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Research Article

Generalized Numerical Index of Function Algebras

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Let X be a complex Banach space and $C_b(\Omega : X)$ be the Banach space of all bounded continuous functions from a Hausdorff space Ω to X, equipped with sup norm. A closed subspace $\mathscr{A}$ of $C_b(\Omega : X)$ is said to be an X-valued function algebra if it satisfies the following three conditions: (i) $A := \{x^* \circ f : f \in \mathscr{A},\ x^* \in X^*\}$ is a closed subalgebra of $C_b(\Omega)$, the Banach space of all bounded complex-valued continuous functions; (ii) $\phi \otimes x \in \mathscr{A}$ for all $\phi \in A$ and $x \in X$; and (iii) $\phi f \in \mathscr{A}$ for every $\phi \in A$ and for every $f \in \mathscr{A}$. It is shown that k-homogeneous polynomial and analytic numerical index of certain X-valued function algebras are the same as those of X.

1. Introduction

In this paper, we consider only complex nontrivial Banach spaces. Given a Banach space X, we denote by B_X and S_X its closed unit ball and unit sphere, respectively. Let X^* be the dual space of X. If X and Y are Banach spaces, a k-*homogeneous polynomial* P from X to Y is a mapping such that there is a k-linear continuous mapping L from X to Y such that $P(x) = L(x, \ldots, x)$ for every x in X. The Banach space of all k-homogeneous polynomials from X to Y is denoted by $\mathscr{P}({}^k X : Y)$ endowed with the polynomial norm $\|P\| = \sup_{x\in B_X}\|P(x)\|$. We refer to [1] for background knowledge on polynomials.

We are mainly interested in the following spaces. For two Banach spaces X, Y and a Hausdorff topological space Ω,

$$\begin{aligned} C_b(\Omega : Y) &:= \{f : \Omega \longrightarrow Y : \\ &\quad f \text{ is a bounded continuous function on } \Omega\}, \\ \mathscr{A}_b(B_X : Y) &:= \{f \in C_b(B_X : Y) : \\ &\quad f \text{ is holomorphic on } B_X^\circ\} \\ \mathscr{A}_u(B_X : Y) &:= \{f \in \mathscr{A}_b(B_X : Y) : \\ &\quad f \text{ is uniformly continuous}\}, \end{aligned} \tag{1}$$

where B_X° is the interior of B_X. Then $C_b(\Omega : Y)$ is a Banach space under the sup norm $\|f\| := \sup\{\|f(t)\|_Y : t \in \Omega\}$ and both $\mathscr{A}_b(B_X : Y)$ and $\mathscr{A}_u(B_X : Y)$ are closed subspaces of $C_b(B_X : Y)$. In case that Y is the complex scalar field $\mathbb{C}$, we just write $C_b(B_X)$, $\mathscr{A}_b(B_X)$, and $\mathscr{A}_u(B_X)$. Let

$$\Pi(X) := \{(x, x^*) : \|x\| = \|x^*\| = 1 = x^*(x)\}. \tag{2}$$

The *spatial numerical range* of f in $C_b(B_X : X)$ is defined by

$$W(f) = \{x^*(f(x)) : (x, x^*) \in \Pi(X)\}, \tag{3}$$

and the *numerical radius* of f is defined by

$$v(f) = \sup\{|\lambda| : \lambda \in W(f)\}. \tag{4}$$

Let X be a Banach space. The k-*homogeneous polynomial numerical index* $n^{(k)}(X)$ is defined in [2] by

$$n^{(k)}(X) := \inf\{v(P) : P \in \mathscr{P}({}^k X : X), \|P\| = 1\}. \tag{5}$$

The b-*analytic numerical index* $n_{ba}(X)$ and u-*analytic index* $n_{ua}(X)$ are defined, respectively, by

$$\begin{aligned} n_{ba}(X) &:= \inf\{v(f) : f \in \mathscr{A}_b(B_X : X), \|f\| = 1\}, \\ n_{ua}(X) &:= \inf\{v(f) : f \in \mathscr{A}_u(B_X : X), \|f\| = 1\}. \end{aligned} \tag{6}$$

It is clear from the definitions that $0 \leqslant n_{ba}(X) \leqslant n_{ua}(X) \leqslant n^{(k)}(X) \leqslant 1$ for all $k \geq 1$.

Choi, García, Kim, and Maestre showed [3] that $n^{(k)}(A) = 1$ and $n_{ua}(A) = 1$ for uniform algebras A. In general, it is not difficult to see that if A is a (unital) function algebra on a Hausdorff space, then, by the Gelfand transform, A is isometric to a (unital) uniform algebra on Δ where Δ is the maximal ideal space of A. We present this fact in Proposition 2 for the completeness of the paper. In this paper, we introduce a X-valued function algebra and the Gelfand transform does not work in this case. In the proof of [3], they used a very useful Urysohn type theorem, which was obtained by Cascales, Guiro, and Kadets [4]. Recently, Kim and the author found [5] that a Urysohn type theorem holds for some function algebras. It plays an important role in the main results of this paper. For some geometric properties on k-homogeneous polynomial (analytic) numerical index, refer to [6, 7].

Let us briefly review some necessary notions. A nontrivial $\|\cdot\|_\infty$-closed subalgebra of A of $C_b(\Omega)$ is called a *function algebra* on a Hausdorff space Ω. For a Banach space X, a nontrivial subspace $\mathscr{A}$ of $C_b(\Omega : X)$ is said to be an *X-valued function algebra* if it satisfies three conditions: (i) $A := \{x^* \circ f : f \in \mathscr{A},\ x^* \in X^*\}$ is a function algebra on Ω; (ii) $A \otimes X = \{\phi \otimes x : \phi \in A, x \in X\}$, where $(\phi \otimes x)(t) = \phi(t)x$ for $t \in \Omega$; and (iii) $\phi f \in \mathscr{A}$ for every $\phi \in A$ and $f \in \mathscr{A}$, where $(\phi f)(t) = \phi(t) f(t)$ for $t \in \Omega$. A subset T of Ω is said to be *norming* for $\mathscr{A}$ if $\|f\| = \sup\{\|f(t)\| : t \in T\}$ holds for all $f \in \mathscr{A}$. By *unital function algebra*, we mean a function algebra containing all constant functions. A function algebra A on a compact Hausdorff space K is said to be a *uniform algebra* if A separates the points of K (that is, for every $x \neq y$ in K, there is $f \in A$ such that $f(x) \neq f(y)$. Note that the definition of function algebra in this paper is different from the usual one in [8].

Let f be an element of an X-valued function algebra $\mathscr{A}$. The f is said to be a *peak function* at t_0 if there exists a unique $t_0 \in \Omega$ such that $\|f\| = \|f(t_0)\|$. A peak function f is said to be a *strong peak function* at $t_0 \in \Omega$ if $\|f\| = \|f(t_0)\|$ and for every open subset V containing t_0 we get

$$\sup\{\|f(t)\| : t \in \Omega \setminus V\} < \|f\|. \tag{7}$$

The corresponding point t_0 is called a *strong peak point* for $\mathscr{A}$. We denote by $\rho\mathscr{A}$ the set of all strong peak points for $\mathscr{A}$. It is easy to see that if Ω is compact, then every peak function is a strong peak function. It is worth remarking that if A is a nontrivial separating separable subalgebra of $C(\Omega)$ on a compact Hausdorff space Ω, then ρA is a norming subset for A [9]. There is a compact Hausdorff space K such that $\rho C(K)$ is an empty set [10]. For more information about peak functions and points, refer to [8, 10].

For an X-valued function algebra $\mathscr{A}$, let $A = \{x^* \circ f : x^* \in X^*,\ f \in \mathscr{A}\}$. Then $\rho\mathscr{A} = \rho A$. Indeed, if $f \in \mathscr{A}$ is a strong peak function at t_0, then choose $x^* \in S_{X^*}$ such that $x^* f(t_0) = \|f(t_0)\| = \|f\|$ and it is clear that $x^* \circ f \in A$ is a strong peak function in A at t_0. Therefore, $\rho\mathscr{A} \subset \rho A$. Conversely, if $g \in A$ is a strong peak function at t, then choose $x \in S_X$. Therefore, $g \otimes x \in \mathscr{A}$ is a strong peak function at t. Hence we have $\rho A \subset \rho\mathscr{A}$. In addition, if $\rho\mathscr{A}$ is norming for $\mathscr{A}$, then it is also norming for A since $g \otimes x$ in $\mathscr{A}$ has the same norm as g for every $g \in A$ and $x \in S_X$.

The following lemma will be useful to get main results. In proofs of the main results, the denseness of the strong peak functions in an X-valued function algebra $\mathscr{A}$ is an important part and equivalent to the fact that the set of strong peak points is norming for $\mathscr{A}$. That means that the fact that every element in $\mathscr{A}$ can be approximated by the sequence of strong peak functions is equivalent to the fact that the norm of every element in $\mathscr{A}$ can be approximated on the set of strong peak points for $\mathscr{A}$. The approximation by strong peak functions will prove to be useful to deal with the geometric properties of function algebras especially those related to generalized numerical indices of Banach spaces.

Lemma 1 (see [5]). *Let A be a function algebra on Ω and fix $\omega_0 \in \rho A$. Then, given $0 < \epsilon < 1$ and for every open subset U containing ω_0, there exists a strong peak function $\phi \in A$ such that $\|\phi\| = 1 = |\phi(\omega_0)|$, $\sup_{\omega \in \Omega \setminus U} |\phi(\omega)| < \epsilon$, and for all $\omega \in \Omega$,*

$$|\phi(\omega)| + (1-\epsilon)|1 - \phi(\omega)| \leqslant 1. \tag{8}$$

2. Main Results

The proof of [3, Theorem 2.1] shows that $n_{ba}(A) = 1$ if A is a uniform algebra. Since a function algebra is isometric to a uniform algebra by the Gelfand transform, we have the following.

Proposition 2. *Let A be a function algebra on a Hausdorff space Ω. Then it is isometric to a uniform algebra on a compact Hausdorff space and $n_{ba}(A) = 1$.*

Proof. Let A be a function algebra and Δ be the set of all nonzero algebra homomorphisms from A to $\mathbb{C}$. The maximal ideal space Δ is a compact Hausdorff space with the Gelfand topology. The Gelfand transform $\widehat{f}$ of $f \in A$ is defined by $\widehat{f}(\phi) = \phi(f)$ for $\phi \in \Delta$. For $t \in \Omega$, let δ_t be the dirac delta function by $\delta_t(f) = f(t)$ for $f \in A$. Fix a nonzero $f \in A$ and let $\Omega_f = \{t : f(t) \neq 0\}$; then $\delta_t \in \Delta$ for all $t \in \Omega_f$ and

$$\begin{aligned} \|\widehat{f}\| &= \sup\{|\widehat{f}(\phi)| : \phi \in \Delta\} \leqslant \|f\| \\ &= \sup\{|f(t)| : t \in \Omega_f\} = \sup\{|\delta_t(f)| : t \in \Omega_f\} \\ &\leqslant \|\widehat{f}\|. \end{aligned} \tag{9}$$

Since the Gelfand transform $f \longmapsto \widehat{f}$ is a homomorphism, A is isometrically isomorphic to the image $\widehat{A}$, where $\widehat{A}$ is the image of the Gelfand transform. Then $\widehat{A}$ is a closed subalgebra of $C(\Delta)$ and it is separating the points of Δ. Thus, it is a uniform algebra on the compact Hausdorff space Δ.

For the second part, the proof used in [3, Theorem 2.1] to show $n_{ua}(A) = 1$ can be applied to show that $n_{ba}(A) = 1$ for uniform algebras A. □

Proposition 2 gives a positive answer to the third question raised by Acosta and Kim [11].

Theorem 3. *Let X be a Banach space and suppose that $\mathscr{A}$ is an X-valued function algebra on a Hausdorff space Ω such that $\rho\mathscr{A}$ is a norming subset for $\mathscr{A}$. Then we have*

(i) $n^{(k)}(\mathscr{A}) \geq n^{(k)}(X)$ *for every* $k \geq 1$,

(ii) $n_{ua}(\mathscr{A}) \geq n_{ua}(X)$ *and*

(iii) $n_{ba}(\mathscr{A}) \geq n_{ba}(X)$.

Proof. We prove $n_{ba}(\mathscr{A}) \geq n_{ba}(X)$ holds. The proofs for the other two cases are exactly the same. It is well-known that $n_{ba}(X) > 0$ for all complex Banach spaces X [12].

Let $A = \{x^* \circ f : f \in \mathscr{A}\}$. Then A is a function algebra. Let $P \in \mathscr{A}_b(B_{\mathscr{A}} : \mathscr{A})$ with $\|P\| = 1$ and $0 < \epsilon < b_{ba}(X)$ be given. Choose $f_0 \in S_{\mathscr{A}}$ so that $\|P(f_0)\| > 1-\epsilon/6$. Since ρA is norming for $\mathscr{A}$, find $t_0 \in \rho A$ such that $\|P(f_0)(t_0)\| > 1-\epsilon/6$. Since P is continuous, there is $0 < \delta < 1$ such that $\|P(f_0) - P(g)\| < \epsilon/6$ for every $g \in B_{\mathscr{A}}$ with $\|f_0 - g\| < \delta$.

Let $W = \{t \in \Omega : \|f_0(t) - f_0(t_0)\| < \delta/6,\ \|P(f_0)(t) - P(f_0)(t_0)\| < \epsilon/3\}$ and W be an open subset of Ω containing t_0. Then by Lemma 1, there is a strong peak function $\phi \in A$ such that $\|\phi\| = \phi(t_0) = 1$ and $|\phi(t)| < \delta/6$ for every $t \in \Omega\backslash W$, and

$$\left|\phi(t)\right| + \left(1 - \frac{\epsilon}{6}\right)\left|1 - \phi(t)\right| \leqslant 1 \tag{10}$$

for every $t \in \Omega$.

Define $\Psi : X \longrightarrow \mathscr{A}$ by $\Psi(x) = (1-\delta/6)(1-\phi)f_0 + \phi \otimes x$ for all $x \in X$. It is easy to check that Ψ is well-defined and $\|\Psi(x)\| \leqslant 1$ for all $x \in B_X$. Then, let $x_0 = f_0(t_0)$,

$$\begin{aligned} \|f_0 - \Psi(x_0)\| &= \sup_{t\in\Omega}\Big\| f_0(t) \\ &\quad - \left(1 - \frac{\delta}{6}\right)(1-\phi(t)) f_0(t) - \phi(t) f_0(t_0)\Big\| \\ &\leqslant \sup_{t\in\Omega}\Big(\frac{\delta}{6}\|f_0(t)\| + |\phi(t)| \|f_0(t) - f_0(t_0)\| \\ &\quad + \frac{\delta}{6}|\phi(t)| \|f_0(t)\|\Big) < \frac{\delta}{6} + \frac{\delta}{3} + \frac{\delta}{6} < \delta. \end{aligned} \tag{11}$$

Then we have the following.

$$\begin{aligned} \|P(\Psi(x_0))(t_0)\| &\geq \|P(f_0)(t_0)\| \\ &\quad - \|P(f_0)(t_0) - P(\Psi(x_0))(t_0)\| \\ &> 1 - \frac{\epsilon}{6} - \frac{\epsilon}{6} > 1 - \epsilon. \end{aligned} \tag{12}$$

Choose $x_0^* \in S_{X^*}$ such that $x_0^*[P(\Psi(x_0)(t_0)] > 1 - \epsilon$ and find a complex number z_0 with $|z_0| \leqslant 1$ and a proper $\tilde{x}_0 \in S_X$ satisfying $x_0 = z_0\tilde{x}_0$. Then the function $\varphi(z) = x_0^*[P(\Psi(z\tilde{x}_0))(t_0)]$ is an element of $\mathscr{A}_u(B_{\mathbb{C}})$. By the maximum modulus theorem, there exists z_1 with $|z_1| = 1$ such that φ takes its maximum modulus on $B_{\mathbb{C}}$. Hence,

$$\begin{aligned} \|P(\Psi(z_1\tilde{x}_0))(t_0)\| &\geq \left|x_0^*[P(\Psi(z_1\tilde{x}_0))(t_0)]\right| \\ &\geq \left|x_0^*[P(\Psi(z_0\tilde{x}_0))(t_0)]\right| \\ &> 1 - \epsilon. \end{aligned} \tag{13}$$

Let $x_1 = z_1\tilde{x}_0$, choose $x_1^* \in S_{X^*}$ with $x_1^*(x_1) = 1$, and define the function $\Phi : X \longrightarrow \mathscr{A}$ by

$$\Phi(x) = x_1^*(x)\left(1 - \frac{\delta}{6}\right)(1-\phi) f_0 + \phi \otimes x \tag{14}$$

for $x \in X$. Then $\Phi(x_1) = \Psi(x_1) = \Psi(z_1\tilde{x}_0)$, and hence $\|P(\Phi(x_1))(t_0)\| > 1 - \epsilon$. Let $Q(x) = P(\Phi(x))(t_0)$ for $x \in X$. Then $Q \in \mathscr{A}_b(B_X : X)$. Then

$$1 - \epsilon < \|P(\Phi(x_1))(t_0)\| = \|Q(x_1)\| \leqslant \|Q\| \leqslant 1. \tag{15}$$

Since $0 < \epsilon < b_{ba}(X)$, there is $(x_2, x_2^*) \in \Pi(X)$ so that

$$\left|x_2^*\left(\frac{Q(x_2)}{\|Q\|}\right)\right| > v\left(\frac{Q}{\|Q\|}\right) - \epsilon \geq b_{ba}(X) - \epsilon > 0. \tag{16}$$

Note that $(\Phi(x_2), x_2^* \circ \delta_{t_0}) \in \Pi(\mathscr{A})$ because $\Phi(x_2)(t_0) = x_2$. Hence we have

$$\begin{aligned} v(P) &\geq \left|\left(x_2^* \circ \delta_{t_0}\right)(P(\Phi(x_2)))\right| \\ &= \left|x_2^*[P(\Phi(x_2))(t_0)]\right| = \left|x_2^* Q(x_2)\right| \\ &\geq \|Q\|\left(b_{ba}(X) - \epsilon\right) \geq (1-\epsilon)\left(b_{ba}(X) - \epsilon\right). \end{aligned} \tag{17}$$

Since $0 < \epsilon < b_{ba}(X)$ is arbitrary, $v(P) \geq b_{ba}(X)$. This holds for all $P \in \mathscr{A}_b(B_{\mathscr{A}}; \mathscr{A})$ with $\|P\| = 1$. Therefore, we get $b_{ba}(\mathscr{A}) \geq b_{ba}(X)$. □

A version of the Bishop-Phelps-Bollobás type theorem for holomorphic functions has been shown [5, 13]. In the following theorem, we present a similar result. However the main focus is the denseness of the set of all strong peak functions, which is different from that of the results in [5].

Theorem 4. *Let X be a Banach space and $\mathscr{A}$ an X-valued function algebra on a Hausdorff space Ω. Then, given $\epsilon > 0$, whenever a norm-one element f in $\mathscr{A}$ and a point ω_0 in $\rho\mathscr{A}$ satisfy $\|f(\omega_0)\| > 1 - \epsilon/5$, there is a norm-one strong peak function $g \in \mathscr{A}$ at $\omega_0 \in \rho\mathscr{A}$ such that $\|f - g\| < \epsilon$.*

Proof. Suppose that f satisfies the prescribed conditions. Then

$$U_1 = \left\{\omega \in \Omega : \left\|\frac{f(\omega_0)}{\|f(\omega_0)\|} - f(\omega)\right\| < \frac{\epsilon}{5}\right\} \tag{18}$$

is an open set containing ω_0. There exists $\omega_1 \in \rho A \cap U_2$. Using Lemma 1, take a strong peak function $\phi \in A$ such that $\phi(\omega_0) = 1 = \|\phi\|$, $\sup\{|\phi(\omega)| : \omega \in \Omega \setminus U_1\} < \epsilon/5$, and

$$\left|\phi(\omega)\right| + \left(1 - \frac{\epsilon}{5}\right)\left|1 - \phi(\omega)\right| \leqslant 1 \tag{19}$$

for all $\omega \in \Omega$. Set

$$g(\omega) = \phi(\omega)\frac{f(\omega_0)}{\|f(\omega_0)\|} + \left(1 - \frac{\epsilon}{5}\right)(1 - \phi(\omega)) f(\omega). \tag{20}$$

It is easy to check that $g \in \mathscr{A}$ and $\|g(\omega_1)\| = 1 = \|g\|$. Moreover, from the inequality

$$\begin{aligned}\|g(\omega)-f(\omega)\| \leqslant |\phi(\omega)| \left\| \frac{f(\omega_0)}{|f(\omega_0)|} - f(\omega) \right\| \\ + \frac{\epsilon}{5}\left|1-\phi(\omega)\right| \|f(\omega)\|,\end{aligned} \tag{21}$$

we have that $\|g(\omega) - f(\omega)\| \leqslant \epsilon/5 + 2\epsilon/5$ for all $\omega \in \Omega$ if we consider two cases $\omega \in U_1$ and $\omega \in \Omega \setminus U_1$. Hence, we get $\|f - g\| < \epsilon$ and complete the proof since we know g is a strongly norm attaining function from the fact that ϕ is a strong peak function. □

From Theorem 4, we have the following consequence.

Corollary 5. *Let X be a Banach space and $\mathscr{A}$ be an X-valued function algebra on a Hausdorff space Ω. Then the set $\rho\mathscr{A}$ is norming if and only if the set of strong peak functions in $\mathscr{A}$ is dense.*

Proof. The necessity is proved by Theorem 4. For the converse, assume that the set of strongly norm attaining functions in $\mathscr{A}$ is dense in $\mathscr{A}$. Given $f \in \mathscr{A}$, there is a sequence $\{f_n\}$ of strong peak functions in $\mathscr{A}$ such that $\lim_{n\longrightarrow\infty}\|f_n - f\| = 0$. For each n, let t_n be the strong peak point corresponding to f_n. Then

$$\begin{aligned}0 &= \lim_{n\longrightarrow\infty}\|f_n - f\| \geq \limsup_{n\longrightarrow\infty}\|f_n(t_n) - f(t_n)\| \\ &\geq \limsup_{n\longrightarrow\infty}\left|\|f_n(t_n)\| - \|f(t_n)\|\right|.\end{aligned} \tag{22}$$

Thus,

$$\begin{aligned}0 &= \lim_{n\longrightarrow\infty}\left|\|f_n(t_n)\| - \|f(t_n)\|\right| \\ &= \lim_{n\longrightarrow\infty}\left|\|f_n\| - \|f(t_n)\|\right|.\end{aligned} \tag{23}$$

This means that $\lim_{n\longrightarrow\infty}\|f_n(t_n)\| = \lim_{n\longrightarrow\infty}\|f_n\| = \|f\|$. This shows that $\rho\mathscr{A}$ is a norming subset of $\mathscr{A}$. □

Theorem 6. *Let X be a Banach space and $\mathscr{A}$ be an X-valued function algebra on a Hausdorff space Ω such that $\rho\mathscr{A}$ is a norming subset for $\mathscr{A}$. Fix $P \in \mathscr{A}_u(B_X : X)$ and define the map $Q_P : B_{\mathscr{A}} \longrightarrow C_b(\Omega : X)$ by $Q_P(f)(t) = P(f(t))$ for $f \in B_{\mathscr{A}}$ and $t \in \Omega$. Suppose that $Q_P(f)$ is an element of $\mathscr{A}$ for every $f \in B_{\mathscr{A}}$ and for every $P \in \mathscr{A}_u(B_X : X)$. Then we have $n_{ua}(\mathscr{A}) = n_{ua}(X)$.*

Proof. By Theorem 3, we have only to show that $n_{ua}(\mathscr{A}) \leqslant n_{ua}(X)$. Consider the set

$$\begin{aligned}L = \{(f, x^* \circ \delta_t) : f \in S_{\mathscr{A}}, t \in \Omega, x^* \\ \in S_{X^*} \text{ and } x^*(f(t)) = 1\}.\end{aligned} \tag{24}$$

Let $\pi_1 : \mathscr{A} \times (\mathscr{A})^* \longrightarrow \mathscr{A}$ be the natural projection. Then since $\rho\mathscr{A}$ is norming for $\mathscr{A}$, Corollary 5 shows that $\pi_1(L)$ is dense in $S_{\mathscr{A}}$. Then, it is shown [14] that for every $Q \in \mathscr{A}_u(B_{\mathscr{A}}; \mathscr{A})$, we have

$$v(Q) = \sup\{|x^*[Q(f)(t)]| : (f, x^* \circ \delta_t) \in L\}. \tag{25}$$

Given $P \in \mathscr{A}_u(B_X : X)$ with $\|P\| = 1$, we have $Q_P \in \mathscr{A}_u(B_{\mathscr{A}}; \mathscr{A})$. Indeed, Q_P is a map from $B_{\mathscr{A}}$ to $B_{\mathscr{A}}$. Since P is uniformly continuous on B_X, given $\epsilon > 0$, there is $\delta > 0$ such that if $x, y \in B_X$ and $\|x - y\| \leqslant \delta$, then $\|P(x) - P(y)\| < \epsilon$. If $f, g \in B_{\mathscr{A}}$ and $\|f - g\| < \delta$, then $\|P(f(t)) - P(g(t))\| < \epsilon$ for all $t \in \Omega$. Hence $\|Q(f) - Q(g)\| \leqslant \epsilon$. This shows that Q_P is uniformly continuous on $B_{\mathscr{A}}$. Now it is enough to show that Q_P is G-holomorphic on $B^\circ_{\mathscr{A}}$ [15]. Fix $f \in B^\circ_{\mathscr{A}}$ and $g \in \mathscr{A}$, and let $U(f, g) = \{z \in \mathbb{C} : f + zg \in B^\circ_{\mathscr{A}}\}$ be an open subset in the complex plane. Let $\varphi(z) = Q_P(f + zg)$ for $z \in U(f, g)$. Then $\varphi(z)$ is a $\mathscr{A}$-valued continuous function on $U(f, g)$. For each $t \in \Omega$, $\varphi(z)(t) = Q_P(f + zg)(t) = P(f(t) + zg(t))$ is holomorphic. Fix $z_0 \in U(f, g)$ and choose $\delta > 0$ such that $z_0 + \delta B_{\mathbb{C}} \subset U(f, g)$. The Cauchy integral formula shows that, for each $t \in \Omega$,

$$\varphi(z_0)(t) = \frac{1}{2\pi i}\int_{|z-z_0|=\delta}\frac{\varphi(z)(t)}{z - z_0}dz. \tag{26}$$

As a result, we have

$$\varphi(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=\delta}\frac{\varphi(z)}{z - z_0}dz, \tag{27}$$

since the continuity of $\varphi(z)$ implies the Bochner integrability of the integral. This means that φ is holomorphic on $U(f, g)$ and Q_P is holomorphic on $B^\circ_{\mathscr{A}}$ [15]. We also have $\|Q_P\| = 1$ since there is a strong peak function $g \in A$ at $t_0 \in \Omega$ such that $g(t_0) = 1 = \|g\|$ and $g \otimes x$ is in $B_{\mathscr{A}}$ for each $x \in B_X$. It is clear that $v(Q_P) \geq n_{ua}(\mathscr{A})$. For every $\epsilon > 0$, there is $(f, x^* \circ \delta_t) \in L$ such that

$$\begin{aligned}n_{ua}(\mathscr{A}) - \epsilon \leqslant v(Q_P) - \epsilon < \left|(x^* \circ \delta_t)(Q_P(f))\right| \\ = \left|x^*(P(f(t)))\right| \leqslant v(P).\end{aligned} \tag{28}$$

Therefore, we get $n_{ua}(\mathscr{A}) \leqslant n_{ua}(X)$. □

The same proof shows the following.

Theorem 7. *Let X be a Banach space and $\mathscr{A}$ be an X-valued function algebra on a Hausdorff space Ω such that $\rho\mathscr{A}$ is a norming subset for $\mathscr{A}$. Fix $P \in \mathscr{P}(^kX : X)$ and define the map $Q_P : \mathscr{A} \longrightarrow C_b(\Omega : X)$ by $Q_P(f)(t) = P(f(t))$ for $f \in \mathscr{A}$ and $t \in \Omega$. Suppose that $Q_P(f)$ is an element of $\mathscr{A}$ for every f in $\mathscr{A}$ and for every $P \in \mathscr{P}(^kX : X)$). Then we have $n^{(k)}(\mathscr{A}) = n^{(k)}(X)$.*

Proof. The main difficulty in the proof of Theorem 7 is to check that Q_P is in $\mathscr{P}(^k\mathscr{A}; \mathscr{A})$. Let L be the corresponding continuous k-linear map defining P. Let $\widetilde{L} : \mathscr{A}^k \longrightarrow \mathscr{A}$ by $\widetilde{L}(f_1, \ldots, f_k)(t) = L(f_1(t), f_2(t), \ldots, f_k(t))$ for $f_i \in \mathscr{A}$ and $t \in \Omega$. Then it is easy to check that $\widetilde{L}$ is a continuous k-linear map and $Q_P(f) = \widetilde{L}(f, \ldots, f)$ for $f \in \mathscr{A}$. The other part of the proof is the same as the proof of Theorem 6. □

Let X, Y be Banach spaces and let $\mathscr{A}(B_X : Y)$ be either $\mathscr{A}_u(B_X : Y)$ or $\mathscr{A}_b(B_X : Y)$. Notice that $\mathscr{A}(B_X : Y)$ are Y-valued function algebras over B_X. If a Banach space X is finite dimensional, $\rho\mathscr{A}(B_X)$ is the set of all complex extreme points of B_X as observed in [16, 17]. A strongly exposed point of B_X is a strong peak point for $\mathscr{A}(B_X)$, so if a strongly exposed point of B_X is dense in S_X, then $\overline{\rho\mathscr{A}(B)} = S_X$ and it is norming for $\mathscr{A}(B_X)$. It is also proved in [17] that if X is locally c-uniformly convex space and it is an order continuous sequence space, then $\rho\mathscr{A}_u(B_X)$ is norming. The typical example of uniformly complex convex sequence space is ℓ_1. For the definitions related to various complex convexities and more examples, we refer to [9, 18–21].

Let C be a closed convex and bounded set in a Banach space X. The set C has the *Radon-Nikodým property* if, for every probability space $(\Omega, \mathscr{B}, \mu)$ and every X-valued countably additive measure τ on $\mathscr{B}$ such that $\tau(A)/\mu(A) \in C$ for every $A \in \mathscr{B}$ with $\mu(A) > 0$, there is a Bochner measurable $f : \Omega \longrightarrow X$ so that

$$\tau(A) = \int_A f(\omega)\, d\mu(\omega), \quad A \in \mathscr{B}. \tag{29}$$

The space X is said to have the *Radon-Nikodým property* if its unit ball B_X has the Radon-Nikodým property [22]. For the basic properties and useful information on the Radon-Nikodým property, see also [22–25]. It has been shown [9] that if X has the Radon-Nikodým property, then $\rho\mathscr{A}(B_X)$ is norming for $\mathscr{A}(B_X)$.

Corollary 8. *Suppose that X satisfies one of the following conditions: (i) X has the Radon-Nikodým property; (ii) X is locally uniformly convex space; (iii) X is a locally c-uniformly convex order continuous sequence space. Then we have*

(i) $n^{(k)}(\mathscr{A}(B_X : Y)) = n^{(k)}(Y)$ *for every* $k \geq 1$,

(ii) $n_{ua}(\mathscr{A}(B_X : Y)) = n_{ua}(Y)$.

Proof. If X satisfies one of the three conditions, $\rho\mathscr{A}(B_X : Y) = \rho\mathscr{A}(B_X)$ and it is norming for $\mathscr{A}(B_X : Y)$. Therefore, Theorem 3 implies that $n^{(k)}(\mathscr{A}(B_X : Y)) \geq n^{(k)}(Y)$, $n_{ua}(\mathscr{A}(B_X : Y)) \geq n_{ua}(Y)$. For the case (ii), fix $P \in \mathscr{A}_u(B_Y : Y)$ and define the map $Q_P : \mathscr{A}(B_X : Y) \longrightarrow C_b(B_X : Y)$ by $Q_P(f)(x) = P(f(x))$ for $f \in \mathscr{A}(B_X : Y)$ and $x \in B_X$. Then $Q_P(f) \in \mathscr{A}(B_X : Y)$. Consequently, Theorem 6 shows that

$$n_{ua}(\mathscr{A}(B_X : Y)) = n_{ua}(Y) \tag{30}$$

and the proof of (ii) is complete. The remaining proof (i) can be finished in the same way by Theorem 7. □

By Theorem 3, we get the following.

Corollary 9. *Let Ω be a Hausdorff topological space and suppose that $\rho C_b(\Omega)$ is norming for $C_b(\Omega)$. If Y is a Banach space with $n_{ba}(Y) = 1$, we have $v(f) = \|f\|$ for all $f \in \mathscr{A}_b(B_{C_b(\Omega:Y)} : C_b(\Omega : Y))$.*

As we show in the next proposition, closed bounded convex sets with the Radon-Nikodým property satisfy the condition of Corollary 9.

Proposition 10. *Suppose that Ω is a nonempty closed bounded convex subset of a Banach space and Ω has the Radon-Nikodým property. Then $\rho C_b(\Omega)$ is norming for $C_b(\Omega)$ and the set of strong peak functions of $C_b(\Omega)$ is dense.*

Proof. It is enough to show that the set of strong peak functions of $C_b(\Omega)$ is dense by Corollary 5. Given $f \in C_b(\Omega)$ and $\epsilon > 0$, the Stegall perturbed optimization theorem [25] shows that there is $x^* \in X^*$ such that the function $\varphi(x) = |f(x)| + |\mathrm{Re}(x^*(x))|$ strongly attains its norm at $x_0 \in \Omega$ and $\|x^*\| < \epsilon$. Choose a complex number $z_0 \in S_{\mathbb{C}}$ such that

$$\left|f(x_0)\right| + \left|\mathrm{Re}\left(x^*(x_0)\right)\right| = \left|f(x_0) + z_0 \mathrm{Re}\left(x^*(x_0)\right)\right|. \tag{31}$$

Then it is easy to check that $g(x) = f(x) + z_0 \mathrm{Re}(x^*(x))$ is a strong peak function at x_0 and $\|f - g\| < \epsilon$. This shows the denseness of the set of strong peak functions on $C_b(\Omega)$. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares no conflicts of interest.

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