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Research Article

The Nehari Manifold for a Fractional p-Kirchhoff System Involving Sign-Changing Weight Function and Concave-Convex Nonlinearities

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In this paper, we are concerned with the following fractional p-Kirchhoff system with sign-changing nonlinearities: $\mathscr{M}(\int_{\mathbb{R}^{2n}}(|u(x)-u(y)|^p/|x-y|^{n+ps})dxdy)(-\Delta)_p^s u = \lambda a(x)|u|^{q-2}u+(\alpha/(\alpha+\beta))f(x)|u|^{\alpha-2}u|v|^\beta$, *in* Ω, $\mathscr{M}(\int_{\mathbb{R}^{2n}}(|v(x)-v(y)|^p/|x-y|^{n+ps})dxdy)(-\Delta)_p^s v = \mu b(x)|v|^{q-2}v + (\beta/(\alpha+\beta))f(x)|u|^\alpha|v|^{\beta-2}v$, *in* Ω, and $u = v = 0$, *in* $\mathbb{R}^n \setminus \Omega$, where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > ps$, $s \in (0,1)$, λ, μ are two real parameters, $1 < q < p < p(h+1) < \alpha+\beta < p_s^* = np/(n-ps)$, $\mathscr{M}$ is a continuous function, given by $\mathscr{M}(t) = k + lt^h$, $k > 0$, $l \geq 0$, $h \geq 1$, $a(x), b(x) \in L^{(\alpha+\beta)/(\alpha+\beta-q)}(\Omega)$ are sign changing and either $a^\pm = \max\{\pm a, 0\} \not\equiv 0$ or $b^\pm = \max\{\pm b, 0\} \not\equiv 0$, $f \in L(\overline{\Omega})$ with $\|f\|_\infty = 1$, and $f \geq 0$. Using Nehari manifold method, we prove that the system has at least two solutions with respect to the pair of parameters (λ, μ).

1. Introduction

In this article, we study the following fractional p-Kirchhoff problem (P) involving concave-convex nonlinearities and sign-changing weight functions:

$$\begin{aligned}
&\mathscr{M}\left(\int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)(-\Delta)_p^s u \\
&\quad = \lambda a(x)|u|^{q-2}u + \frac{\alpha}{\alpha+\beta} f(x)|u|^{\alpha-2}u|v|^\beta, \quad in\ \Omega, \\
&\mathscr{M}\left(\int_{\mathbb{R}^{2n}} \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}} dxdy\right)(-\Delta)_p^s v \\
&\quad = \mu b(x)|v|^{q-2}v + \frac{\beta}{\alpha+\beta} f(x)|u|^\alpha|v|^{\beta-2}v, \quad in\ \Omega, \\
&u = v = 0, \quad in\ \mathbb{R}^n \setminus \Omega,
\end{aligned} \tag{1}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$, $n > ps$, $s \in (0,1)$, λ, μ are two real parameters, $1 < q < p < p(h+1) < \alpha+\beta < p_s^* = np/(n-ps)$, $\mathscr{M}$ is a continuous function, given by $\mathscr{M}(t) = k + lt^h$, $k > 0$, $l > 0$, $h \geq 1$, and $(-\Delta)_p^s u$ is the fractional p-Laplacian operator, which is defined as follows:

$$\begin{aligned}
(-\Delta)_p^s u(x) &= 2 \\
&\cdot \lim_{\varepsilon \to 0^+} \int_{\mathbb{R}^n \setminus B_\varepsilon(x)} \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))}{|x-y|^{n+ps}} dy, \\
&\qquad x \in \mathbb{R}^n,
\end{aligned} \tag{2}$$

where $B_\varepsilon(x) = \{y \in \mathbb{R}^n : |x-y| < \varepsilon\}$, $a(x)$, $b(x)$, $f(x)$ satisfy the following assumptions: set $\gamma = (\alpha+\beta)/(\alpha+\beta-q)$,

(A_1) $a(x), b(x) \in L^\gamma(\Omega)$, and either $a^\pm = \max\{\pm a, 0\} \not\equiv 0$ or $b^\pm = \max\{\pm b, 0\} \not\equiv 0$;

(A_2) $f(x) \in L(\overline{\Omega})$ with $\|f\|_\infty = 1$ and $f \geq 0$.

In recent years, the Kirchhoff equations have received extensive attention from many scholars because of its wide application in many fields such as mathematical finance, continuum mechanics, etc. (see [1, 2]). There are many excellent and interesting results about the existence and multiplicity of solutions for nonlocal fractional problems. We can look up the literature [3, 4] for Laplace operator and [5–8] for the p-Laplacian case.

In addition, for a single equation with sign-changing weights functions, in [9, 10], the authors studied the existence and multiplicity of nonnegative solutions in subcritical and critical cases respectively. In the special case of $p = 2$, $s = 1$, and $\mathscr{M} = 1$, Tsung-Fang Wu [11] proved that system has least two nontrivial nonnegative solutions by using the Nehari manifold. Moreover, when $\mathscr{M}$ is not a constants, the authors [12] investigated the fractional p-Kirchhoff system with sign-changing nonlinearities, which is given by $\mathscr{M} = a + bt$. For a more general case $\mathscr{M} = k + lt^{\tau}$, Yang and An [13] show the system has at least two solutions with the help of Nehari manifold, but without considering sign-changing weights functions. Hence, inspired by above works, combining [12, 13], in this paper we will consider the new multiplicity result of the problem. Our conclusions can be seen as an extension of [12, 13].

To illustrate our result, we need to introduce some notations. Set Ω be an open set in $\mathbb{R}^n$, $0 < s < 1$, and $p \geq 1$. Define the usual fractional Soblev Space $W^{s,p}(\Omega)$ and its norm

$$\|u\|_{W^{s,p}(\Omega)} = \|u\|_{L^p(\Omega)} + \left(\int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{1/p}. \tag{3}$$

Let $K = \mathbb{R}^{2n} \setminus (\mathscr{C}\Omega \times \mathscr{C}\Omega)$ and $\mathscr{C}\Omega = \mathbb{R}^n \setminus \Omega$. Define the space X as

$$X = \left\{u \mid u : \mathbb{R}^n \longrightarrow \mathbb{R} \text{ is measurable}, \ u|_{\Omega} \in L^p(\Omega), \left(\frac{u(x)-u(y)}{(x-y)^{n/p+s}}\right) \in L^p(K)\right\}. \tag{4}$$

Then the space X of norm is defined by

$$\|u\|_X = \|u\|_{L^p(\Omega)} + \left(\int_K \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{1/p}. \tag{5}$$

Set Banach space X_0 to be the completion of the space $C_0^{\infty}(\Omega)$ in X, which is can be defined as the norm

$$\|u\|_{X_0} = \left(\int_K \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{1/p}. \tag{6}$$

Clearly, (6) is equivalent to the (3), as $u = 0$ a.e. in $\mathbb{R}^n \setminus \Omega$, we obtain that the integral in (3), (5), and (6) can be extended to the full space $\mathbb{R}^n$. According to the literature [14, 15], we know that $X_0 \hookrightarrow L^r(\Omega)$ is a continuous embedding for any $r \in [1, p^*]$, and compact whenever $r \in [1, p^*)$. When $\alpha + \beta \in (p, p^*)$, then, for any $u \in X_0$, we get that

$$\|u\|_{L^{\alpha+\beta}(\Omega)} \leq S \|u\|_{X_0}. \tag{7}$$

More about the properties of X and X_0, please consult [16] and the references therein. The reflexive Banach space $H = X_0 \times X_0$ is the Cartesian product of two spaces, which is endowed with the norm

$$\begin{aligned} \|(u,v)\| &= \left(\|u\|_{X_0}^p + \|v\|_{X_0}^p\right)^{1/p} \\ &= \left(\int_K \frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}} dxdy \right. \\ &\quad \left. + \int_K \frac{|v(x)-v(y)|^p}{|x-y|^{n+ps}} dxdy\right)^{1/p}. \end{aligned} \tag{8}$$

Definition 1. A pair of functions $(u, v) \in H$ is called weak solution of problem (P) if for all $(\phi, \varphi) \in H$ one has

$$\begin{aligned} &\mathscr{M}\left(\|u\|_{X_0}^p\right) \\ &\cdot \int_K \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+ps}} dxdy \\ &+ \mathscr{M}\left(\|v\|_{X_0}^p\right) \\ &\cdot \int_K \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}} dxdy \\ &= \lambda \int_\Omega a(x)|u|^{q-2} u\phi(x)\,dx + \mu \int_\Omega b(x)|v|^{q-2} v\varphi(x)\,dx \\ &+ \frac{\alpha}{\alpha+\beta} \int_\Omega f(x)|u|^{\alpha-2} u|v|^{\beta} \phi(x)\,dx + \frac{\beta}{\alpha+\beta} \\ &\cdot \int_\Omega f(x)|u|^{\alpha}|v|^{\beta-2} v\varphi(x)\,dx. \end{aligned} \tag{9}$$

We introduce the set

$$\begin{aligned} G_\Gamma = \Big\{&(\lambda,\mu) \in \mathbb{R}^2 \setminus \{(0,0)\} : \\ &\left(\left(|\lambda|\,\|a\|_\gamma\right)^{p/(p-q)} + \left(|\mu|\,\|b\|_\gamma\right)^{p/(p-q)}\right)^{(p-q)/p} \\ &< \Gamma(\alpha,\beta,q,S)\Big\}. \end{aligned} \tag{10}$$

Then we give our result as follows.

Theorem 2. *Assume that the conditions (A_1) and (A_2) hold. If $1 < q < p < p(h+1) < \alpha + \beta < p_s^*$, then there is an explicit constant $\Gamma > 0$ such that the problem (P) has at least two nonnegative solutions for $(\lambda, \mu) \in G_{(q/p)\Gamma}$.*

The rest of the paper is organized as follows. In Section 2, we give some notations and preliminaries about Nehari manifold and fibering maps. In Section 3, we prove Theorem 2.

2. The Variational Setting and Preliminaries

Define energy functional associated with problem (P) as follows:

$$T(u,v) = \frac{k}{p}\|(u,v)\|^p + \frac{l}{\xi}\|(u,v)\|^\xi - \frac{1}{m}\int_\Omega f(x)|u|^\alpha |v|^\beta dx - \frac{1}{q}Q(u,v), \tag{11}$$

where $\xi = p(h+1)$ and $m = \alpha + \beta$ and

$$Q(u,v) = \lambda\int_\Omega a(x)|u|^q dx + \mu\int_\Omega b(x)|v|^q dx. \tag{12}$$

By a direct calculation we obtain that $T(u,v) \in C^1(H,R)$, and for all $(\phi,\varphi) \in H$, we have

$$\begin{aligned}\left\langle T'(u,v),(\phi,\varphi)\right\rangle &= \mathscr{M}\left(\|u\|_{X_0}^p\right)\\ &\cdot\int_K \frac{|u(x)-u(y)|^{p-2}(u(x)-u(y))(\phi(x)-\phi(y))}{|x-y|^{n+ps}}dxdy\\ &+\mathscr{M}\left(\|v\|_{X_0}^p\right)\\ &\cdot\int_K \frac{|v(x)-v(y)|^{p-2}(v(x)-v(y))(\varphi(x)-\varphi(y))}{|x-y|^{n+ps}}dxdy\\ &-\lambda\int_\Omega a(x)|u|^{q-2}u\phi(x)dx - \mu\int_\Omega b(x)|v|^{q-2}v\varphi(x)dx\\ &-\frac{\alpha}{m}\int_\Omega f(x)|u|^{\alpha-2}u|v|^\beta\phi(x)dx - \frac{\beta}{m}\\ &\cdot\int_\Omega f(x)|u|^\alpha|v|^{\beta-2}v\varphi(x)dx.\end{aligned} \tag{13}$$

Then, the weak solution (u,v) is equivalent to being a critical point of T. Since T is not bounded below on H, therefore, we consider the Nehari manifold

$$N = \left\{(u,v) \in H\setminus(0,0) \mid \left\langle T'(u,v),(\phi,\varphi)\right\rangle = 0\right\}. \tag{14}$$

By (13), we get

$$\left\langle T'(u,v),(u,v)\right\rangle = k\|u,v\|^p + l\|u,v\|^\xi - \int_\Omega f(x)|u|^\alpha|v|^\beta dx - Q(u,v). \tag{15}$$

Hence, $(u,v) \in N$ if and only if

$$k\|u,v\|^p + l\|u,v\|^\xi - \int_\Omega f(x)|u|^\alpha|v|^\beta dx - Q(u,v) = 0. \tag{16}$$

Moreover, for any $(u,v) \in N$, the following equality holds:

$$\begin{aligned}T(u,v) &= \left(\frac{1}{p}-\frac{1}{m}\right)k\|(u,v)\|^p + \left(\frac{1}{\xi}-\frac{1}{m}\right)l\|(u,v)\|^\xi - \left(\frac{1}{q}-\frac{1}{m}\right)Q(u,v)\\ &= \left(\frac{1}{p}-\frac{1}{q}\right)k\|(u,v)\|^p + \left(\frac{1}{\xi}-\frac{1}{q}\right)l\|(u,v)\|^\xi - \left(\frac{1}{m}-\frac{1}{q}\right)\int_\Omega f(x)|u|^\alpha|v|^\beta dx.\end{aligned} \tag{17}$$

Obviously, the solution of the problem (P) depends on N. N is a set which we find that is smaller than X_0, so it is easier to study on N. Therefore, define our familiar fiber maps: $\psi_{u,v}: t \longrightarrow T(tu,tv)$ as follows:

$$\begin{aligned}\psi_{u,v}(t) &= T(tu,tv)\\ &= \frac{k}{p}t^p\|u,v\|^p + \frac{l}{\xi}t^\xi\|u,v\|^\xi - \frac{1}{m}t^m\int_\Omega f(x)|u|^\alpha|v|^\beta dx - \frac{1}{q}t^qQ(u,v),\end{aligned} \tag{18}$$

$$\psi'_{u,v}(1) = k\|u,v\|^p + l\|u,v\|^\xi - \int_\Omega f(x)|u|^\alpha|v|^\beta dx - Q(u,v), \tag{19}$$

$$\psi''_{u,v}(1) = (p-1)k\|(u,v)\|^p + (\xi-1)l\|(u,v)\|^\xi - (m-1)\int_\Omega f(x)|u|^\alpha|v|^\beta dx - (q-1)Q(u,v). \tag{20}$$

It follows from (19) that $(u,v) \in N$ if and only if $\psi'_{u,v}(1) = 0$. So it is natural that we divide N into three parts: local minima, local maxima, and points of inflection. For this, we let

$$\begin{aligned}N^\pm &:= \left\{(u,v) \in N : \psi''_{u,v}(1) \gtrless 0\right\},\\ N^0 &:= \left\{(u,v) \in N : \psi''_{u,v}(1) = 0\right\}.\end{aligned} \tag{21}$$

Define

$$\Phi(u,v) = \left\langle T'(u,v),(u,v)\right\rangle, \quad \forall (u,v) \in H. \tag{22}$$

Moreover, for every $(u,v) \in N$, from (16), we also have

$$\left\langle \Phi'(u,v),(u,v)\right\rangle = k(p-m)\|u,v\|^p + l(\xi-m)\|u,v\|^\xi - (q-m)Q(u,v)$$

$$= k(p-q)\|u,v\|^p + l(\xi-q)\|u,v\|^\xi - (m-q)\int_\Omega f(x)|u|^\alpha |v|^\beta dx. \tag{23}$$

Now, we give some preliminaries about main result.

Lemma 3. *If (u_0, v_0) is a minimizer of T on N and $(u_0, v_0) \notin N^0$. For every $(\lambda, \mu) \in G_\Gamma$, then we have $T'(u_0, v_0) = 0$ in H^*.*

Proof. For the detailed process of certification we can refer to the literature ([17], Theorem 2.3). For the convenience of the reader we give its completeness. If (u_0, v_0) is a local minimizer on N to T, by the theory of Lagrange multipliers, there is a constant $\theta \in \mathbb{R}$ such that

$$T'(u_0, v_0) = \theta\Phi'(u_0, v_0). \tag{24}$$

So

$$\begin{aligned}\left\langle T'(u_0, v_0), (u_0, v_0)\right\rangle &= \theta\left\langle \Phi'(u_0, v_0), (u_0, v_0)\right\rangle \\ &= \theta\psi''_{u,v}(1) = 0.\end{aligned} \tag{25}$$

But $\langle \Phi'(u_0, v_0), (u_0, v_0)\rangle \neq 0$, because of $(u_0, v_0) \notin N^0$. Therefore $\theta = 0$. This completes the proof. □

Lemma 4. *T is coercive and bounded below on N.*

Proof. By the Sobolev inequalities and Hölder inequalities, we get

$$\begin{aligned}Q(u,v) &\le |\lambda|\,\|a\|_{L^\gamma}\|u\|^q_{L^m} + |\mu|\,\|b\|_{L^\gamma}\|v\|^q_{L^m} \\ &\le S^q\left(\left(|\lambda|\,\|a\|_\gamma\right)^{p/(p-q)} + \left(|\mu|\,\|b\|_\gamma\right)^{p/(p-q)}\right)^{(p-q)/p} \\ &\quad\cdot \|(u,v)\|^q.\end{aligned} \tag{26}$$

Also, according to (17) and (26), we have

$$\begin{aligned}T(u,v) &= \left(\frac{1}{p}-\frac{1}{m}\right)k\|(u,v)\|^p + \left(\frac{1}{\xi}-\frac{1}{m}\right) \\ &\quad\cdot l\|(u,v)\|^\xi - \left(\frac{1}{q}-\frac{1}{m}\right)Q(u,v) \ge \left(\frac{1}{p}-\frac{1}{m}\right) \\ &\quad\cdot k\|(u,v)\|^p + \left(\frac{1}{\xi}-\frac{1}{m}\right)l\|(u,v)\|^\xi - \left(\frac{1}{q}-\frac{1}{m}\right) \\ &\quad\cdot S^q\left(\left(|\lambda|\,\|a\|_\gamma\right)^{p/(p-q)} + \left(|\mu|\,\|b\|_\gamma\right)^{p/(p-q)}\right)^{(p-q)/p} \\ &\quad\cdot \|(u,v)\|^q.\end{aligned} \tag{27}$$

As $q < p < \xi < m$, from above inequality, we can conclude T is coercive and bounded below on N. We complete this proof. □

Lemma 5. *Under condition (A_2), there exists $\Gamma > 0$, given by*

$$\Gamma = \frac{k(m-p)}{(m-q)S^q}\left(\frac{k(p-q)}{(m-q)S^m}\right)^{(p-q)/(m-p)}, \tag{28}$$

such that, for any $(\lambda, \mu) \in G_\Gamma$, we have $N^0 = \emptyset$.

Proof. We argue by contradiction, moreover dividing the following two cases: assume that there exist $(\lambda, \mu) \in G_\Gamma$ such that $N^0 \neq \emptyset$. Then for $(u, v) \in N^0$, we have

$$\begin{aligned}\Phi(u,v) &= 0, \\ \left\langle \Phi'(u,v), (u,v)\right\rangle &= 0.\end{aligned} \tag{29}$$

□

Case 1. $Q(u, v) = 0$. From (29), (19), and (20), we have

$$\begin{aligned}0 &= (p-1)k\|(u,v)\|^p + (\xi-1)l\|(u,v)\|^\xi \\ &\quad - (m-1)\int_\Omega f(x)|u|^\alpha|v|^\beta dx \\ &= (p-m)k\|(u,v)\|^p + (\xi-m)l\|(u,v)\|^\xi < 0,\end{aligned} \tag{30}$$

which is a contradiction.

Case 2. $Q(u, v) \neq 0$, then it follows from (29), (23), and (26) that

$$\|(u,v)\| \le \left(\frac{(m-q)S^q\left(\left(|\lambda|\,\|a\|_\Gamma\right)^{p/(p-q)} + \left(|\mu|\,\|b\|_\Gamma\right)^{p/(p-q)}\right)^{(p-q)/p}}{k(m-p)}\right)^{1/(p-q)}. \tag{31}$$

In addition, by condition (A_2) and Young's inequality, we get

$$\begin{aligned}\int_\Omega f(x)|u|^\alpha|v|^\beta dx &\le \frac{\alpha}{m}\int_\Omega f(x)|u|^m dx \\ &\quad + \frac{\beta}{m}\int_\Omega f(x)|v|^m dx \\ &\le \frac{\alpha}{m}\|f\|_\infty S^m\|u\|^m_{X_0}\end{aligned}$$

$$+\frac{\beta}{m}\|f\|_{\infty}S^{m}\|v\|_{X_0}^{m}\leq S^{m}\|(u,v)\|^{m}. \tag{32}$$

By (29), (32), and (23), we have

$$\begin{aligned}k(p-q)\|(u,v)\|^{p}&\leq(m-q)\int_{\Omega}f(x)|u|^{\alpha}|v|^{\beta}dx\\&\leq(m-q)S^{m}\|(u,v)\|^{m},\end{aligned} \tag{33}$$

and hence, we obtain

$$\|(u,v)\|\geq\left(\frac{k(p-q)}{(m-q)S^{m}}\right)^{1/(m-p)}. \tag{34}$$

From (31) and (34), we get

$$\begin{aligned}&\left(\left(|\lambda|\|a\|_{\gamma}\right)^{p/(p-q)}+\left(|\mu|\|b\|_{\gamma}\right)^{p/(p-q)}\right)^{(p-q)/p}\\&\geq\frac{k(m-p)}{(m-q)S^{q}}\left(\frac{k(p-q)}{(m-q)S^{m}}\right)^{(p-q)/(m-p)}=\Gamma,\end{aligned} \tag{35}$$

which contradicts $(\lambda,\mu)\in G_{\Gamma}$. We have completed the proof of this lemma.

According to Lemma 4 and Lemma 5, we know $N=N^{+}+N^{-}$ for $(\lambda,\mu)\in G_{\Gamma}$, and we set

$$\begin{aligned}Z^{+}&=\inf_{(u,v)\in N^{+}}T(u,v),\\Z^{-}&=\inf_{(u,v)\in N^{-}}T(u,v).\end{aligned} \tag{36}$$

We introduce the following lemma.

Lemma 6. *Assume that $(\lambda,\mu)\in G_{(q/p)\Gamma}$. Then, we have*
(i) $Z^{+}<0$,
(ii) $Z^{-}>e_0$ for some $e_0>0$.

Proof. (i) Set $(u,v)\in N^{+}$, we know $\langle\Phi'(u,v),(u,v)\rangle>0$, and from (23), we have

$$\begin{aligned}\int_{\Omega}f(x)|u|^{\alpha}|v|^{\beta}dx&<\frac{k(p-q)}{m-q}\|(u,v)\|^{p}\\&\quad+\frac{l(\xi-q)}{m-q}\|(u,v)\|^{\xi}.\end{aligned} \tag{37}$$

Put (37) into (17),

$$\begin{aligned}T(u,v)&<-\frac{k(p-q)}{mqp}\|(u,v)\|^{p}\\&\quad-\frac{l(m-p)(p-q)}{mqp}\|(u,v)\|^{\xi}<0,\end{aligned} \tag{38}$$

which implies $Z^{+}=\inf_{(u,v)\in N^{+}}T(u,v)<0$.

(ii) It follows from (17) and (23) that

$$\begin{aligned}T(u,v)&\geq\frac{(m-p)k}{mp}\|(u,v)\|^{p}-\frac{m-q}{mq}Q(u,v)\\&\geq\frac{(m-p)k}{mp}\|(u,v)\|^{p}-\frac{m-q}{mq}\\&\quad\cdot S^{q}\left(\left(|\lambda|\|a\|_{\gamma}\right)^{p/(p-q)}+\left(|\mu|\|b\|_{\gamma}\right)^{p/(p-q)}\right)^{(p-q)/p}\\&\quad\cdot\|(u,v)\|^{q}=\|(u,v)\|^{q}\left(\frac{(m-p)k}{mp}\|(u,v)\|^{p-q}\right.\\&\quad-\frac{m-q}{mq}S^{q}\left(\left(|\lambda|\|a\|_{\gamma}\right)^{p/(p-q)}\right.\\&\quad\left.\left.+\left(|\mu|\|b\|_{\gamma}\right)^{p/(p-q)}\right)^{(p-q)/p}\right).\end{aligned} \tag{39}$$

Since $(u,v)\in N^{-}$, then $\langle\Phi'(u,v),(u,v)\rangle<0$. Hence, according to (17), we obtain (34), combining above inequality and (34), and we get

$$\begin{aligned}T(u,v)&\geq\left(\frac{k(p-q)}{(m-q)S^{m}}\right)^{q/(m-p)}\\&\quad\cdot\left(\frac{(m-p)k}{mp}\left(\frac{k(p-q)}{(m-q)S^{m}}\right)^{(p-q)/(m-p)}-\frac{m-q}{mq}\right.\\&\quad\cdot S^{q}\left(\left(|\lambda|\|a\|_{\gamma}\right)^{p/(p-q)}\right.\\&\quad\left.\left.+\left(|\mu|\|b\|_{\gamma}\right)^{p/(p-q)}\right)^{(p-q)/p}\right).\end{aligned} \tag{40}$$

Obviously, if $((|\lambda|\|a\|_{\gamma})^{p/(p-q)}+(|\mu|\|b\|_{\gamma})^{p/(p-q)})^{(p-q)/p}<(q/p)\Gamma$, then there is a constant $e_0>0$ such that $Z^{-}>e_0$. □

Under condition (A_2), we study the behavior of the fibering map with respect to the sign of $Q(u,v)$. We divide into two cases.

Case 1 ($Q(u,v)>0$). Fix $(u,v)\in H$. Let

$$\begin{aligned}O_{u,v}(t)&=kt^{p-q}\|(u,v)\|^{p}+lt^{\xi-q}\|(uv)\|^{\xi}\\&\quad-t^{m-q}\int_{\Omega}f(x)|u|^{\alpha}|v|^{\beta}dx.\end{aligned} \tag{41}$$

Clearly, $(tu,tv)\in N$ if and only if $O_{u,v}(t)=Q(u,v)$.

$$\begin{aligned}O'_{u,v}(t)&=kt^{p-q-1}\|(u,v)\|^{p}+lt^{\xi-q-1}\|(u,v)\|^{\xi}\\&\quad-t^{m-q-1}\int_{\Omega}f(x)|u|^{\alpha}|v|^{\beta}dx.\end{aligned} \tag{42}$$

It is obvious that $O_{u,v}(t)\longrightarrow-\infty$ as $t\longrightarrow\infty$. By (42), we know that

$$\begin{aligned}\lim_{t\longrightarrow0^{+}}O'_{u,v}(t)&>0,\\\lim_{t\longrightarrow\infty}O'_{u,v}(t)&<0.\end{aligned} \tag{43}$$

Thus, there is a unique $t^* = t^*(u,v) > 0$, such that $O_{u,v}(t)$ is increasing on $(0,t^*)$, decreasing on (t^*,∞). Moreover, $O'_{u,v}(t^*) = 0$, in addition,

$$O_{u,v}(t^*) = (t^*)^{-q}\left(k(t^*)^p \|(u,v)\|^p + l(t^*)^\xi \|(u,v)\|^\xi - (t^*)^m \int_\Omega f(x)|u|^\alpha |v|^\beta dx\right), \tag{44}$$

where t^* is the root of

$$k(p-q)(t^*)^p \|(u,v)\|^p + l(\xi-q)(t^*)^\xi \|(u,v)\|^\xi - (m-q)(t^*)^m \int_\Omega f(x)|u|^\alpha |v|^\beta dx = 0. \tag{45}$$

From above the equality, we obtain

$$t^* \geq \left(\frac{k(p-q)\|(u,v)\|^p}{(m-q)\int_\Omega f(x)|u|^\alpha |v|^\beta dx}\right)^{1/(m-p)} := t_0. \tag{46}$$

Since $(\lambda,\mu) \in G_{(q/p)\Gamma}$, according to (26) and (46), we get

$$\begin{aligned} O_{u,v}(t^*) &\geq O_{u,v}(t_0) \geq k t_0^{p-q} \|(u,v)\|^p - t_0^{m-p}\int_\Omega f(x)|u|^\alpha |v|^\beta dx \geq \|(u,v)\|^q \\ &\cdot \frac{k(m-p)}{(m-q)}\left(\frac{k(p-q)}{(m-q)S^m}\right)^{(p-q)/(m-p)} \geq \|(u,v)\|^q \\ &\cdot S^q \frac{q}{p}\frac{k(m-p)}{(m-q)S^q}\left(\frac{k(p-q)}{(m-q)S^m}\right)^{(p-q)/(m-p)} \\ &\geq S^q\left(\left(|\lambda|\,\|a\|_\gamma\right)^{p/(p-q)} + \left(|\mu|\,\|b\|_\gamma\right)^{p/(p-q)}\right)^{(p-q)/p} \\ &\cdot \|(u,v)\|^q \geq Q(u,v). \end{aligned} \tag{47}$$

Hence, there are unique $t^+ < t^*$ and $t^- > t^*$ such that $O_{u,v}(t^+) = O_{u,v}(t^-) = Q(u,v)$. It means that $(t^+u,t^+v) \in N$ and $(t^-u,t^-v) \in N$. Moreover it is easy to see that $\langle \Phi'(tu,tv),(tu,tv)\rangle = t^{q+1}O'_{u,v}(t)$, also $O'_{u,v}(t^+) > 0$ and $O'_{u,v}(t^-) < 0$ imply $(t^+u,t^+v) \in N^+$ and $(t^-u,t^-v) \in N^-$. Since $\psi'_{u,v}(t) = t^q(O_{u,v}(t) - Q(u,v))$, then $\psi'_{u,v}(t) < 0$ for any $t \in [0,t^+)$ and $\psi'_{u,v}(t) > 0$ for any $t \in (t^+,t^-)$. Thus, $T(t^+u,t^+v) = \inf_{0<t<t^-} T(tu,tv)$. Furthermore, $\psi'_{u,v}(t) > 0$ for any $t \in [t^+,t^-)$, $\psi'_{u,v}(t^-) = 0$, and $\psi'_{u,v}(t) < 0$ for any $t \in (t^-,\infty)$ imply that $T(t^-u,t^-v) = \sup_{t>t^*} T(tu,tv)$.

Case 2 ($Q(u,v) \leq 0$). As we know that $O_{u,v}(t) \longrightarrow -\infty$ as $t \longrightarrow \infty$, therefore for any $(\lambda,\mu) \in \mathbb{R}$, there is a unique $t_* > 0$ such that $O_{u,v}(t_*) = Q(u,v)$; moreover $\psi'_{u,v}(t) > 0$ for any $t \in [0,t_*)$ and $\psi'_{u,v}(t) < 0$ for any $t \in (t_*,\infty)$, which implies that $(t_*u,t_*v) \in N^-$ and $T(t_*u,t_*v) = \sup_{t\geq 0} T(tu,tv)$.

Thus according to the above discussion we obtain that the following lemma.

Lemma 7. *Under condition (A_2), for any $(\lambda,\mu) \in G_{(q/p)\Gamma}$, we have the following:*

(i) If $Q(u,v) \leq 0$, then there is a unique $t_* > 0$ such that $(t_*u,t_*v) \in N^-$, and

$$T(t_*u,t_*v) = \sup_{t\geq 0} T(tu,tv). \tag{48}$$

(ii) If $Q(u,v) > 0$, then there exist a unique $t^* = t^*(u,v) > 0$ and unique $t^+(u,v) < t^* < t^-(u,v)$ such that $(t^+u,t^+v) \in N^+$, $(t^-u,t^-v) \in N^-$, and

$$\begin{aligned} T(t^+u,t^+v) &= \inf_{0<t<t^-} T(tu,tv), \\ T(t^-u,t^-v) &= \sup_{t>t^*} T(tu,tv). \end{aligned} \tag{49}$$

3. Proof of the Main Result

In this section, we establish the existence of minimizers in N^+ and N^-.

Proposition 8. *Under condition (A_2), if $(\lambda,\mu) \in G_{(q/p)\Gamma}$, then the functional T has a minimizer (u_0^+,v_0^+) in N^+ and fulfills the following:*

(i) $T(u_0^+,v_0^+) = Z^+ < 0$,

(ii) (u_0^+,v_0^+) is a solution of problem (P).

Proof. Since T is bounded from below on N^+, there is a minimizing sequence $\{(u_k,v_k)\} \in N^+$ such that

$$\lim_{k\longrightarrow\infty} T(u_k,v_k) = \inf_{(u,v)\in N^+} T(u,v) = Z^+. \tag{50}$$

Hence, by Lemma 4, then (u_k,v_k) is bounded on H. So there exists $(u_0^+,v_0^+) \in H$, up to a subsequence, such that

$$\begin{aligned} u_k &\rightharpoonup u_0^+, \\ v_k &\rightharpoonup v_0^+ \end{aligned} \quad \text{weakly in } H \text{ as } k \longrightarrow \infty. \tag{51}$$

Moreover, according to ([3], lemma 8),

$$\begin{aligned} u_k &\longrightarrow u_0^+, \\ v_k &\longrightarrow v_0^+ \end{aligned} \quad \text{strongly in } L^r(\Omega) \text{ as } k \longrightarrow \infty, \tag{52}$$

$$\begin{aligned} u_k(x) &\longrightarrow u_0^+(x), \\ v_k(x) &\longrightarrow v_0^+(x) \end{aligned} \quad \text{a.e. in } \Omega \text{ as } k \longrightarrow \infty.$$

For each $1 \leq r < p^*$, by ([18], Theorem IV-9), there exists $s(x) \in L^r(\mathbb{R}^n)$ such that

$$\begin{aligned} |u_k(x)| &\leq s(x), \\ |v_k(x)| &\leq s(x), \end{aligned} \tag{53}$$

$$\text{almost everywhere in } \mathbb{R}^n.$$

Hence, by the dominated convergence theorem, we get

$$\begin{aligned}\lim_{k\longrightarrow\infty}&\left(\lambda\int_\Omega a(x)|u_k|^q\,dx+\mu\int_\Omega b(x)|v_k|^q\,dx\right)\\&=\int_\Omega\lim_{k\longrightarrow\infty}\left(a(x)|u_k|^q+b(x)|v_k|^q\right)dx\\&=\int_\Omega\left(a(x)|u_0^+|^q+b(x)|v_0^+|^q\right)dx,\end{aligned}\tag{54}$$

and

$$\begin{aligned}\lim_{k\longrightarrow\infty}&\int_\Omega f(x)|u_k|^\alpha|v_k|^\beta\,dx\\&=\int_\Omega f(x)|u_0^+|^\alpha|v_0^+|^\beta\,dx,\end{aligned}\tag{55}$$

as $n\longrightarrow\infty$. Now, on N, from (17), we have

$$\begin{aligned}\left(\frac{1}{q}-\frac{1}{m}\right)Q(u_k,v_k)&=\left(\frac{1}{p}-\frac{1}{m}\right)k\|(u_k,v_k\|^p\\&\quad+\left(\frac{1}{\xi}-\frac{1}{m}\right)l\|(u_k,v_k)\|^\xi\\&\quad-T(u_k,v_k).\end{aligned}\tag{56}$$

Letting $k\longrightarrow\infty$, since $q<p<\xi<m$, from Lemma 6, (50), and (54), we get

$$Q(u_0^+,v_0^+)>0.\tag{57}$$

From Lemma 7, there exists $t^+<t^*$ such that $(t^+u_0^+,t^+v_0^+)\in N^+$ and $\langle T'(t^+u_0^+,t^+v_0^+),(t^+u_0^+,t^+v_0^+)\rangle=0$. Next we show that $(u_k,v_k)\longrightarrow(u_0^+,v_0^+)$ strongly in H. If this is not true, then

$$\|(u_0^+,v_0^+)\|<\liminf_{k\longrightarrow\infty}\|(u_k,v_k)\|.\tag{58}$$

Since

$$\begin{aligned}&\left\langle T'(t^+u_k,t^+v_k),(t^+u_k,t^+v_k)\right\rangle\\&\quad=k(t^+)^p\|(u_k,v_k)\|^p+l(t^+)^\xi\|(u_k,v_k)\|^\xi\\&\qquad-(t^+)^m\int_\Omega f(x)|u_k|^\alpha|v_k|^\beta\,dx\\&\qquad-(t^+)^qQ(u_k,v_k),\end{aligned}\tag{59}$$

and

$$\begin{aligned}&\left\langle T'(t^+u_0^+,t^+v_0^+),(t^+u_0^+,t^+v_0^+)\right\rangle\\&\quad=k(t^+)^p\|(u_0^+,v_0^+)\|^p+l(t^+)^\xi\|(u_0^+,v_0^+)\|^\xi\\&\qquad-(t^+)^m\int_\Omega f(x)|u_0^+|^\alpha|v_0^+|^\beta\,dx\\&\qquad-(t^+)^qQ(u_0^+,v_0^+),\end{aligned}\tag{60}$$

we obtain

$$\begin{aligned}&\lim_{k\longrightarrow\infty}\left\langle T'(t^+u_k,t^+v_k),(t^+u_k,t^+v_k)\right\rangle\\&\quad>\left\langle T'(t^+u_0^+,t^+v_0^+),(t^+u_0^+,t^+v_0^+)\right\rangle=0.\end{aligned}\tag{61}$$

It means that $\langle T'(t^+u_k,t^+v_k),(t^+u_k,t^+v_k)\rangle>0$ for n large enough. As $\{(u_k,v_k\}\in N^+$, it is obvious to know that $\langle T'(u_k,v_k),(u_k,v_k)\rangle=0$, and $\langle T'(tu_k,tv_k),(tu_k,tv_k)\rangle<0$ for $0<t<1$. Thus we have $t^+>1$. On the other hand, $T(tu_0^+,tv_0^+)$ is decreasing on $(0,t^+)$, so

$$\begin{aligned}T(t^+u_0^+,t^+v_0^+)&\le T(u_0^+,v_0^+)<\lim_{k\longrightarrow\infty}T(u_k,v_k)\\&=\inf_{(u,v)\in N^+}T(u,v)=Z^+,\end{aligned}\tag{62}$$

which is a contradiction. Thus $(u_k,v_k)\longrightarrow(u_0^+,u_0^+)$ strongly in H. This means

$$\begin{aligned}T(u_k,v_k)\longrightarrow T(u_0^+,v_0^+)&=\inf_{(u,v)\in N^+}T(u,v)=Z^+,\\&\qquad k\longrightarrow\infty.\end{aligned}\tag{63}$$

That is, (u_0^+,v_0^+) is a minimizer of T on N^+; using Lemma 3, (u_0^+,v_0^+) is a solution of problem (P). □

Proposition 9. *Under condition (A_2), if $(\lambda,\mu)\in G_{(q/p)\Gamma}$, then the functional T has a minimizer (u_0^-,v_0^-) in N^- and fulfills the following:*

(i) $T(u_0^-,v_0^-)=Z^->0$,

(ii) (u_0^-,v_0^-) is a solution of problem (P).

Proof. T is bounded from below such that

$$\lim_{k\longrightarrow\infty}T(\widetilde{u_k},\widetilde{v_k})=\inf_{(u,v)\in N^-}T(u,v)=Z^-.\tag{64}$$

It is similar to the proof of the Proposition 8, so there exists $(u_0^-,v_0^-)\in H$, up to a subsequence, such that

$$\begin{aligned}&\widetilde{u_k}\rightharpoonup u_0^-,\\&\widetilde{v_k}\rightharpoonup v_0^-\\&\qquad\text{weakly in } H \text{ as } k\longrightarrow\infty.\end{aligned}\tag{65}$$

Moreover,

$$\begin{aligned}&\widetilde{u_k}\longrightarrow u_0^-,\\&\widetilde{v_k}\longrightarrow v_0^-\\&\qquad\text{strongly in } L^r(\Omega) \text{ as } k\longrightarrow\infty,\\&\widetilde{u_k}(x)\longrightarrow u_0^-(x),\\&\widetilde{v_k}(x)\longrightarrow v_0^-(x)\\&\qquad\text{a.e. in } \Omega \text{ as } k\longrightarrow\infty.\end{aligned}\tag{66}$$

For each $1 \le r < p^*$, by the dominated convergence theorem, we also get

$$\begin{aligned}\lim_{k\longrightarrow\infty}&\left(\lambda\int_\Omega a(x)\left|\widetilde{u_k}\right|^q dx+\mu\int_\Omega b(x)\left|\widetilde{v_k}\right|^q dx\right)\\&=\int_\Omega\lim_{k\longrightarrow\infty}\left(a(x)\left|\widetilde{u_k}\right|^q+b(x)\left|\widetilde{v_k}\right|^q\right)dx\\&=\int_\Omega\left(a(x)\left|u_0^-\right|^q+b(x)\left|v_0^-\right|^q\right)dx,\end{aligned}\tag{67}$$

and

$$\begin{aligned}\lim_{k\longrightarrow\infty}&\int_\Omega f(x)\left|\widetilde{u_k}\right|^\alpha\left|\widetilde{v_k}\right|^\beta dx\\&=\int_\Omega f(x)\left|u_0^-\right|^\alpha\left|v_0^-\right|^\beta dx,\end{aligned}\tag{68}$$

and similarly, by Lemma 7, there is a t^- such that $(t^-u_0^-, t^-v_0^-) \in N^-$. Next we show that $(\widetilde{u_k}, \widetilde{v_k}) \longrightarrow (u_0^-, v_0^-)$ strongly in H. Suppose that this is not true, then

$$\left\|(u_0^-, v_0^-)\right\| < \liminf_{k\longrightarrow\infty}\left\|(\widetilde{u_k}, \widetilde{v_k})\right\|.\tag{69}$$

Since $\{(\widetilde{u_k}, \widetilde{v_k})\} \in N^-$ and $T(\widetilde{u_k}, \widetilde{v_k}) \ge T(t\widetilde{u_k}, t\widetilde{v_k})$ for any $t > 0$, we also have

$$\begin{aligned}T\left(t^-u_0^-, t^-v_0^-\right)&<\lim_{k\longrightarrow\infty}T\left(t^-\widetilde{u_k}, t^-\widetilde{v_k}\right)\\&=\inf_{(u,v)\in N^-}T\left(\widetilde{u_k}, \widetilde{v_k}\right)=Z^-,\end{aligned}\tag{70}$$

which is a contradiction. Hence $(\widetilde{u_k}, \widetilde{v_k}) \longrightarrow (u_0^-, v_0^-)$ strongly in H. This implies

$$T\left(\widetilde{u_k}, \widetilde{v_k}\right)\longrightarrow T\left(u_0^-, v_0^-\right)=\inf_{(u,v)\in N^-}T(u,v)=Z^-,\quad k\longrightarrow\infty.\tag{71}$$

Namely, (u_0^-, v_0^-) is a minimizer of T on N^-; by Lemma 3, (u_0^-, v_0^-) is a solution of problem (P). □

Proof of Theorem 2. According to Propositions 8 and 9, we obtain that, for $(\lambda, \mu) \in G_{(q/p)\Gamma}$, problem (P) has two solutions $(u_0^+, v_0^+) \in N^+$ and $(u_0^-, v_0^-) \in N^-$ in H. Since

$$\begin{aligned}&T\left(u_0^\pm, v_0^\pm\right)=T\left(\left|u_0^\pm\right|, \left|v_0^\pm\right|\right),\\&\left(\left|u_0^\pm\right|, \left|v_0^\pm\right|\right)\in N^\pm,\end{aligned}\tag{72}$$

moreover $N^+ \cap N^- = \emptyset$, so we get that $(u_0^\pm, v_0^\pm)$ are distinct nonnegative solutions. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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