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*Research Article*

# Solutions for a Class of Hadamard Fractional Boundary Value Problems with Sign-Changing Nonlinearity

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Using fixed point methods we establish some existence theorems of positive (nontrivial) solutions for a class of Hadamard fractional boundary value problems with sign-changing nonlinearity.

## 1. Introduction

In this paper, using fixed point methods we study the existence of positive (nontrivial) solutions for the Hadamard fractional boundary value problems with sign-changing nonlinearity:

$$\begin{aligned} -D^{\alpha}u(t) &= f(t,u(t)), \quad t\in[1,e], \\ u(1) &= \delta u(1) = \delta u(e) = 0, \end{aligned} \tag{1}$$

where $\alpha \in (2,3]$ is a real number, $D^{\alpha}$ is the left-sided Hadamard fractional derivative of order $\alpha$, $\delta u(t) = tdu(t)/dt$, and $f \in C([1,e]\times\mathbb{R}^{+},\mathbb{R})$ is a sign-changing function; i.e., there exists a constant $M > 0$ such that

(H0) $f(t,u) + M \geq 0$ for all $(t,u) \in [1,e]\times\mathbb{R}^{+}$.

As is known, fractional differential equations have been paid special attention by many researchers for the reason that they serve as an excellent tool for wide applications in various disciplines of science and engineering such as mechanics, electricity, chemistry, and control theory; for more details, we refer to books [1–3]. In recent years, there have been a large number of papers dealing with the existence of solutions of nonlinear initial (boundary) value problems of fractional differential equations by using some techniques of nonlinear analysis, such as fixed-point results [4–13], iterative methods [14–23], the topological degree [24–29], the Leray-Schauder alternative [30, 31], and stability [32].

In [4], the authors studied the following abstract evolution of the system for HIV-1 population dynamics, which takes the form in fractional sense:

$$\begin{aligned} &D_t^{\alpha}u(t) + \lambda f\left(t,u(t),D_t^{\beta}u(t),v(t)\right) = 0, \\ &D_t^{\gamma}v(t) + \lambda g(t,u(t)) = 0, \quad 0<t<1, \\ &D_t^{\beta}u(0) = D_t^{\beta+1}u(0) = 0, \\ &D_t^{\beta}u(1) = \int_0^1 D_t^{\beta}u(s)\,dA(s), \\ &v(0) = v'(0) = 0, \\ &v(1) = \int_0^1 v(s)\,dB(s), \end{aligned} \tag{2}$$

where $f:(0,1)\times[0,+\infty)^3 \longrightarrow (-\infty,+\infty)$ and $g:(0,1)\times[0,+\infty) \longrightarrow (-\infty,+\infty)$ are two semipositone functions. By using the Guo-Krasnosel'skii fixed point theorem, they not only obtained the existence of positive solutions for (2) but also discussed the effect of parameters $\lambda$ on the existence of solutions.

In [14], the authors adopted generalized $\alpha$-contractive map to study some fractional integro-differential equations with the Caputo-Fabrizio derivation and obtained the

approximate solutions for these problems by using of some appropriate Lipschitz conditions for their nonlinearities.

In [15], Cui used the convergence of Cauchy sequences in complete spaces to obtain the unique solution for the fractional boundary value problems:

$$\begin{aligned} D_t^p x(t) + p(t) f(t, x(t)) + q(t) = 0, \quad t \in (0,1), \\ x(0) = x'(0) = 0, \\ x(1) = 0, \end{aligned} \tag{3}$$

where $f$ is a Lipschitz continuous function, with the Lipschitz constant associated with the first eigenvalue for the relevant operator. This method can also be applied in papers [16, 17] and references therein.

However, as a generalization of fractional calculus by Riemann and Liouville, Hadamard fractional equations have seldom been studied in the literature; we only refer to [8–10, 22, 23, 29, 32] and references therein. In [8], the authors used the Guo-Krasnosel'skii fixed point theorem on cones to establish the existence and nonexistence of positive solutions for (1) with nonnegative nonlinearity $\lambda a(t) f(u)$ and considered solvability for the influence of the parameter intervals.

In [32], the authors used Banach and Schauder fixed point theorem to obtain the existence and Hyers-Ulam stability of solutions for Hadamard fractional impulsive Cauchy problems of the form

$$\begin{aligned} {}_H D_{1^+}^{\alpha} u(t) = f(t, u(t)), \alpha \in (0,1), \\ t \in (1,e] \setminus \{t_1, t_2, \ldots, t_m\}, \\ \Delta u(t_i) = {}_H J_{1^+}^{1-\alpha} u(t_i^+) - {}_H J_{1^+}^{1-\alpha} u(t_i^-) = p_i, \\ p_i \in \mathbb{R}, \ i = 1, 2, \ldots, m, \\ {}_H J_{1^+}^{1-\alpha} u(1^+) = u_0, \quad u_0 \in \mathbb{R}, \end{aligned} \tag{4}$$

where $f$ satisfies a Lipschitz condition.

In this paper, motivated by works aforementioned, we used fixed point methods to study the existence of solutions for (1) with sign-changing nonlinearity. We have the main results: (i) when the nonlinear term $f$ grows both superlinearly and sublinearly at $\infty$, we use the fixed point index theory to obtain two existence theorems of positive solutions for (1); (ii) when $f$ satisfies an appropriate Lipschitz condition, we obtain a unique solution for (1) and establish a sequence of iterations uniformly converges to the unique solution.

## 2. Preliminaries

*Definition 1* (see [1–3]). The left-sided Hadamard fractional derivative of order $\alpha \in [n-1, n)$, $n \in \mathbb{Z}_+$, of a function $f$ is given by

$$D^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \left(t \frac{d}{dt}\right)^n \int_1^t \left(\ln \frac{t}{s}\right)^{n-\alpha+1} f(s) \frac{ds}{s}, \quad 1 \le t \le e, \tag{5}$$

where $\Gamma(\cdot)$ is the Gamma function.

We now offer Green's function for (1). From Lemma 2.1 of [8], (1) is equivalent to the integral equation

$$u(t) = \int_1^e G(t,s) f(s, u(s)) \frac{ds}{s}, \quad t \in [1,e], \tag{6}$$

where

$$G(t,s) = \frac{1}{\Gamma(\alpha)} \cdot \begin{cases} (1-\ln s)^{\alpha-2} (\ln t)^{\alpha-1}, & 1 \le t \le s \le e, \\ (1-\ln s)^{\alpha-2} (\ln t)^{\alpha-1} - \left(\ln\left(\frac{t}{s}\right)\right)^{\alpha-1}, & 1 \le s \le t \le e. \end{cases} \tag{7}$$

**Lemma 2** (see [10, Lemma 2.2]). *Let $G$ be defined by (7). Then the following inequalities are satisfied:*

$$\begin{aligned} G(t,s) \ge 0, \\ (\ln t)^{\alpha-1} G(e,s) \le G(t,s) \le G(e,s), \\ \forall (t,s) \in [1,e] \times [1,e], \end{aligned} \tag{8}$$

*where $G(e,s) = (1-\ln s)^{\alpha-2} - (1-\ln s)^{\alpha-1}$ and $\forall s \in [1,e]$.*

**Lemma 3.** *Let $\varphi(t) = (1-\ln t)^{\alpha-2} - (1-\ln t)^{\alpha-1}$ and $\forall t \in [1,e]$. Then the following inequalities are satisfied:*

$$\begin{aligned} \int_1^e (\ln t)^{\alpha-1} \varphi(t) \frac{dt}{t} \cdot \varphi(s) \le \int_1^e G(t,s) \varphi(t) \frac{dt}{t} \\ \le \int_1^e \varphi(t) \frac{dt}{t} \cdot \varphi(s), \quad \forall s \in [1,e]. \end{aligned} \tag{9}$$

*This is a direct result from Lemma 2, so we omit its proof. Moreover, for convenience, let*

$$\begin{aligned} \kappa_1 = \int_1^e (\ln t)^{\alpha-1} \varphi(t) \frac{dt}{t}, \\ \kappa_2 = \int_1^e \varphi(t) \frac{dt}{t}. \end{aligned} \tag{10}$$

*Let $E := C[1,e]$, $\|u\| := \sup_{t\in[1,e]} |u(t)|$, and $P := \{u \in E : u(t) \ge 0, \ \forall t \in [1,e]\}$. Then $(E, \|\cdot\|)$ becomes a real Banach space and $P$ is a cone on $E$. Define $B_\rho := \{u \in E : \|u\| < \rho\}$ for $\rho > 0$ in the sequel.*

*Define $P_0 = \{u \in P : u(t) \ge (\ln t)^{\alpha-1} \|u\|$ and $\forall t \in [1,e]\}$. Then $P_0$ is also a cone on $E$. In what follows, we verify that when $u \in P_0$, $\|u\| \ge (M/\Gamma(\alpha)) \int_1^e (1-\ln s)^{\alpha-2} (ds/s)$; we have $u(t) - w(t) \ge 0$, $\forall t \in [1,e]$, where $w$ is a solution for the problem*

$$\begin{aligned} -D^{\alpha} u(t) = M, \quad t \in [1,e], \\ u(1) = \delta u(1) = \delta u(e) = 0, \end{aligned} \tag{11}$$

*where $M$ is defined by (H0). From (1) and (6), $w$ takes the form as follows:*

$$w(t) = M \int_1^e G(t,s) \frac{ds}{s}, \quad t \in [1,e], \tag{12}$$

*where $G$ is defined by (7). Indeed, when $u \in P_0$, from (7) we have*

$$\begin{aligned} u(t) - w(t) &\geq (\ln t)^{\alpha-1} \|u\| - M \int_1^e G(t,s) \frac{ds}{s} \\ &\geq (\ln t)^{\alpha-1} \|u\| - \frac{M}{\Gamma(\alpha)} \int_1^e (1-\ln s)^{\alpha-2} (\ln t)^{\alpha-1} \frac{ds}{s} \\ &= (\ln t)^{\alpha-1} \left( \|u\| - \frac{M}{\Gamma(\alpha)} \int_1^e (1-\ln s)^{\alpha-2} \frac{ds}{s} \right) \geq 0, \\ &\qquad \forall t \in [1,e]. \end{aligned} \tag{13}$$

*For semipositone condition (H0), we need to construct an appropriate operator with which to study problem (1). Hence, we consider the modified problem*

$$\begin{aligned} -D^{\alpha} u(t) &= f(t, \max\{u(t) - w(t), 0\}) + M, \\ &\qquad t \in [1,e], \\ u(1) &= \delta u(1) = \delta u(e) = 0, \end{aligned} \tag{14}$$

*where $w$ is a solution for (11). Clearly, we are easy to show that if $u$ solves (14), $w$ solves (11), and $u(t) - w(t) \geq 0 (\not\equiv 0)$ and $\forall t \in [1,e]$; then $u(t) - w(t)$ is a positive solution for (1). Consequently, we turn to study the modified problem (14). From (1) and (6), (14) is equivalent to the integral equation*

$$\begin{aligned} u(t) &= \int_1^e G(t,s) \left[ f(s, \max\{u(s) - w(s), 0\}) + M \right] \frac{ds}{s}, \\ &\qquad t \in [1,e], \ u \in E. \end{aligned} \tag{15}$$

*Hence, we can define an operator $A : P \longrightarrow P$ as follows:*

$$\begin{aligned} (Au)(t) &= \int_1^e G(t,s) \left[ f(s, \max\{u(s) - w(s), 0\}) + M \right] \frac{ds}{s}, \\ &\qquad t \in [1,e], \ u \in E. \end{aligned} \tag{16}$$

*It is not difficult to prove that if $Au_0 = u_0$, then $u_0$ is a solution for (14). Moreover, from Lemma 2 we easily have $A(P) \subset P_0$.*

Now, we offer some basic theorems for fixed point methods used in our problem.

**Lemma 4** (see [33]). *Let $E$ be a real Banach space and $P$ a cone on $E$. Suppose that $\Omega \subset E$ is a bounded open set and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If there exists $\omega_0 \in P \backslash \{0\}$ such that*

$$\omega - A\omega \neq \lambda \omega_0, \quad \forall \lambda \geq 0, \ \omega \in \partial\Omega \cap P, \tag{17}$$

*then $i(A, \Omega \cap P, P) = 0$, where $i$ denotes the fixed point index on $P$.*

**Lemma 5** (see [33]). *Let $E$ be a real Banach space and $P$ a cone on $E$. Suppose that $\Omega \subset E$ is a bounded open set with $0 \in \Omega$ and that $A : \overline{\Omega} \cap P \longrightarrow P$ is a continuous compact operator. If*

$$\omega - \lambda A\omega \neq 0, \quad \forall \lambda \in [0,1], \ \omega \in \partial\Omega \cap P, \tag{18}$$

*then $i(A, \Omega \cap P, P) = 1$.*

## 3. Positive Solutions for (1)

Let $\lambda_1 = \kappa_1^{-1}, \lambda_2 = \kappa_2^{-1}$, and $\mathcal{N} = (M/\Gamma(\alpha)) \int_1^e (1 - \ln s)^{\alpha-2} (ds/s)$. Then we give some assumptions for nonlinear term $f$.

(H1) $\liminf_{u \longrightarrow +\infty} (f(t,u)/u) > \lambda_1$ uniformly on $t \in [1,e]$,

(H2) there exist $Q : [1,e] \longrightarrow \mathbb{R}^+$ such that

$$\begin{aligned} &f(t, \max\{u(t) - w(t), 0\}) + M \leq Q(t), \\ &\qquad \forall (t,u) \in [1,e] \times \left[ 0, \frac{M}{\Gamma(\alpha)} \int_1^e (1-\ln s)^{\alpha-2} \frac{ds}{s} \right], \end{aligned} \tag{19}$$

where $\int_1^e \varphi(t) Q(t) (dt/t) < \mathcal{N}$,

(H3) $\limsup_{u \longrightarrow +\infty} (f(t,u)/u) < \lambda_2$ uniformly on $t \in [1,e]$,

(H4) there exist $l \in (1,e)$, $\widetilde{Q} : [1,e] \longrightarrow \mathbb{R}^+$ such that

$$\begin{aligned} &f(t, \max\{u(t) - w(t), 0\}) + M \geq \widetilde{Q}(t), \\ &\qquad \forall (t,u) \in [1,e] \times \left[ 0, \frac{M}{\Gamma(\alpha)} \int_1^e (1-\ln s)^{\alpha-2} \frac{ds}{s} \right], \end{aligned} \tag{20}$$

where $\int_1^e (\ln l)^{\alpha-1} \varphi(t) \widetilde{Q}(t) (dt/t) > \mathcal{N}$,

(H5) $f \in C([1,e] \times \mathbb{R}, \mathbb{R})$, and there exists $k \in (0,1)$ such that $|f(t,u) - f(t,v)| \leq k\lambda_2 |u - v|$ for $t \in [1,e]$, $u, v \in \mathbb{R}$,

(H6) $f(t,0) \not\equiv 0$ for $t \in [1,e]$.

We now state our main results and offer their proofs.

**Theorem 6.** *Suppose that (H0)-(H2) hold. Then (1) has at least a positive solution.*

*Proof.* We first prove that there exist $R > \mathcal{N}$ large enough such that

$$u - Au \neq \lambda \varphi_0^*, \quad \forall \lambda \geq 0, \ u \in \partial B_R \cap P, \tag{21}$$

where $\varphi_0^* \in P_0$ is a given element. If false, there exits $u \in \partial B_R \cap P, \lambda_0 \geq 0$ such that $u - Au = \lambda_0 \varphi_0^*$. Note that $A(P) \subset P_0$; then $Au \in P_0$ for $u \in P$, and thus $u \in P_0$. This also implies that $u(t) \geq$ for $t \in [1,e]$.

Note that $u \in \partial B_R \cap P$; then $\|u\| = R > \mathcal{N}$, and $u(t) - w(t) \geq 0$ for $u \in P_0, t \in [1,e]$. From (H1) we have

$$\begin{aligned} &\liminf_{u \longrightarrow +\infty} \frac{f(t, \max\{u - w, 0\}) + M}{u - w} \\ &\qquad = \liminf_{u \longrightarrow +\infty} \frac{f(t, u - w) + M}{u - w} > \lambda_1 \end{aligned} \tag{22}$$

uniformly on $t \in [1,e]$. As a result, there exist $\varepsilon_1 > 0$ and $c_1 > 0$ such that

$$\begin{aligned} &f(t, u(t) - w(t)) + M \\ &\qquad \geq (\lambda_1 + \varepsilon_1)(u(t) - w(t)) - c_1, \quad \forall t \in [1,e]. \end{aligned} \tag{23}$$

This implies that

$$\begin{aligned} u(t) &\geq (Au)(t) \\ &\geq \int_1^e G(t,s)\left[(\lambda_1+\varepsilon_1)(u(s)-w(s))-c_1\right]\frac{ds}{s} \\ &\geq (\lambda_1+\varepsilon_1)\int_1^e G(t,s)(u(s)-w(s))\frac{ds}{s}-c_1\kappa_2. \end{aligned} \tag{24}$$

Note that (12) is multiplied by $\varphi(t)$ on both sides of the above and integrated over $[1,e]$ and use Lemma 3 to obtain

$$\begin{aligned} &\int_1^e u(t)\varphi(t)\frac{dt}{t} \geq \int_1^e \varphi(t) \\ &\cdot\left[(\lambda_1+\varepsilon_1)\int_1^e G(t,s)(u(s)-w(s))\frac{ds}{s}\right. \\ &\left.-c_1\kappa_2\right]\frac{dt}{t} \geq \frac{\lambda_1+\varepsilon_1}{\lambda_1}\int_1^e \varphi(t)(u(t) \\ &-w(t))\frac{dt}{t}-c_1\kappa_2^2=(1+\varepsilon_1\kappa_1)\int_1^e u(t) \\ &\cdot\varphi(t)\frac{dt}{t}-(1+\varepsilon_1\kappa_1)\int_1^e \varphi(t) \\ &\cdot M\int_1^e G(t,s)\frac{ds}{s}\frac{dt}{t}-c_1\kappa_2^2 \geq (1+\varepsilon_1\kappa_1) \\ &\cdot\int_1^e u(t)\varphi(t)\frac{dt}{t}-M(1+\varepsilon_1\kappa_1)\kappa_2^2-c_1\kappa_2^2. \end{aligned} \tag{25}$$

Consequently, we have

$$\int_1^e u(t)\varphi(t)\frac{dt}{t} \leq (\varepsilon_1\kappa_1)^{-1}(M(1+\varepsilon_1\kappa_1)+c_1)\kappa_2^2. \tag{26}$$

Noting that $u \in P_0$, we have

$$\begin{aligned} &\|u\|\int_1^e (\ln t)^{\alpha-1}\varphi(t)\frac{dt} \\ &\leq (\varepsilon_1\kappa_1)^{-1}(M(1+\varepsilon_1\kappa_1)+c_1)\kappa_2^2, \\ &\text{and } \|u\| \leq \varepsilon_1^{-1}\kappa_1^{-2}(M(1+\varepsilon_1\kappa_1)+c_1)\kappa_2^2. \end{aligned} \tag{27}$$

Therefore, if we choose $R > \max\{\mathcal{N}, \varepsilon_1^{-1}\kappa_1^{-2}(M(1+\varepsilon_1\kappa_1)+c_1)\kappa_2^2\}$, then (21) holds true. From Lemma 4 we have

$$i(A, B_R \cap P, P) = 0. \tag{28}$$

On the other hand, we prove that

$$u \neq \lambda Au, \quad \forall\lambda \in [0,1],\ u \in \partial B_{\mathcal{N}} \cap P. \tag{29}$$

If false, there exist $u \in \partial B_{\mathcal{N}} \cap P, \lambda_1^* \in [0,1]$ such that $u = \lambda_1^* Au$; this implies $u(t) \leq (Au)(t)$, $\forall t \in [1,e]$, and $\|u\| \leq \|Au\|$. However, from (H2) we have

$$\begin{aligned} &(Au)(t) \\ &= \int_1^e G(t,s)\left[f(s,\max\{u(s)-w(s),0\})+M\right]\frac{ds}{s} \\ &\leq \int_1^e \varphi(s)Q(s)\frac{ds}{s} < \mathcal{N} = \|u\|, \\ &\qquad\qquad \text{for } u \in \partial B_{\mathcal{N}} \cap P. \end{aligned} \tag{30}$$

This indicates that $\|Au\| < \|u\|$ for $u \in \partial B_{\mathcal{N}} \cap P$. This has a contradiction, and thus (29) holds true. From Lemma 5 we have

$$i(A, B_{\mathcal{N}} \cap P, P) = 1. \tag{31}$$

From (28) and (31), we obtain

$$\begin{aligned} i(A, (B_R \backslash \overline{B}_{\mathcal{N}}) \cap P, P) &= i(A, B_R \cap P, P) \\ &\quad - i(A, B_{\mathcal{N}} \cap P, P) = -1. \end{aligned} \tag{32}$$

Therefore the operator $A$ has at least one fixed point $u$ in $(B_R \backslash \overline{B}_{\mathcal{N}}) \cap P$ with $\|u\| \geq \mathcal{N}$, and then $u(t) - w(t)$ is a positive solution for (1). This completes the proof. □

**Theorem 7.** *Suppose that (H0), (H3), and (H4) hold. Then (1) has at least a positive solution.*

*Proof.* We first prove that there exist $R > \mathcal{N}$ large enough such that

$$u \neq \lambda Au, \quad \forall\lambda \in [0,1],\ u \in \partial B_R \cap P. \tag{33}$$

If false, there exist $u \in \partial B_R \cap P, \lambda_2^* \in [0,1]$ such that $u = \lambda_2^* Au$, and $u \in P_0$ for the fact that $Au \in P_0$ when $u \in P$. This also implies $u(t) \leq (Au)(t)$, $t \in [1,e]$.

Note that $u \in \partial B_R \cap P$; then $\|u\| = R > \mathcal{N}$, and $u(t) - w(t) \geq 0$ for $u \in P_0, t \in [1,e]$. From (H3) we have

$$\begin{aligned} &\limsup_{u\longrightarrow+\infty}\frac{f(t,\max\{u-w,0\})+M}{u-w} \\ &= \limsup_{u\longrightarrow+\infty}\frac{f(t,u-w)+M}{u-w} < \lambda_2 \end{aligned} \tag{34}$$

uniformly on $t \in [1,e]$. As a result, there exist $\varepsilon_2 \in (0,\lambda_2)$ and $c_2 > 0$ such that

$$\begin{aligned} &f(t,u(t)-w(t))+M \\ &\leq (\lambda_2-\varepsilon_2)(u(t)-w(t))+c_2, \quad \forall t \in [1,e]. \end{aligned} \tag{35}$$

This implies that

$$\begin{aligned} u(t) &\leq (Au)(t) \\ &\leq \int_1^e G(t,s)\left[(\lambda_2-\varepsilon_2)(u(s)-w(s))+c_2\right]\frac{ds}{s} \\ &\leq (\lambda_2-\varepsilon_2)\int_1^e G(t,s)(u(s)-w(s))\frac{ds}{s}+c_2\kappa_2. \end{aligned} \tag{36}$$

Note that (12) is multiplied by $\varphi(t)$ on both sides of the above and integrated over $[1,e]$ and use Lemma 3 to obtain

$$\begin{aligned}\int_1^e u(t)\varphi(t)\frac{dt}{t} &\le \int_1^e \varphi(t)\\ &\cdot\left[(\lambda_2-\varepsilon_2)\int_1^e G(t,s)(u(s)-w(s))\frac{ds}{s}\right.\\ &\left.+c_2\kappa_2\right]\frac{dt}{t} \le \frac{\lambda_2-\varepsilon_2}{\lambda_2}\int_1^e \varphi(t)(u(t)\\ &-w(t))\frac{dt}{t}+c_2\kappa_2^2 \le (1-\varepsilon_2\kappa_2)\int_1^e u(t)\\ &\cdot\varphi(t)\frac{dt}{t}+(1-\varepsilon_2\kappa_2)\int_1^e \varphi(t)\\ &\cdot M\int_1^e G(t,s)\frac{ds}{s}\frac{dt}{t}+c_2\kappa_2^2 \le (1-\varepsilon_2\kappa_2)\\ &\cdot\int_1^e u(t)\varphi(t)\frac{dt}{t}+M(1-\varepsilon_2\kappa_2)\kappa_2^2+c_2\kappa_2^2.\end{aligned} \tag{37}$$

Consequently, we have

$$\int_1^e u(t)\varphi(t)\frac{dt}{t} \le (\varepsilon_2\kappa_2)^{-1}(M(1-\varepsilon_2\kappa_2)+c_2)\kappa_2^2. \tag{38}$$

Noting that $u\in P_0$, we obtain

$$\begin{aligned}&\|u\|\int_1^e (\ln t)^{\alpha-1}\varphi(t)\frac{dt}{t}\\ &\quad\le (\varepsilon_2\kappa_2)^{-1}(M(1-\varepsilon_2\kappa_2)+c_2)\kappa_2^2,\\ &\text{and } \|u\| \le (\varepsilon_2\kappa_1\kappa_2)^{-1}(M(1-\varepsilon_2\kappa_2)+c_2)\kappa_2^2.\end{aligned} \tag{39}$$

Taking $R>\max\{\mathcal{N},(\varepsilon_2\kappa_1\kappa_2)^{-1}(M(1-\varepsilon_2\kappa_2)+c_2)\kappa_2^2\}$, then (33) is satisfied. From Lemma 5 we have

$$i(A,B_R\cap P,P)=1. \tag{40}$$

On the other hand, we prove that

$$u-Au\ne\lambda\varphi_1^*,\quad \forall\lambda\ge 0,\ u\in\partial B_{\mathcal{N}}\cap P, \tag{41}$$

where $\varphi_1^*\in P$ is a given element. If false, there exist $u\in\partial B_{\mathcal{N}}\cap P,\lambda_3\ge 0$ such that $u-Au=\lambda_3\varphi_1^*$; this implies $u(t)\ge(Au)(t)$, $t\in[1,e]$, and thus $\|u\|\ge\|Au\|$. However, from (H4) we have

$$\begin{aligned}\|Au\| &= \max_{t\in[1,e]}(Au)(t)\\ &\ge \int_1^e G(l,s)[f(s,\max\{u(s)-w(s),0\})+M]\frac{ds}{s}\\ &\ge \int_1^e (\ln l)^{\alpha-1}\varphi(s)\widetilde{Q}(s)\frac{ds}{s} > \mathcal{N}=\|u\|\end{aligned} \tag{42}$$

for $u\in\partial B_{\mathcal{N}}\cap P$. This has a contradiction, and thus (41) holds true. From Lemma 4 we have

$$i(A,B_{\mathcal{N}}\cap P,P)=0. \tag{43}$$

From (40) and (43), we obtain

$$\begin{aligned}i(A,(B_R\backslash\overline{B}_{\mathcal{N}})\cap P,P) &= i(A,B_R\cap P,P)\\ &\quad - i(A,B_{\mathcal{N}}\cap P,P)=1.\end{aligned} \tag{44}$$

Therefore the operator $A$ has at least one fixed point $u$ in $(B_R\backslash\overline{B}_{\mathcal{N}})\cap P$ with $\|u\|\ge\mathcal{N}$, and then $u(t)-w(t)$ is a positive solution for (1). This completes the proof. □

From (6), we define an operator $T:E\longrightarrow E$ as follows:

$$(Tu)(t)=\int_1^e G(t,s)f(s,u(s))\frac{ds}{s},\quad u\in E. \tag{45}$$

Then $T$ is a completely continuous operator, and $u$ is a solution for (1) if and only if $u$ is a fixed point of $T$.

**Theorem 8.** *Suppose that (H5)-(H6) hold. Then (1) has only a nontrivial solution, denoted by $u^*$, and for all $u_0\in E$, $u_0(t)\not\equiv 0, t\in[1,e]$, the sequence $u_n=Tu_{n-1}$ $(n=1,2,\ldots)$ uniformly converges to $u^*$.*

*Proof.* (H6) ensures that 0 is not a solution for (1). Then if (1) has a solution, this solution is nontrivial. For all $n\in\mathbb{N}$, from Lemma 3 we have

$$\begin{aligned}|u_{n+1}(t)-u_n(t)| &= |(Tu_n)(t)-(Tu_{n-1})(t)|\\ &= \left|\int_1^e G(t,s)(f(s,u_n(s))-f(s,u_{n-1}(s)))\frac{ds}{s}\right|\\ &\le \int_1^e G(t,s)|f(s,u_n(s))-f(s,u_{n-1}(s))|\frac{ds}{s}\\ &\le k\lambda_2\int_1^e \varphi(s)|u_n(s)-u_{n-1}(s)|\frac{ds}{s}\\ &= k\lambda_2\int_1^e \varphi(t)|(Tu_{n-1})(t)-(Tu_{n-2})(t)|\frac{dt}{t}\\ &\le k\lambda_2\int_1^e \varphi(t)\int_1^e G(t,s)\\ &\quad\cdot|f(s,u_{n-1}(s))-f(s,u_{n-2}(s))|\frac{ds}{s}\frac{dt}{t}\\ &\le k^2\lambda_2\int_1^e \varphi(t)|u_{n-1}(t)-u_{n-2}(t)|\frac{dt}{t}\\ &\quad\vdots\\ &\le k^n\lambda_2\int_1^e \varphi(t)|u_1(t)-u_0(t)|\frac{dt}{t}.\end{aligned} \tag{46}$$

On the other hand, letting $\nu=0$ and $f_0=\max_{t\in[1,e]}|f(t,0)|$ in (H5), we have

$$|f(t,u)|\le k\lambda_2|u|+f_0,\quad \forall u\in\mathbb{R},\ t\in[1,e]. \tag{47}$$

Noting that $u_1 = Tu_0$, we obtain

$$\begin{aligned}&\int_1^e \varphi(t)\left|u_1(t)-u_0(t)\right|\frac{dt}{t}\\&\quad\le (1+k)\int_1^e \varphi(t)\left|u_0(t)\right|\frac{dt}{t}+\kappa_2^2 f_0 := \beta_1.\end{aligned}\tag{48}$$

Therefore, for all $n \in \mathbb{N}$, we have

$$\left|u_{n+1}(t)-u_n(t)\right| \le k^n\beta_1\lambda_2, \quad \forall t\in[1,e]. \tag{49}$$

Consequently, for all $n, m \in \mathbb{N}$, we have

$$\begin{aligned}&\left|u_{n+m}(t)-u_n(t)\right|\\&\quad\le \left|u_{n+m}(t)-u_{n+m-1}(t)\right|\\&\qquad+\left|u_{n+m-1}(t)-u_{n+m-2}(t)\right|+\cdots\\&\qquad+\left|u_{n+1}(t)-u_n(t)\right|\\&\quad\le \beta_1\lambda_2\left(k^{n+m-1}+k^{n+m-2}+\cdots+k^n\right)\le k^n\frac{\beta_1\lambda_2}{1-k}\\&\quad\longrightarrow 0, \quad \text{when } n\longrightarrow\infty.\end{aligned}\tag{50}$$

This implies $\{u_n\}$ is a Cauchy sequence, and from $E$'s completeness, there exists $u^* \in E$ such that $\lim_{n\longrightarrow\infty} u_n = u^*$. Taking the limits for sequence $u_n = Tu_{n-1}$ and we have $Tu^* = u^*$; i.e., $u^*$ is a nontrivial solution for (1).

Next we prove that (1) has only a solution. If $u, v \in E$ are solutions for (1) and $u \ne v$, then $T^n u = u$ and $T^n v = v$ for all $n \in \mathbb{N}$. By (H5) we obtain

$$\begin{aligned}&|u(t)-v(t)| = \left|(T^n u)(t)-(T^n v)(t)\right| = \left|T\left(T^{n-1}u\right)\right.\\&\quad\left.\cdot(t)-T\left(T^{n-1}v\right)(t)\right|\\&\le \int_1^e G(t,s)\left|f\left(s,\left(T^{n-1}u\right)(s)\right)\right.\\&\quad\left.-f\left(s,\left(T^{n-1}v\right)(s)\right)\right|\frac{ds}{s}\\&\le k\lambda_2\int_1^e \varphi(s)\left|\left(T^{n-1}u\right)(s)-\left(T^{n-1}v\right)(s)\right|\frac{ds}{s}\\&\le k\lambda_2\int_1^e \varphi(t)\int_1^e G(t,s)\\&\quad\cdot\left|f\left(s,\left(T^{n-2}u\right)(s)\right)-f\left(s,\left(T^{n-1}v\right)(s)\right)\right|\frac{ds}{s}\frac{dt}{t}\\&\le k^2\lambda_2\int_1^e \varphi(t)\left|\left(T^{n-2}u\right)(t)-\left(T^{n-1}v\right)(t)\right|\frac{dt}{t}\\&\qquad\vdots\\&\le k^{n-1}\lambda_2\int_1^e \varphi(t)\left|(Tu)(t)-(Tv)(t)\right|\frac{dt}{t}\\&= k^{n-1}\lambda_2\int_1^e \varphi(t)\left|u(t)-v(t)\right|\frac{dt}{t}\\&\le k^{n-1}\|u-v\|.\end{aligned}\tag{51}$$

This implies that

$$\|u-v\| \le k^{n-1}\|u-v\|, \quad \text{for all } n\in\mathbb{N}. \tag{52}$$

Noting that $k \in (0,1)$, then there exists $N \in \mathbb{N}$, when $n > N$, $k^{n-1} < 1$, and thus a contradiction for the above inequality. This obtains the uniqueness of solutions for (1). This completes the proof. □

In what follows, we offer some examples for our main results. Let $\alpha = 2.5$, $l = \sqrt{e}$. Then $\kappa_1 = \int_1^e (\ln t)^{\alpha-1}\varphi(t)(dt/t) \approx 0.1227$, $\kappa_2 = \int_1^e \varphi(t)(dt/t) = 4/15 \approx 0.2667$, and $\mathcal{N} = (M/\Gamma(\alpha))\int_1^e (1-\ln s)^{\alpha-2}(ds/s) \approx 0.5015M$.

*Example 9.* Let $f(t,u) = (1/Me^{\ln t})u^2 - M$ for all $t \in [1,e]$, $u \in \mathbb{R}^+$, and $Q \equiv 1.8M$. Then $\liminf_{u\longrightarrow+\infty}(f(t,u)/u) = \liminf_{u\longrightarrow+\infty}(((1/Me^{\ln t})u^2 - M)/u) = +\infty > \lambda_1$ uniformly on $t \in [1,e]$, and when $(t,u) \in [1,e]\times[0,\mathcal{N}]$, $f(t,u) + M \le (1/M)0.2515M^2 + M \approx 1.2515M \le Q(t)$. Moreover, $\int_1^e \varphi(t)Q(t)(dt/t) \approx 0.48M < \mathcal{N}$. Therefore, (H1) and (H2) hold.

*Example 10.* Let $f(t,u) = (6M/e^{-0.5015M})e^{-u} - M$ for all $t \in [1,e]$, $u \in \mathbb{R}^+$, and $Q \equiv 5.4M$. Then $\limsup_{u\longrightarrow+\infty}(f(t,u)/u) = \limsup_{u\longrightarrow+\infty}(((6M/e^{-0.5015M})e^{-u} - M)/u) = 0 < \lambda_2$ uniformly on $t \in [1,e]$, and when $(t,u) \in [1,e]\times[0,\mathcal{N}]$, $f(t,u) + M \ge (6M/e^{-0.5015M})e^{-0.5015M} - M + M = 6M \ge Q(t)$. Moreover, $\int_1^e (0.5)^{1.5}\varphi(t)5.4M(dt/t) \approx 0.5093M > \mathcal{N}$. Therefore, (H3) and (H4) hold.

*Example 11.* Let $f(t,u) = 3.7ku + g(t)$, where $k \in (0,1)$ and $g \in C([1,e],\mathbb{R})$ with $g(t) \not\equiv 0$ for $t \in [1,e]$. Therefore, (H5) and (H6) hold.

## Data Availability

No data were used to support this study

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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