_2(x, y) \leq 2\omega^{(1)}(f; \delta_n), \tag{51}$$

where δ_n is given in (45). Combining (50) and (51), we obtain (43).

Now, by using linearity and the monotonicity of the operators, and taking into account (40), we have

$$\begin{aligned}
& \left| \overline{C}_{n,m,s}^{(\alpha,\beta)}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y) \right| \\
& \leq \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \omega\left(f; \sqrt{\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n - x\right)^2 + \left(\frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m - y\right)^2}\right) \times \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right) \\
& \leq \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \left| f\left(\frac{p_n^{n+s-k}[k]_{p_n,q_n} + \alpha}{[n]_{p_n,q_n} + \beta}a_n, \frac{p_m^{m+s-l}[l]_{p_m,q_m} + \alpha}{[m]_{p_m,q_m} + \beta}b_m\right) - f(x, y) \right| \times \Phi_{k_n,p_n,q_n}\left(\frac{x}{a_n}\right) \Phi_{k_m,p_m,q_m}\left(\frac{y}{b_m}\right)
\end{aligned}$$

$$\begin{aligned}
&\leq 1 + \frac{1}{\delta} \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} \omega \left(f; \sqrt{\left(\frac{p_n^{n+s-k} [k]_{p_n, q_n} + \alpha}{[n]_{p_n, q_n} + \beta} a_n - x \right)^2 + \left(\frac{p_m^{m+s-l} [l]_{p_m, q_m} + \alpha}{[m]_{p_m, q_m} + \beta} b_m - y \right)^2} \right) \\
&\quad \times \Phi_{k_n, p_n, q_n} \left(\frac{x}{a_n} \right) \Phi_{k_m, p_m, q_m} \left(\frac{y}{b_m} \right)
\end{aligned} \tag{52}$$

Using (41) and the Cauchy-Schwartz inequality, we get (44). $\square$

5. Generalization of the Bivariate Chlodowsky Variant of Bernstein-Schurer Operators Based on (p, q) -Integers

In this section, we introduce generalization of bivariate Chlodowsky variant of Bernstein-Schurer operators based on (p, q) -integers. The generalized operators help us to approximate continuous functions defined on more general weighted

spaces. Note that this kind of generalization was considered earlier for the Chlodowsky-Bernstein polynomials [12].

For $x \geq 0$, consider any continuous function $\omega(x, y) \geq 1$ and define

$$G_f(t, s) = f(t, s) \frac{1 + t^2 + s^2}{w(t, s)}. \tag{53}$$

Let us consider the generalization of the $\overline{C}_{n, m, s}^{(\alpha, \beta)}(f; p_n, q_n, p_m, q_m; x, y)$ as follows:

$$\begin{aligned}
&L_{n, p}^{\alpha, \beta} \left(\overline{C}_{n, m, s}^{(\alpha, \beta)}(f; p_n, q_n, p_m, q_m; x, y) \right) \\
&= \left\{ \frac{w(x, y)}{1 + x^2 + y^2} \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} G_f \left(\frac{p_n^{n+s-k} [k]_{p_n, q_n} + \alpha}{[n]_{p_n, q_n} + \beta} a_n, \frac{p_m^{m+s-l} [l]_{p_m, q_m} + \alpha}{[m]_{p_m, q_m} + \beta} b_m \right) \times \Phi_{k_n, p_n, q_n} \left(\frac{x}{a_n} \right) \Phi_{k_m, p_m, q_m} \left(\frac{y}{b_m} \right), \right. \\
&\quad \left. (x, y) \in \mathbb{D}_{a_n, b_n}, \right. \\
&\quad \left. \mathbb{R}_+^2 \setminus \mathbb{D}_{a_n, b_n} \right\}, \tag{54}
\end{aligned}$$

where $(x, y) \in \mathbb{D}_{a_n, b_n}$ and $\{a_n\}$ and $\{b_m\}$ have the same properties of bivariate Chlodowsky variant of Bernstein-Schurer operators based on (p, q) -integers.

Theorem 8. For all continuous functions f satisfying $|f(x, y)| \leq M_f w(x, y)$, $x, y \geq 0$, and $\lim_{x, y \rightarrow \infty} (f(x, y)/w(x, y)) = 0$, we have

$$\lim_{n, m \rightarrow \infty} \|L_{n, p}^{\alpha, \beta}(f; p_n, q_n, p_m, q_m; \cdot, \cdot) - f(\cdot, \cdot)\|_w = 0 \tag{55}$$

where $w(x, y) = 1 + x^2 + y^2$.

Proof. Clearly,

$$\begin{aligned}
&\left| L_{n, p}^{\alpha, \beta}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y) \right| \\
&= \frac{w(x, y)}{1 + x^2 + y^2} \left| \sum_{k=0}^{n+s} \sum_{l=0}^{m+s} G_f \left(\frac{p_n^{n+s-k} [k]_{p_n, q_n} + \alpha}{[n]_{p_n, q_n} + \beta} a_n, \right. \right. \\
&\quad \left. \left. \frac{p_m^{m+s-l} [l]_{p_m, q_m} + \alpha}{[m]_{p_m, q_m} + \beta} b_m \right) \right. \\
&\quad \left. \times \Phi_{k_n, p_n, q_n} \left(\frac{x}{a_n} \right) \Phi_{k_m, p_m, q_m} \left(\frac{y}{b_m} \right) - G_f(x, y) \right|, \tag{56}
\end{aligned}$$

thus

$$\begin{aligned}
&\|L_{n, p}^{\alpha, \beta}(f; p_n, q_n, p_m, q_m; \cdot, \cdot) - f(\cdot, \cdot)\|_w \\
&= \sup_{x, y \in \mathbb{R}_+^2} \frac{|L_{n, p}^{\alpha, \beta}(f; p_n, q_n, p_m, q_m; x, y) - f(x, y)|}{w(x, y)} \\
&= \sup_{x, y \in \mathbb{R}_+^2} \frac{|T_{n, m}(G_f; p_n, q_n, p_m, q_m; x, y) - G_f(x, y)|}{1 + x^2 + y^2}. \tag{57}
\end{aligned}$$

Since $|f(x, y)| \leq M_f w(x, y)$, then $|G_f(x, y)| \leq M_f w(x, y)$ for $x, y \geq 0$ and $G_f(x, y)$ is continuous function on $\mathbb{R}_+^2$. Furthermore, from $\lim_{x, y \rightarrow \infty} (f(x, y)/w(x, y)) = 0$, we have

$$\lim_{x, y \rightarrow \infty} \frac{G_f(x, y)}{w(x, y)} = 0. \tag{58}$$

Thus, from Theorem 6 we get the result. $\square$

Finally, note that taking $w(x, y) = 1 + x^2 + y^2$, then the operators $L_{n, p}^{\alpha, \beta}(f; p_n, q_n, p_m, q_m; x, y)$ reduce $T_{n, m}(G_f; p_n, q_n, p_m, q_m; x, y)$.

TABLE 1: $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ parameters used for numerical analysis.

Parameters	Values
n	20
m	5
a_n	$(n^2 + 4) / (n^2 + 10n)$
b_m	$(m^2 + 4) (m^2 + 10m)$
p_n	0.989
q_n	0.99
p_m	0.989
q_m	0.99
s	1
α, β	Variable

6. Numerical Results and Discussions

In this section, numerical results are presented in order to demonstrate the effectiveness and validate the accuracy of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to $f(x, y)$ with different values of parameters. The parameters used in the mathematical calculations are given in Table 1.

Figures 1 and 2 are given for $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ as a function of x and y for different α and β values considering two different multivariable functions $f(x, y) = e^{\sin xy}$ and $f(x, y) = x^2 y^2$ accordingly.

Figure 1 demonstrates the convergence of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to the multivariable function $f(x, y) = e^{\sin xy}$. For different x and y values, various α and β values are taken into account. According to the results obtained, the $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ behaves like the function $f(x, y) = e^{\sin xy}$ for smaller α and β values. Therefore, the error of approximation of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to the function $f(x, y) = e^{\sin xy}$ is minimised for $\alpha = 0.02$ and $\beta = 0.03$, where $s=1$, $n=20$, and $m=5$ for the sequences a_n and b_m as provided in Table 1.

Figure 2 shows the convergence $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to the function $f(x, y) = x^2 y^2$ but this time for different α and β values, while keeping all the other parameters constant. Unlike in Figure 1, increasing α and β values upto 0.3 and 0.5 respectively provided better approximate results. It is evident that, similar to Figure 1, α and β values have a significant effect on the convergence of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to the multivariable function $f(x, y) = x^2 y^2$, which provide better approximate results.

Comparative results are given in Tables 2 and 3, for the errors of the approximation of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to the functions $f(x, y) = e^{\sin xy}$ and $f(x, y) = x^2 y^2$.

FIGURE 1: Convergence of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to multivariable function $f(x, y) = e^{\sin xy}$.

FIGURE 2: Convergence of $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to multivariable function $f(x, y) = x^2 y^2$.

TABLE 2: Errors of approximation $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to $f(x, y) = e^{\sin xy}$, $(|f(x, y) - \overline{C}_{n,m,s}^{0.02,0.03}(f; p_n, q_n, p_m, q_m; x, y)|)$.

x	y										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0.0015	0.0039	0.0075	0.0112	0.0149	0.0186	0.0222	0.0258	0.0294	0.0329	0.0364
0.1	0.0044	0.0015	0.0022	0.0032	0.0041	0.0049	0.0056	0.0063	0.0069	0.0074	0.0078
0.2	0.0086	0.0023	0.0034	0.0049	0.0063	0.0075	0.0086	0.0096	0.0104	0.011	0.0116
0.3	0.0128	0.0033	0.0046	0.0066	0.0083	0.0099	0.0114	0.0128	0.0141	0.0153	0.0164
0.4	0.0169	0.0038	0.0051	0.0072	0.0092	0.0109	0.0127	0.0145	0.0163	0.0183	0.0206
0.5	0.0208	0.004	0.0048	0.0066	0.0082	0.0099	0.0117	0.0138	0.0164	0.0198	0.0243
0.6	0.0245	0.0035	0.0033	0.0042	0.0051	0.0061	0.0076	0.01	0.0137	0.0193	0.0274
0.7	0.028	0.0025	0.0006	0.0002	0.0009	0.0012	0.0004	0.0021	0.0074	0.0164	0.0303
0.8	0.0313	0.0007	0.0038	0.0071	0.0104	0.0128	0.0133	0.0105	0.0031	0.0108	0.0332
0.9	0.0342	0.002	0.0101	0.0171	0.0239	0.0294	0.0317	0.0288	0.0185	0.0023	0.0366
1	0.0368	0.0057	0.0184	0.0303	0.0421	0.0517	0.0566	0.0537	0.0393	0.0093	0.0407

TABLE 3: Errors of approximation $\overline{C}_{n,m,s}^{\alpha,\beta}(f; p_n, q_n, p_m, q_m; x, y)$ to $f(x, y) = x^2 y^2$, $|f(x, y) - \overline{C}_{n,m,s}^{0.3,0.5}(f; p_n, q_n, p_m, q_m; x, y)|$.

x	y										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0	0	0	0.0001	0.0002	0.0004	0.0005	0.0007	0.001	0.0013	0.0017	0.0021
0.1	0.0001	0.0002	0.0004	0.0007	0.0012	0.0016	0.0022	0.0029	0.0037	0.0048	0.0066
0.2	0.0003	0.0005	0.001	0.0017	0.0025	0.0033	0.0043	0.0055	0.0069	0.0093	0.0134
0.3	0.0006	0.001	0.0019	0.0029	0.004	0.0052	0.0065	0.0079	0.0099	0.0136	0.0209
0.4	0.001	0.0017	0.003	0.0044	0.0058	0.0072	0.0086	0.0102	0.0127	0.0177	0.0289
0.5	0.0015	0.0024	0.0043	0.0061	0.0078	0.0093	0.0108	0.0124	0.0151	0.0217	0.0374
0.6	0.0021	0.0034	0.0058	0.0081	0.01	0.0116	0.0129	0.0143	0.0173	0.0256	0.0466
0.7	0.0028	0.0045	0.0075	0.0103	0.0124	0.014	0.0151	0.0162	0.0192	0.0292	0.0563
0.8	0.0036	0.0058	0.0094	0.0127	0.0151	0.0166	0.0173	0.0179	0.0209	0.0327	0.0665
0.9	0.0046	0.0072	0.0116	0.0154	0.018	0.0193	0.0195	0.0194	0.0222	0.0361	0.0773
1	0.0056	0.0087	0.014	0.0183	0.0211	0.0221	0.0217	0.0207	0.0233	0.0392	0.0887

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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