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Research Article

The Growth on the Maximum Modulus of Double Dirichlet Series

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The main purpose of this paper is to investigate the growth of several entire functions represented by double Dirichlet series of finite logarithmic order, h-order. Besides, we also study some properties on the maximum modulus of double Dirichlet series and its partial derivative. Our results are extension and improvement of previous results given by Huo and Liang.

1. Introduction and Basic Notes

For Dirichlet series

$$f(s)=\sum_{n=1}^{\infty}a_n e^{\lambda_n s},\quad s=\sigma+it, \tag{1}$$

where

$$\begin{aligned} &0\le\lambda_1<\lambda_2<\cdots<\lambda_n<\cdots,\\ &\qquad\lambda_n\longrightarrow+\infty\ as\ n\longrightarrow+\infty; \end{aligned} \tag{2}$$

$s=\sigma+it$ (σ,t are real variables); a_n are nonzero complex numbers.

It is an interesting topic to study some properties of Dirichlet series in the fields of complex analysis;particularly, considerable attention has been paid to the analytic function and entire funtcions represented by Dirichlet series in the half-plane and whole plane, and a number of interesting and important results can be found in [1]. For example, J. R. Yu, G. Srivastava, P. V. Filevich, Z. S. Gao, D. C. Sun, etc. studied the growth and value distribution of Dirichlet series and random Dirichlet series (see [2–17]); Y. Y. Kong, G. T. Deng investigated the growth of Dirichlet-Hadamard product (see [18, 19]); A. R. Reddy, M. N. Sheremeta, C. F. Yi, and H. Y. Xu studied the approximation of Dirichlet series (see [20–24]), and so on.

In 1962, J. R. Yu [25] had made some pioneering research for the growth of double Dirichlet series as follows:

$$f(s_1,s_2)=\sum_{m,n=0}^{+\infty}a_{m,n}\exp\{\lambda_m s_1+\mu_n s_2\}, \tag{3}$$

where $s_1=\sigma_1+it_1$, $s_2=\sigma_2+it_2$, $a_{m,n}\in\mathbb{C}$, and

$$\begin{aligned} &0=\lambda_0<\lambda_1<\cdots<\lambda_m<\cdots\uparrow+\infty,\\ &0=\mu_0<\mu_1<\cdots<\mu_n<\cdots\uparrow+\infty,\quad(n\longrightarrow+\infty). \end{aligned} \tag{4}$$

However, there were few results about double Dirichlet series because the research involves complex two-dimensional space. In 2009, J. Liu and Z. S. Gao [26] discussed the problem on θ-order of entire function represented by the double Dirichlet series in the double horizontal line; recently, G. N. Gao [27] further studied the problem about θ-order of double Dirichlet series by using the Knopp-Kojima method. In this paper, we further investigate the growth of entire functions represented by double Dirichlet series, such as the logarithmic order, h-order, and some properties of the maximum modulus of double Dirichlet series and its partial derivatives.

If double Dirichlet series satisfies

$$\limsup_{m+n\longrightarrow+\infty}\frac{\log(m+n)}{\lambda_m+\mu_n}=E<+\infty,$$
$$\limsup_{m+n\longrightarrow+\infty}\frac{\log|a_{m,n}|}{\lambda_m+\mu_n}=-\infty, \tag{5}$$

then we call that $f(s_1,s_2)$ is analytic in the double whole plane, i.e., the entire Dirichlet series of double whole planes. Let D be the set of all entire function represented by double Dirichlet series (3) satisfying (4)-(5). Let

$$M(\sigma_1,\sigma_2,f)=\sup_{-\infty<t_1,t_2<+\infty}\{|f(\sigma_1+it_1,\sigma_2+it_2)|,\ Res_1\leq\sigma_1,\ Res_2\leq\sigma_2\}, \tag{6}$$

be the maximum modulus of $f(s_1,s_2)$ in $Res_1\leq\sigma_1$, $Res_2\leq\sigma_2$. From the definition of the maximum modulus, if $f(s_1,s_2)$ is nonconstant with respect to s_1,s_2, we have

$$M(\sigma_1',\sigma_2,f)>M(\sigma_1,\sigma_2,f),\quad \sigma_1'>\sigma_1;$$
$$M(\sigma_1,\sigma_2')>M(\sigma_1,\sigma_2),\quad \sigma_2'>\sigma_2; \tag{7}$$

and

$$M(\sigma_1',\sigma_2',f)>M(\sigma_1,\sigma_2,f),\quad \sigma_1'>\sigma_1,\ \sigma_2'>\sigma_2. \tag{8}$$

To state our results, we can introduce the following definitions.

Definition 1. Suppose $f(s_1,s_2)\in D$; let L be the set of $f(s_1,s_2)(\in D)$ with finite logarithmic order and $f(s_1,s_2)$ satisfy the following conditions:

(i) For any fixed value of $\sigma_2>0$, there exists $\sigma_1^0=\sigma_1^0(K_1,\beta_1,\sigma_2)$ such that

$$M(\sigma_1,\sigma_2,f)<\exp\{K_1\sigma_1^{\beta_1}\},\quad for\ \sigma_1\geq\sigma_1^0, \tag{9}$$

(ii) For any fixed value of $\sigma_1>0$, there exists $\sigma_2^0=\sigma_2^0(K_2,\beta_2,\sigma_1)$ such that

$$M(\sigma_1,\sigma_2,f)<\exp\{K_2\sigma_2^{\beta_2}\},\quad for\ \sigma_2\geq\sigma_2^0, \tag{10}$$

where $K_1>0,\beta_1>0;K_2>0,\beta_2>0$ are constants, so there exists $\sigma=\sigma(K_1,\beta_1,K_2,\beta_2)$ such that

$$M(\sigma_1,\sigma_2,f)<\exp\{K_1\sigma_1^{\beta_1}+K_2\sigma_2^{\beta_2}\},\quad for\ \sigma_1,\sigma_2\geq\sigma. \tag{11}$$

Definition 2. Suppose that $f(s_1,s_2)\in D$; for any small $\varepsilon>0$ and fixed value $\sigma_2>0$, there exists $\sigma^{(1)}=\sigma^{(1)}(\varepsilon,\sigma_2)$ such that

$$M(\sigma_1,\sigma_2,f)<\exp\{\sigma_1^{\rho_1^L+\varepsilon}\},\quad for\ \sigma_1\geq\sigma^{(1)}, \tag{12}$$

and there exists at least a real value $\sigma_2^0(\varepsilon)$ and sufficiently large number σ_{1i} such that

$$M(\sigma_{1i},\sigma_2^0(\varepsilon),f)>\exp\{\sigma_{1i}^{\rho_1^L-\varepsilon}\}, \tag{13}$$

then we say that $f(s_1,s_2)$ has partial logarithmic order ρ_1^L with respect to s_1; similarly, for any small $\varepsilon>0$ and fixed value $\sigma_1>0$, there exists $\sigma^{(2)}=\sigma^{(2)}(\varepsilon,\sigma_2)$ such that

$$M(\sigma_1,\sigma_2,f)<\exp\{\sigma_2^{\rho_2^L+\varepsilon}\},\quad for\ \sigma_2\geq\sigma^{(2)}, \tag{14}$$

and there exists at least a real value $\sigma_1^0(\varepsilon)$ and sufficiently large number σ_{2j} such that

$$M(\sigma_{2j},\sigma_1^0(\varepsilon),f)>\exp\{\sigma_{2j}^{\rho_2^L-\varepsilon}\}, \tag{15}$$

then we say that $f(s_1,s_2)$ has partial logarithmic order ρ_2^L with respect to s_2.

Remark 3. If $f(s_1,s_2)$ has partial logarithmic order ρ_1^L with respect to s_1, that is,

$$\limsup_{\sigma_2\longrightarrow+\infty}\left\{\limsup_{\sigma_1\longrightarrow+\infty}\frac{\log\log M(\sigma_1,\sigma_2,f)}{\log\sigma_1}\right\}=\rho_1^L; \tag{16}$$

and if $f(s_1,s_2)$ has partial logarithmic order ρ_2 with respect to s_2, that is,

$$\limsup_{\sigma_1\longrightarrow+\infty}\left\{\limsup_{\sigma_2\longrightarrow+\infty}\frac{\log\log M(\sigma_1,\sigma_2,f)}{\log\sigma_2}\right\}=\rho_2^L. \tag{17}$$

Definition 4. Let $f(s_1,s_2)\in D$, $0<\rho_1^L,\rho_2^L<\infty$, and satisfy the following conditions:

(i) $f(s_1,s_2)\in L$;

(ii) $f(s_1,s_2)$ has partial logarithmic order ρ_1^L with respect to s_1 and partial logarithmic order ρ_2^L with respect to s_2;

(iii) for any small number $\varepsilon>0$, there exists $\sigma=\sigma(\varepsilon)$ such that

$$M(\sigma_1,\sigma_2,f)<\exp\{\sigma_1^{\rho_1^L+\varepsilon}+\sigma_2^{\rho_2^L+\varepsilon}\},\quad as\ \sigma_1,\sigma_2>\sigma, \tag{18}$$

then we say that $f(s_1,s_2)$ has finite logarithmic order (ρ_1^L,ρ_2^L).

For the logarithmic order of double Dirichlet series $f(s_1,s_2)$, we have the following.

Theorem 5. *Let $f(s_1,s_2)\in D$ be of finite logarithmic order (ρ_1^L,ρ_2^L), $1<\rho_1^L,\rho_2^L<+\infty$; then*

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\frac{\log\log M(\sigma_1,\sigma_2,f)}{\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2}=1. \tag{19}$$

To estimate the growth of $f(s_1,s_2)$ more precisely, we will give the logarithmic type of $f(s_1,s_2)$ as follows.

Definition 6. Suppose $f(s_1, s_2) \in D$; let L_T be the set of $f(s_1, s_2) \in L$ given by double Dirichlet series of finite logarithmic order (ρ_1^L, ρ_2^L) has finite logarithmic type and $f(s_1, s_2)$ satisfy the following conditions:

(i) For any fixed value of $\sigma_2 > 0$, there exists $\sigma_1^0 = \sigma_1^0(K_1, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{K_1 \rho_1^L \sigma_1\right\}, \quad for\ \sigma_1 \geq \sigma_1^0, \tag{20}$$

(ii) For any fixed value of $\sigma_1 > 0$, there exists $\sigma_2^0 = \sigma_2^0(K_2, \sigma_1)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{K_2 \rho_2^L \sigma_2\right\}, \quad for\ \sigma_2 \geq \sigma_2^0, \tag{21}$$

where $K_1 > 0, K_2 > 0$, are constants, so there exists $\sigma = \sigma(K_1, K_2,)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{K_1 \rho_1^L \sigma_1 + K_2 \rho_2^L \sigma_2\right\}, \quad for\ \sigma_1, \sigma_2 \geq \sigma. \tag{22}$$

Definition 7. Suppose that $f(s_1, s_2) \in D$ is of finite logarithmic (ρ_1^L, ρ_2^L), and for any small $\varepsilon > 0$ and fixed value $\sigma_2 > 0$, there exists $\sigma^{(1)} = \sigma^{(1)}(\varepsilon, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\left(\tau_1^L + \varepsilon\right) \rho_1^L \sigma_1\right\}, \quad for\ \sigma_1 \geq \sigma^{(1)}, \tag{23}$$

and there exist at least a real value $\sigma_2^0(\varepsilon)$ and sufficiently large number σ_{1i} such that

$$M\left(\sigma_{1i}, \sigma_2^0(\varepsilon), f\right) > \exp\left\{\left(\tau_1^L - \varepsilon\right) \rho_1^L \sigma_{1i}\right\}, \tag{24}$$

then we say that $f(s_1, s_2)$ has partial logarithmic type τ_1^L with respect to s_1; similarly, for any small $\varepsilon > 0$ and fixed value $\sigma_1 > 0$, there exists $\sigma^{(2)} = \sigma^{(2)}(\varepsilon, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\left(\tau_2^L + \varepsilon\right) \rho_2^L \sigma_2\right\}, \quad for\ \sigma_2 \geq \sigma^{(2)}, \tag{25}$$

and there exists at least a real value $\sigma_1^0(\varepsilon)$ and sufficiently large number σ_{2j} such that

$$M\left(\sigma_{2j}, \sigma_1^0(\varepsilon), f\right) > \exp\left\{\left(\tau_2^L - \varepsilon\right) \rho_2^L \sigma_{2j}\right\}, \tag{26}$$

then we say that $f(s_1, s_2)$ have partial logarithmic type τ_2^L with respect to s_2.

Remark 8. If $f(s_1, s_2)$ is of finite logarithmic order (ρ_1^L, ρ_2^L) having partial logarithmic type τ_1^L with respect to s_1, that is,

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{\limsup_{\sigma_1 \longrightarrow +\infty} \frac{\log M(\sigma_1, \sigma_2, f)}{\rho_1^L \sigma_1}\right\} = \tau_1^L; \tag{27}$$

and if $f(s_1, s_2)$ has partial logarithmic order ρ_2 with respect to s_2, that is,

$$\limsup_{\sigma_1 \longrightarrow +\infty} \left\{\limsup_{\sigma_2 \longrightarrow +\infty} \frac{\log M(\sigma_1, \sigma_2, f)}{\rho_2^L \sigma_2}\right\} = \tau_2^L. \tag{28}$$

Definition 9. Let $f(s_1, s_2) \in D$ be of finite logarithmic order $(\rho_1^L, \rho_2^L), 0 < \rho_1^L, \rho_2^L < \infty$, and satisfy the following conditions:

(i) $f(s_1, s_2) \in L_T$;

(ii) $f(s_1, s_2)$ has partial logarithmic type τ_1^L with respect to s_1 and partial logarithmic type τ_2^L with respect to s_2;

(iii) for any small number $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\left(\tau_1^L + \varepsilon\right) \rho_1^L \sigma_1 + \left(\tau_2^L + \varepsilon\right) \rho_2^L \sigma_2\right\}, \quad as\ \sigma_1, \sigma_2 > \sigma, \tag{29}$$

then we say that $f(s_1, s_2)$ has finite logarithmic type (τ_1^L, τ_2^L).

For the logarithmic type of double Dirichlet series $f(s_1, s_2)$, we have the following.

Theorem 10. *Let $f(s_1, s_2) \in D$ be of finite logarithmic order (ρ_1^L, ρ_2^L), $1 < \rho_1^L, \rho_2^L < +\infty$, and of finite logarithmic type (τ_1^L, τ_2^L); then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \frac{\log M(\sigma_1, \sigma_2, f)}{\tau_1^L \rho_1^L \sigma_1 + \tau_2^L \rho_2^L \sigma_2} = 1. \tag{30}$$

In this paper, we also deal with the growth of double Dirichlet series of finite order and infinite order by using a class of functions to reduce $M(\sigma_1, \sigma_2, f)$ which is better than the previous form. So, we firstly give the definition of h-order of double Dirichlet series as follows, which is an extension of [6, 9].

Let $\mathfrak{F}$ be the class of all functions $h(x)$ satisfying the following conditions:

(i) $h(x)$ is defined on $[a, +\infty)$ and is positive, strictly increasing, and differentiable and tends to $+\infty$ as $x \longrightarrow +\infty$;

(ii) $h(x) \sim \log(x)$ as $x \longrightarrow +\infty$ for $p = 1$, and $\lim_{x \longrightarrow +\infty}(d(h(x))/d(\log^p x)) = 0$, $p > 1, p \in N^+$, where $\log^0 x = x, \log^1 x = \log x$ and $\log^p x = \log(\log^{p-1} x)$.

Similar to the above definitions, we give the h-order of double Dirichlet series as follows.

Definition 11. Suppose $f(s_1, s_2) \in D$ and $h(x) \in \mathfrak{F}$; let L^h be the set of $f(s_1, s_2)(\in D)$ with finite h-order, and $f(s_1, s_2)$ satisfies the following conditions:

(i) For any fixed value of $\sigma_2 > 0$, there exists $\sigma_1^0 = \sigma_1^0(K_1, \beta_1, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\gamma\left(K_1 \sigma_1^{\beta_1}\right)\right\}, \quad for\ \sigma_1 \geq \sigma_1^0, \tag{31}$$

(ii) For any fixed value of $\sigma_1 > 0$, there exists $\sigma_2^0 = \sigma_2^0(K_2, \beta_2, \sigma_1)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\gamma\left(K_2 \sigma_2^{\beta_2}\right)\right\}, \quad for\ \sigma_2 \geq \sigma_2^0, \tag{32}$$

where $K_1 > 0, \beta_1 > 0; K_2 > 0, \beta_2 > 0$ are constants, so there exists $\sigma = \sigma(K_1, \beta_1, K_2, \beta_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\gamma\left(K_1 \sigma_1^{\beta_1} + K_2 \sigma_2^{\beta_2}\right)\right\}, \quad \text{for } \sigma_1, \sigma_2 \geq \sigma, \tag{33}$$

where $\gamma(x)$ is the inverse function of $h(x)$, especially $\gamma(x) = e^x$ as $p = 1$.

Definition 12. Suppose that $f(s_1, s_2) \in D$; for any small $\varepsilon > 0$ and fixed value $\sigma_2 > 0$, there exists $\sigma^{(1)} = \sigma^{(1)}(\varepsilon, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\gamma\left(\sigma_1^{\rho_1^h + \varepsilon}\right)\right\}, \quad \text{for } \sigma_1 \geq \sigma^{(1)}, \tag{34}$$

and there exist at least a real value $\sigma_2^0(\varepsilon)$ and sufficiently large number σ_{1i} such that

$$M\left(\sigma_{1i}, \sigma_2^0(\varepsilon), f\right) > \exp\left\{\gamma\left(\sigma_{1i}^{\rho_1^h - \varepsilon}\right)\right\}, \tag{35}$$

then we say that $f(s_1, s_2)$ has partial h-order ρ_1^h with respect to s_1; similarly, for any small $\varepsilon > 0$ and fixed value $\sigma_1 > 0$, there exists $\sigma^{(2)} = \sigma^{(2)}(\varepsilon, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\gamma\left(\sigma_2^{\rho_2^h + \varepsilon}\right)\right\}, \quad \text{for } \sigma_2 \geq \sigma^{(2)}, \tag{36}$$

and there exist at least a real value $\sigma_1^0(\varepsilon)$ and sufficiently large number σ_{2j} such that

$$M\left(\sigma_{2j}, \sigma_1^0(\varepsilon), f\right) > \exp\left\{\gamma\left(\sigma_{2j}^{\rho_2^h - \varepsilon}\right)\right\}, \tag{37}$$

then we say that $f(s_1, s_2)$ has partial h-order ρ_2^h with respect to s_2.

Remark 13. If $f(s_1, s_2)$ has partial h-order ρ_1^h with respect to s_1, that is,

$$\limsup_{\sigma_2 \to +\infty} \left\{\limsup_{\sigma_1 \to +\infty} \frac{h(\log M(\sigma_1, \sigma_2, f))}{\sigma_1}\right\} = \rho_1^h; \tag{38}$$

and if $f(s_1, s_2)$ has partial h-order ρ_2^h with respect to s_2, that is,

$$\limsup_{\sigma_1 \to +\infty} \left\{\limsup_{\sigma_2 \to +\infty} \frac{h(\log M(\sigma_1, \sigma_2, f))}{\sigma_2}\right\} = \rho_2^h. \tag{39}$$

Definition 14. Let $f(s_1, s_2) \in D$, $0 < \rho_1^h, \rho_2^h < \infty$, and satisfy the following conditions:

(i) $f(s_1, s_2) \in L^h$;

(ii) $f(s_1, s_2)$ has partial h-order ρ_1^h with respect to s_1 and partial h-order ρ_2^h with respect to s_2;

(iii) for any small number $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon)$ such that

$$M(\sigma_1, \sigma_2, f) < \exp\left\{\gamma\left(\sigma_1^{\rho_1^h + \varepsilon} + \sigma_2^{\rho_2^h + \varepsilon}\right)\right\}, \quad \text{as } \sigma_1, \sigma_2 > \sigma, \tag{40}$$

then we say that $f(s_1, s_2)$ has finite h-order (ρ_1^h, ρ_2^h).

For h-order of double Dirichlet series, we have the following.

Theorem 15. *Let $f(s_1, s_2) \in D$ and be of finite h-order (ρ_1^h, ρ_2^h), $0 < \rho_1^h, \rho_2^h < +\infty$; then*

$$\limsup_{\sigma_1, \sigma_2 \to +\infty} \frac{h(\log M(\sigma_1, \sigma_2, f))}{\rho_1^h \sigma_1 + \rho_2^h \sigma_2} = 1. \tag{41}$$

In fact, when $h(x) = \log(x)$, then h-order is called finite order. Thus, we obtain the following conclusion.

Corollary 16. *Let $f(s_1, s_2) \in D$ be of finite order (ρ_1, ρ_2), $0 < \rho_1, \rho_2 < \infty$; then*

$$\limsup_{\sigma_1, \sigma_2 \to +\infty} \frac{\log\log M(\sigma_1, \sigma_2, f)}{\rho_1 \sigma_1 + \rho_2 \sigma_2} = 1. \tag{42}$$

And when $h(x) = \log^p x$, $p \geq 2$ and $p \in N_+$, then h-order is called the p-order. So, we have the following.

Corollary 17. *Let $f(s_1, s_2) \in D$ be of finite p-order (ρ_1^p, ρ_2), $0 < \rho_1^p, \rho_2^p < +\infty$; then*

$$\limsup_{\sigma_1, \sigma_2 \to +\infty} \frac{\log_{p+1} M(\sigma_1, \sigma_2, f)}{\rho_1^p \sigma_1 + \rho_2^p \sigma_2} = 1. \tag{43}$$

Similarly, we give the h-type of double Dirichlet series $f(s_1, s_2)$ as follows.

Definition 18. Suppose $f(s_1, s_2) \in D$ and $h(x) \in \mathfrak{F}$; let L_T^h be the set of $f(s_1, s_2) \in L^h$ given by double Dirichlet series of finite logarithmic order (ρ_1^L, ρ_2^L) having finite h-order, and $f(s_1, s_2)$ satisfies the following conditions:

(i) For any fixed value of $\sigma_2 > 0$, there exists $\sigma_1^0 = \sigma_1^0(K_1, \beta_1, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \gamma\left(K_1 \exp\left\{\rho_1^h \sigma_1\right\}\right), \quad \text{for } \sigma_1 \geq \sigma_1^0, \tag{44}$$

(ii) For any fixed value of $\sigma_1 > 0$, there exists $\sigma_2^0 = \sigma_2^0(K_2, \beta_2, \sigma_1)$ such that

$$M(\sigma_1, \sigma_2, f) < \gamma\left(K_2 \exp\left\{\rho_2^h \sigma_2\right\}\right), \quad \text{for } \sigma_2 \geq \sigma_2^0, \tag{45}$$

where $K_1 > 0, \beta_1 > 0; K_2 > 0, \beta_2 > 0$ are constants, so there exists $\sigma = \sigma(K_1, \beta_1, K_2, \beta_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \gamma\left(K_1 \exp\left\{\rho_1^h \sigma_1\right\} + K_2 \exp\left\{\rho_2^h \sigma_2\right\}\right), \quad \text{for } \sigma_1, \sigma_2 \geq \sigma. \tag{46}$$

Definition 19. Suppose that $f(s_1, s_2) \in D$; for any small $\varepsilon > 0$ and fixed value $\sigma_2 > 0$, there exists $\sigma^{(1)} = \sigma^{(1)}(\varepsilon, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \gamma\left(\left(\tau_1^h + \varepsilon\right) \exp\left\{\rho_1^h \sigma_1\right\}\right), \quad \text{for } \sigma_1 \geq \sigma^{(1)}, \tag{47}$$

and there exist at least a real value $\sigma_2^0(\varepsilon)$ and sufficiently large number σ_{1i} such that

$$M\left(\sigma_{1i}, \sigma_2^0(\varepsilon), f\right) > \exp\left\{\gamma(\left(\tau_1^h - \varepsilon\right) \exp\left(\rho_1^h \sigma_{1i}\right)\right\}, \tag{48}$$

then we say that $f(s_1, s_2)$ has partial h-type ρ_1^h with respect to s_1; similarly, for any small $\varepsilon > 0$ and fixed value $\sigma_1 > 0$, there exists $\sigma^{(2)} = \sigma^{(2)}(\varepsilon, \sigma_2)$ such that

$$M(\sigma_1, \sigma_2, f) < \gamma\left(\left(\tau_2^h + \varepsilon\right) \exp\left\{\rho_2^h \sigma_2\right\}\right), \quad \text{for } \sigma_2 \geq \sigma^{(2)}, \tag{49}$$

and there exist at least a real value $\sigma_1^0(\varepsilon)$ and sufficiently large number σ_{2j} such that

$$M\left(\sigma_{2j}, \sigma_1^0(\varepsilon), f\right) > \gamma\left(\left(\tau_2^h - \varepsilon\right) \exp\left\{\rho_2^h \sigma_{2i}\right\}\right), \tag{50}$$

then we say that $f(s_1, s_2)$ has partial h-type ρ_2^h with respect to s_2.

Remark 20. If $f(s_1, s_2)$ has partial h-type τ_1^h with respect to s_1, that is,

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{\limsup_{\sigma_1 \longrightarrow +\infty} \frac{h(M(\sigma_1, \sigma_2, f))}{\exp\{\rho_1^h \sigma_1\}}\right\} = \tau_1^h; \tag{51}$$

and if $f(s_1, s_2)$ has partial h-order ρ_2^h with respect to s_2, that is,

$$\limsup_{\sigma_1 \longrightarrow +\infty} \left\{\limsup_{\sigma_2 \longrightarrow +\infty} \frac{h(M(\sigma_1, \sigma_2, f))}{\exp\{\rho_2^h \sigma_2\}}\right\} = \tau_2^h. \tag{52}$$

Definition 21. Let $f(s_1, s_2) \in D$ be of finite h-order (ρ_1^h, ρ_2^h), $0 < \rho_1^h, \rho_2^h < +\infty$, and satisfy the following conditions:

(i) $f(s_1, s_2) \in L^h$;

(ii) $f(s_1, s_2)$ has partial h-type τ_1^h with respect to s_1 and partial h-type τ_2^h with respect to s_2;

(iii) for any small number $\varepsilon > 0$, there exists $\sigma = \sigma(\varepsilon)$ such that

$$\begin{aligned} &M(\sigma_1, \sigma_2, f) \\ &< \gamma\left(\left(\tau_1^h + \varepsilon\right) \exp\left\{\rho_1^h \sigma_1\right\} + \left(\tau_2^h + \varepsilon\right) \exp\left\{\rho_2^h \sigma_2\right\}\right), \\ &\qquad \text{as } \sigma_1, \sigma_2 > \sigma, \end{aligned} \tag{53}$$

then we say that $f(s_1, s_2)$ has finite h-type (τ_1^h, τ_2^h).

For h-type of double Dirichlet series $f(s_1, s_2)$, we have the following.

Theorem 22. *Let $f(s_1, s_2) \in D$ and be of finite h-order (ρ_1^h, ρ_2^h), $0 < \rho_1^h, \rho_2^h < +\infty$, and of finite h-type (τ_1^h, τ_2^h); then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \frac{h(M(\sigma_1, \sigma_2, f))}{\tau_1^h \exp(\rho_1^h \sigma_1) + \tau_2^h \exp(\rho_2^h \sigma_2)} = 1. \tag{54}$$

Particularly, we can get the following corollaries.

Corollary 23. *Let $f(s_1, s_2) \in D$ and be of finite order (ρ_1, ρ_2) and of finite type (τ_1, τ_2); then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \frac{\log M(\sigma_1, \sigma_2, f)}{\tau_1 \exp(\rho_1 \sigma_1) + \tau_2 \exp(\rho_2 \sigma_2)} = 1. \tag{55}$$

Corollary 24. *Let $f(s_1, s_2) \in D$ and be of finite p-order (ρ_1^p, ρ_2^p), $0 < \rho_1^p, \rho_2^p < +\infty$, and of finite p-type (τ_1^p, τ_2^p) as $p \geq 2$; then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \frac{\log_p M(\sigma_1, \sigma_2, f)\ )}{\tau_1^p \exp\left(\rho_1^p \sigma_1\right) + \tau_2^p \exp\left(\rho_2^p \sigma_2\right)} = 1. \tag{56}$$

Remark 25. In 2010 and 2015, Liang and Gao [28–30] further investigated the growth and convergence of n-multiple Dirichlet series. We only listed some Liang's definitions as $n = 2$; Liang defined the order of multiple Dirichlet series $f(s_1, s_2)$ by

$$\rho = \limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \frac{\log\log M(\sigma_1, \sigma_2, f)}{\log(e^{\sigma_1} + e^{\sigma_2})}, \tag{57}$$

and if $\rho \in (0, +\infty)$, then the type of multiple Dirichlet series $f(s_1, s_2)$

$$T = \limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \frac{\log M(\sigma_1, \sigma_2, f)}{\log(e^{\rho\sigma_1} + e^{\rho\sigma_2})}. \tag{58}$$

Obviously, our definitions about the order and type of double Dirichlet series are more general than Liang's.

The other purpose of this paper is to investigate some relation between the partial derivatives $f_{s_j}(s_1, s_2), (j = 1, 2)$ and growth of $f(s_1, s_2)$. In order to state our results, we first give some notations as follows. Let

$$f_{s_j}(s_1, s_2) = \frac{\partial}{\partial s_j} f(s_1, s_2), \tag{59}$$

and

$$\begin{aligned} &M^{(j)}(\sigma_1, \sigma_2, f) \\ &\quad = \sup_{-\infty < t_1, t_2 < +\infty} \left|f_{s_j}(\sigma_1 + it_1, \sigma_2 + it_2)\right|, \end{aligned} \tag{60}$$

for $j = 1, 2$.

Theorem 26. *Let $f(s_1, s_2) \in D$ be of finite logarithmic order (ρ_1^L, ρ_2^L), $1 < \rho_1^L, \rho_2^L < +\infty$; then*

$$\limsup_{\sigma_2 \longrightarrow \infty} \left\{ \limsup_{\sigma_1 \longrightarrow +\infty} \frac{\log M^{(1)}(\sigma_1, \sigma_2, f) - \log M(\sigma_1, \sigma_2, f)}{\log \sigma_1} \right\} = \rho_1^L - 1, \tag{61}$$

and

$$\limsup_{\sigma_1 \longrightarrow +\infty} \left\{ \limsup_{\sigma_2 \longrightarrow +\infty} \frac{\log M^{(2)}(\sigma_1, \sigma_2, f) - \log M(\sigma_1, \sigma_2, f)}{\log \sigma_2} \right\} = \rho_2^L - 1. \tag{62}$$

Theorem 27. *Let $f(s_1, s_2) \in D$ be of finite logarithmic order (ρ_1^L, ρ_2^L), $1 < \rho_1^L, \rho_2^L < +\infty$; then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \left\{ \frac{\log\left(M^{(1)}(\sigma_1, \sigma_2, f) + M^{(2)}(\sigma_1, \sigma_2, f)\right) - \log M(\sigma_1, \sigma_2, f)}{(\rho_1^L - 1)\log \sigma_1 + (\rho_2^L - 1)\log \sigma_2} \right\} = 1. \tag{63}$$

Theorem 28. *Let $f(s_1, s_2) \in D$ be of finite h-order (ρ_1^h, ρ_2^h), $0 < \rho_1^h, \rho_2^h < +\infty$; then*

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{ \limsup_{\sigma_1 \longrightarrow +\infty} \frac{h\left(M^{(1)}(\sigma_1, \sigma_2, f) / M(\sigma_1, \sigma_2, f)\right)}{\sigma_1} \right\} = \rho_1^h, \tag{64}$$

and

$$\limsup_{\sigma_1 \longrightarrow +\infty} \left\{ \limsup_{\sigma_2 \longrightarrow +\infty} \frac{h\left(M^{(2)}(\sigma_1, \sigma_2, f) / M(\sigma_1, \sigma_2, f)\right)}{\sigma_2} \right\} = \rho_2^h. \tag{65}$$

Theorem 29. *Let $f(s_1, s_2) \in D$ be of finite h-order (ρ_1^h, ρ_2^h), $0 < \rho_1^h, \rho_2^h < +\infty$; then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \left\{ \frac{h\left(\left[M^{(1)}(\sigma_1, \sigma_2, f) + M^{(2)}(\sigma_1, \sigma_2, f)\right] / M(\sigma_1, \sigma_2, f)\right)}{\rho_1^h \sigma_1 + \rho_2^h \sigma_2} \right\} = 1. \tag{66}$$

For finite order and finite p-order, some corollaries are obtained below.

Corollary 30. *Let $f(s_1, s_2) \in D$ be of finite order (ρ_1, ρ_2), $0 < \rho_1, \rho_2 < +\infty$; then*

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{ \limsup_{\sigma_1 \longrightarrow +\infty} \frac{\log M^{(1)}(\sigma_1, \sigma_2, f) - \log M(\sigma_1, \sigma_2, f)}{\sigma_1} \right\} = \rho_1, \tag{67}$$

and

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{ \limsup_{\sigma_1 \longrightarrow +\infty} \frac{\log M^{(2)}(\sigma_1, \sigma_2, f) - \log M(\sigma_1, \sigma_2, f)}{\sigma_2} \right\} = \rho_2. \tag{68}$$

Corollary 31. *Let $f(s_1, s_2) \in D$ be of finite order (ρ_1, ρ_2), $0 < \rho_1, \rho_2 < +\infty$; then*

$$\limsup_{\sigma_1, \sigma_2 \longrightarrow +\infty} \left\{ \frac{\log\left(M^{(1)}(\sigma_1, \sigma_2, f) + M^{(2)}(\sigma_1, \sigma_2, f)\right) - \log M(\sigma_1, \sigma_2, f)}{\rho_1 \sigma_1 + \rho_2 \sigma_2} \right\} = 1. \tag{69}$$

Corollary 32. *Let $f(s_1, s_2) \in D$ be of finite p-order (ρ_1^p, ρ_2^p), $0 < \rho_1^p, \rho_2^p < +\infty$, $p \geq 2$; then*

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{ \limsup_{\sigma_1 \longrightarrow +\infty} \frac{\log_p\left(M^{(1)}(\sigma_1, \sigma_2, f) / M(\sigma_1, \sigma_2, f)\right)}{\sigma_1} \right\} = \rho_1^p, \tag{70}$$

and

$$\limsup_{\sigma_2 \longrightarrow +\infty} \left\{ \limsup_{\sigma_1 \longrightarrow +\infty} \frac{\log_p\left(M^{(2)}(\sigma_1, \sigma_2, f) / M(\sigma_1, \sigma_2, f)\right)}{\sigma_2} \right\} = \rho_2^p. \tag{71}$$

Corollary 33. *Let $f(s_1, s_2) \in D$ be of finite p-order (ρ_1^p, ρ_2^p), $0 < \rho_1^p, \rho_2^p < +\infty, p \geq 2$; then*

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\left\{\frac{\log_p\left(\left[M^{(1)}(\sigma_1,\sigma_2,f)+M^{(2)}(\sigma_1,\sigma_2,f)\right]/M(\sigma_1,\sigma_2,f)\right)}{\rho_1^p\sigma_1+\rho_2^p\sigma_2}\right\}=1. \tag{72}$$

2. Proofs of Theorems 5–22

2.1. The Proof of Theorem 5. From (18), let $\mu_1 > \rho_1^L, \mu_2 > \rho_2^L$; it follows

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\frac{\log\log M(\sigma_1,\sigma_2)}{\mu_1\log\sigma_1+\mu_2\log\sigma_2}\leq 1. \tag{73}$$

When $\sigma_1 > 0, \sigma_2 > 0$, let $\mu=\max\{\mu_1,\mu_2\}$, $\rho^L=\min\{\rho_1^L,\rho_2^L\}$; then

$$\begin{aligned}\frac{\log\log M(\sigma_1,\sigma_2,f)}{\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2}&\leq\frac{\log\log M(\sigma_1,\sigma_2,f)}{\rho^L(\log\sigma_1+\log\sigma_2)}\\&=\frac{\log\log M(\sigma_1,\sigma_2,f)}{\mu_1\log\sigma_1+\mu_2\log\sigma_2}\frac{\mu_1\log\sigma_1+\mu_2\log\sigma_2}{\rho^L(\log\sigma_1+\log\sigma_2)}\\&\leq\frac{\mu}{\rho^L}\frac{\log\log M(\sigma_1,\sigma_2,f)}{\mu_1\log\sigma_1+\mu_2\log\sigma_2},\end{aligned} \tag{74}$$

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\frac{\log\log M(\sigma_1,\sigma_2,f)}{\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2}:=\alpha\leq\frac{\mu}{\rho^L}\leq+\infty. \tag{75}$$

Next, we continue to prove that $\alpha=1$.

Case 1. Suppose that $\alpha > 1$; then there exist two constants α_1,α_2 such that $1<\alpha_2<\alpha_1<\alpha$ and $(\alpha_2-1)\rho_1^L>\varepsilon>0$. From (18), there exist two sequences $\{\sigma_{1i}\}$, $\{\sigma_{2j}\}$ such that

$$\frac{\log\log M\left(\sigma_{1i},\sigma_{2j},f\right)}{\rho_1^L\log\sigma_{1i}+\rho_2^L\log\sigma_{2j}}>\alpha_1,\quad i,j=1,2,\ldots \tag{76}$$

that is,

$$\begin{aligned}&M\left(\sigma_{1i},\sigma_{2j},f\right)\\&\quad>\exp\exp\left\{\alpha_1\left(\rho_1^L\log\sigma_{1i}+\rho_2^L\log\sigma_{2j}\right)\right\},\\&\qquad i,j=1,2,\ldots\end{aligned} \tag{77}$$

Hence, when j is fixed at j_0 and $i>i_0$, we have

$$\begin{aligned}M\left(\sigma_{1i},\sigma_{2j},f\right)&>\exp\exp\left\{\alpha_2\left(\rho_1^L\log\sigma_{1i}\right)\right\}\\&=\exp\left\{\sigma_{1i}^{\rho_1^L\alpha_2}\right\},\end{aligned} \tag{78}$$

Since $(\alpha_2-1)\rho_1^L>\varepsilon>0$, we can obtain a contradiction with the assumption of Theorem 5 from the above inequality.

Case 2. Suppose that $\alpha<1$; then there exist two constants α_1,α_2 such that $\alpha<\gamma_1<\gamma_2<1$ and $(1-\gamma_2)\rho_1^L>\varepsilon>0$. From (75), there exists σ such that for

$$\frac{\log\log M(\sigma_1,\sigma_2,f)}{\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2}<\gamma_1, \tag{79}$$

for $\sigma_1,\sigma_2>\sigma$, that is,

$$\begin{aligned}M(\sigma_1,\sigma_2,f)&<\exp\exp\left\{\gamma_1\left(\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2\right)\right\},\\&\qquad\sigma_1,\sigma_2>\sigma.\end{aligned} \tag{80}$$

Then it follows

$$\begin{aligned}M(\sigma_1,\sigma_2,f)&<\exp\exp\left\{\gamma_1\left(\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_0\right)\right\},\\&\qquad\sigma_1\geq\sigma_0,\ 0<\sigma_2\leq\sigma_0.\end{aligned} \tag{81}$$

Thus, we can choose $\sigma^*\geq\sigma_0$ such that

$$\gamma_1\left(\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2\right)<\gamma_2\rho_1^L\log\sigma_1,\quad\sigma_1\geq\sigma^*, \tag{82}$$

then for $0<\sigma_2\leq\sigma_0$, we have

$$\begin{aligned}M(\sigma_1,\sigma_2,f)&<\exp\exp\left\{\gamma_2\left(\rho_1^L\log\sigma_1\right)\right\}\\&=\exp\left\{\sigma_1^{\rho_1^L\gamma_2}\right\}.\end{aligned} \tag{83}$$

If $\sigma_2>\sigma_0$, we can choose $\sigma^*\geq\sigma_0$ such that $\sigma_1\geq\sigma^*$ and

$$\gamma_1\left(\rho_1^L\log\sigma_1+\rho_2^L\log\sigma_2\right)<\gamma_2\rho_1^L\log\sigma_1, \tag{84}$$

that is,

$$\begin{aligned}M(\sigma_1,\sigma_2,f)&<\exp\exp\left\{\gamma_2\left(\rho_1^L\log\sigma_1\right)\right\}\\&=\exp\left\{\sigma_1^{\rho_1^L\gamma_2}\right\},\end{aligned} \tag{85}$$

as $\sigma_1\geq\sigma^*(\sigma_2)$.

Since ε is arbitrary and $(1-\gamma_2)\rho_1^L>\varepsilon>0$, from (83) and (85), we can get a contradiction with the assumptions of $f(s_1,s_2)$ being of finite logarithmic order (ρ_1^L,ρ_2^L).

Therefore, this completes the proof of Theroem 5 from Cases 1 and 2.

2.2. The Proof of Theorem 10. The proof of Theorem 10 is similar to the argument as in Theorem 5; in order to facilitate the readers, we still give the proof of Theorem 10 as follows.

From (29), take $\mu_1>\tau_1^L,\mu_2>\tau_2^L$; then

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\frac{\log M(\sigma_1,\sigma_2,f)}{\mu_1\sigma_1^{\rho_1^L}+\mu_2\sigma_2^{\rho_2^L}}\leq 1. \tag{86}$$

Let $\mu = \max\{\mu_1, \mu_2\}$, $\tau^L = \min\{\tau_1^L, \tau_2^L\}$; then

$$\begin{aligned}\frac{\log M(\sigma_1,\sigma_2,f)}{\tau_1^L\sigma_1^{\rho_1^L}+\tau_2^L\sigma_2^{\rho_2^L}} &\le \frac{\log M(\sigma_1,\sigma_2,f)}{\tau^L\left(\sigma_1^{\rho_1^L}+\sigma_2^{\rho_2^L}\right)} \\ &= \frac{\log M(\sigma_1,\sigma_2,f)}{\mu_1\sigma_1^{\rho_1^L}+\mu_2\sigma_2^{\rho_2^L}}\frac{\mu_1\sigma_1^{\rho_1^L}+\mu_2\sigma_2^{\rho_2^L}}{\tau^L\left(\sigma_1^{\rho_1^L}+\sigma_2^{\rho_2^L}\right)} \\ &\le \frac{\mu}{\tau^L}\frac{\log M(\sigma_1,\sigma_2,f)}{\mu_1\sigma_1^{\rho_1^L}+\mu_2\sigma_2^{\rho_2^L}}.\end{aligned} \tag{87}$$

Thus we have

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\frac{\log M(\sigma_1,\sigma_2,f)}{\tau_1^L\sigma_1^{\rho_1^L}+\tau_2^L\sigma_2^{\rho_2^L}} := A \le \frac{\mu}{\tau^L} \le +\infty. \tag{88}$$

Now, we are going to prove that $A = 1$.

Case 1. Suppose that $A > 1$; then there exist two constants A_1, A_2 satisfying $1 < A_2 < A_1 < A$ and $(A_2 - 1)\tau_1 > \varepsilon > 0$. Thus, from (88), there exist two sequences $\{\sigma_{1i}\}, \{\sigma_{2j}\}$ such that

$$\frac{\log M(\sigma_{1i},\sigma_{2j},f)}{\tau_1^L\sigma_{1i}^{\rho_1^L}+\tau_2^L\sigma_{2j}^{\rho_2^L}} > A_1, \quad i,j = 1,2,\ldots \tag{89}$$

i.e.,

$$M(\sigma_{1i},\sigma_{2j},f) > \exp\left\{A_1\left(\tau_1^L\sigma_{1i}^{\rho_1^L}+\tau_2^L\sigma_{2j}^{\rho_2^L}\right)\right\}, \quad i,j = 1,2,\ldots \tag{90}$$

then when j is fixed at j_0 and $i > i_0$, it yields

$$\begin{aligned}M(\sigma_{1i},\sigma_{2j},f) &> \exp\left\{A_2\left(\tau_1^L\sigma_{1i}^{\rho_1^L}\right)\right\} \\ &= \exp\left\{A_2\tau_1^L\sigma_{1i}^{\rho_1^L}\right\},\end{aligned} \tag{91}$$

Since $(A_2 - 1)\tau_1 > \varepsilon > 0$, we can get a contradiction with the assumptions of $f(s_1, s_2)$ being of finite logarithmic type (τ_1^L, τ_2^L).

Case 2. Suppose that $A < 1$; then there exist two constants A_1, A_2 satisfying $A < A_1 < A_2 < 1$ and $(1 - A_2)\tau_1 > \varepsilon > 0$. From (88), there exists σ such that

$$\frac{\log M(\sigma_1,\sigma_2,f)}{\tau_1^L\sigma_1^{\rho_1^L}+\tau_2^L\sigma_2^{\rho_2^L}} < A_1, \quad \sigma_1,\sigma_2 > \sigma, \tag{92}$$

that is,

$$M(\sigma_1,\sigma_2,f) < \exp\left\{A_1\left(\tau_1^L\sigma_1^{\rho_1^L}+\tau_2^L\sigma_2^{\rho_2^L}\right)\right\}, \quad \sigma_1,\sigma_2 > \sigma. \tag{93}$$

From (93), there exists σ_0 such that

$$M(\sigma_1,\sigma_2,f) < \exp\left\{A_1\left(\tau_1\sigma_1^{\rho_1^L}+\tau_2\sigma_0^{\rho_2^L}\right)\right\}, \quad \sigma_1 \ge \sigma_0,\ 0 < \sigma_2 \le \sigma_0. \tag{94}$$

We choose $\sigma^* \ge \sigma_0$ such that

$$A_1\left(\tau_1^L\sigma_1^{\rho_1^L}+\tau_2^L\sigma_2^{\rho_2^L}\right) < A_2\tau_1^L\sigma_1^{\rho_1^L}, \quad \sigma_1 \ge \sigma^*, \tag{95}$$

Then, for $0 < \sigma_2 \le \sigma_0$, it follows

$$M(\sigma_1,\sigma_2,f) < \exp\left\{A_2\left(\tau_1^L\sigma_1^{\rho_1^L}\right)\right\}. \tag{96}$$

If $\sigma_2 > \sigma_0$, we can choose $\sigma^* \ge \sigma_0$ such that for $\sigma_1 \ge \sigma^*$

$$A_1\left(\tau_1^L\sigma_1^{\rho_1^L}+\tau_2^L\sigma_2^{\rho_2^L}\right) < A_2\tau_1^L\sigma_1^{\rho_1^L}, \tag{97}$$

that is,

$$M(\sigma_1,\sigma_2,f) < \exp\left\{\gamma_2\left(\tau_1^L\sigma_1^{\rho_1^L}\right)\right\}, \quad \sigma_1 \ge \sigma^*(\sigma_2). \tag{98}$$

Since ε is arbitrary and $(1 - A_2)\tau_1 > \varepsilon > 0$, in view of (96) and (98), it yields a contradiction with the assumption of $f(s_1, s_2)$ being of finite logarithmic type (τ_1^L, τ_2^L).

Therefore, we completes the proof of Theorem 10 from Cases 1 and 2.

2.3. Proofs of Theorems 15 and 22. Similar to the same argument as in the proofs of Theorems 5 and 10, we can complete the proofs of Theorems 15 and 22 easily.

3. Proofs of Theorems 26 and 27

3.1. The proof of Theorem 26. For a fixed value of $\sigma_2 > 0$, let

$$G(\sigma_1,\sigma_2,f) = \frac{\log M(\sigma_1,\sigma_2,f)}{\sigma_1}, \tag{99}$$

it can easily be shown that $G(\sigma_1, \sigma_2)$ is monotonic increasing for $\sigma_1 \ge \sigma^{(1)} = \sigma^{(1)}(f, \sigma_2)$. Set ζ_1 such that $\Re\zeta_1 = \sigma_1$ and $|f(\zeta_1, s_2)| = M(\sigma_1, \sigma_2)$; then we have

$$\begin{aligned}M^{(1)}(\sigma_1,\sigma_2,f) &\ge \left|f_{\zeta_1}(\zeta_1,s_2)\right| \\ &= \left|\lim_{t\longrightarrow0}\frac{f(\zeta_1,s_2)-f(\zeta_1-t,s_2)}{t}\right| \\ &\ge \lim_{t\longrightarrow0}\frac{M(\sigma_1,\sigma_2,f)-M(\sigma_1-t,\sigma_2,f)}{t} \\ &= \lim_{t\longrightarrow0}\frac{\exp\{\sigma_1G(\sigma_1,\sigma_2,f)\}-\exp\{(\sigma_1-t)G(\sigma_1-t,\sigma_2,t)\}}{t} \\ &\ge \lim_{t\longrightarrow0}\frac{\exp\{\sigma_1G(\sigma_1,\sigma_2,f)\}-\exp\{(\sigma_1-t)G(\sigma_1,\sigma_2,t)\}}{t} \\ &= G(\sigma_1,\sigma_2,f)\exp\{\sigma_1G(\sigma_1,\sigma_2,f)\},\end{aligned} \tag{100}$$

it follows

$$\frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} \geq \frac{\log M(\sigma_1,\sigma_2,f)}{\sigma_1}, \quad \textit{for } \sigma_1 \geq \sigma^{(1)}. \tag{101}$$

Similar, there exists $\sigma^{(2)} > 0$ such that

$$\frac{M^{(2)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} \geq \frac{\log M(\sigma_1,\sigma_2,f)}{\sigma_2}, \quad \textit{for } \sigma_2 \geq \sigma^{(2)}. \tag{102}$$

From (101) and (102), we have

$$\log M^{(1)}(\sigma_1,\sigma_2,f) - \log M(\sigma_1,\sigma_2,f) \geq \log\log M(\sigma_1,\sigma_2,f) - \log\sigma_1, \quad \textit{for } \sigma_1 \geq \sigma^{(1)}. \tag{103}$$

$$\log M^{(2)}(\sigma_1,\sigma_2,f) - \log M(\sigma_1,\sigma_2,f) \geq \log\log M(\sigma_1,\sigma_2,f) - \log\sigma_2, \quad \textit{for } \sigma_2 \geq \sigma^{(2)}. \tag{104}$$

Since $f(s_1,s_2)$ is of finite logarithmic order (ρ_1^L,ρ_2^L), $1 < \rho_1^L,\rho_2^L < +\infty$, it follows from (103) and (104) that

$$\limsup_{\sigma_2\longrightarrow+\infty}\left\{\limsup_{\sigma_1\longrightarrow+\infty}\frac{\log M^{(1)}(\sigma_1,\sigma_2,f) - \log M(\sigma_1,\sigma_2,f)}{\log\sigma_1}\right\} \geq \rho_1^L - 1, \tag{105}$$

and

$$\limsup_{\sigma_1\longrightarrow+\infty}\left\{\limsup_{\sigma_2\longrightarrow+\infty}\frac{\log M^{(2)}(\sigma_1,\sigma_2,f) - \log M(\sigma_1,\sigma_2,f)}{\log\sigma_2}\right\} \geq \rho_2^L - 1. \tag{106}$$

On the other hand, for any fixed value of $\sigma_2 > 0$, $\log M(\sigma_1,\sigma_2,f)$ is an increasing convex function of σ_1; then

$$\begin{aligned}\log M(2\sigma_1,\sigma_2,f) &= \log M(\sigma_1,\sigma_2,f) + \int_{\sigma_1}^{2\sigma_1}\frac{(\partial/\partial t_1)M(t_1,\sigma_2,f)}{M(t_1,\sigma_2,f)}dt_1 \\ &\geq 2\sigma_1\frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)}.\end{aligned} \tag{107}$$

Hence, from (107) and the assumptions of Theorem 26, we can easily get

$$\limsup_{\sigma_2\longrightarrow+\infty}\left\{\limsup_{\sigma_1\longrightarrow+\infty}\frac{\log M^{(1)}(\sigma_1,\sigma_2,f) - \log M(\sigma_1,\sigma_2,f)}{\log\sigma_1}\right\} \leq \rho_1^L - 1, \tag{108}$$

Similarly,

$$\limsup_{\sigma_1\longrightarrow+\infty}\left\{\limsup_{\sigma_2\longrightarrow+\infty}\frac{\log M^{(2)}(\sigma_1,\sigma_2,f) - \log M(\sigma_1,\sigma_2,f)}{\log\sigma_2}\right\} \geq \rho_2^L - 1. \tag{109}$$

Therefore, the conclusion of Theorem 26 follows from (105), (106), (108), and (109).

Thus, this completes the proof of Theorem 26.

3.2. The Proof of Theorem 27. Since $f(s_1,s_2)$ is of finite logarithmic order (ρ_1^L,ρ_2^L), $1 < \rho_1^L,\rho_2^L < +\infty$, from Theorem 26, we have that for any $\varepsilon > 0$, and $\sigma_2 > 0$, there exists a real number $\sigma^{(1)} := \sigma^{(1)}(\varepsilon,\sigma_2)$ such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} < \exp\left\{\left(\rho_1^L+\varepsilon\right)\log\sigma_1\right\}, \quad \textit{for } \sigma_1 \geq \sigma^{(1)}, \tag{110}$$

and for any $\sigma_1 > 0$, there exists a real number $\sigma^{(2)} := \sigma^{(2)}(\varepsilon,\sigma_2)$ such that

$$\frac{M^{(2)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} < \exp\left\{\left(\rho_2^L+\varepsilon\right)\log\sigma_2\right\}, \quad \textit{for } \sigma_2 \geq \sigma^{(2)}. \tag{111}$$

Thus, there exists a real number $\sigma = \sigma(\varepsilon) \geq \max\{\sigma^{(1)},\sigma^{(2)}\}$ such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} + \frac{M^{(2)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} < \exp\left\{\left(\rho_1^L+\varepsilon\right)\log\sigma_1 + \left(\rho_2^L+\varepsilon\right)\log\sigma_2\right\}, \tag{112}$$

for $\sigma_1,\sigma_2 > \sigma$. Then it follows

$$\limsup_{\sigma_1,\sigma_2\longrightarrow+\infty}\left\{\frac{\log\left(M^{(1)}(\sigma_1,\sigma_2,f) + M^{(2)}(\sigma_1,\sigma_2,f)\right) - \log M(\sigma_1,\sigma_2,f)}{(\rho_1^L-1)\log\sigma_1 + (\rho_2^L-1)\log\sigma_2}\right\} := B \leq 1. \tag{113}$$

Here we prove that $B = 1$. If $B < 1$, there exists two constants $B < B_1 < B_2 < 1$ such that $0 < \varepsilon < (\rho_1^L - 1)(1 - B_2)$ and

$$\frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} + \frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < \exp\left\{B_1\left[\left(\rho_1^L - 1\right)\log\sigma_1 + \left(\rho_2^L - 1\right)\log\sigma_2\right]\right\}, \tag{114}$$

for $\sigma_1, \sigma_2 \geq \sigma$. Thus, from (114), we obtain that, for any $\sigma_2 > 0$, there exists a real number $\sigma^{(1)} := \sigma^{(1)}(\sigma_2)$ such that

$$\frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < \exp\left\{B_2\left(\rho_1^L - 1\right)\log\sigma_1\right\}, \quad \text{for } \sigma_1 \geq \sigma^{(1)}. \tag{115}$$

Hence, since $0 < \varepsilon < (\rho_1^L - 1)(1 - B_2)$, from Theorem 26, we can easily obtain a contradiction with the assumption that $f(s_1, s_2)$ is of finite partial logarithmic order ρ_1^L with respect to s_1.

Thus, $B = 1$; this completes the proof of Theorem 27.

4. Proofs of Theorems 28 and 29

From the definition of $h(x)$, it follows that

$$\lim_{x \to +\infty} \frac{h(cx)}{h(x)} = 1, \qquad \lim_{x \to +\infty} \frac{h(x + c)}{h(x)} = 1, \tag{116}$$

for any finite constant $c > 0$. Moreover, we have the following.

Lemma 34 (see [6, 9, 24]). *Let $h(x) \in \mathfrak{F}$, $p \geq 2$ and $\varphi(x)$ be the nonconstant function satisfying $\varphi(x) \longrightarrow +\infty$ as $x \longrightarrow +\infty$ and*

$$\limsup_{x \to +\infty} \frac{\log^+ \varphi(x)}{x} = \varrho, \quad (0 \leq \varrho < \infty), \tag{117}$$

if $M(x)$ satisfies $\limsup_{x \to +\infty}(h(\log M(x))/x) = \nu(> 0)$. Then we have

$$\limsup_{x \to +\infty} \frac{h(\varphi(x)\log M(x))}{x} = \nu. \tag{118}$$

Remark 35. Under the assumptions of Lemma 34, for $p = 1$, we have

$$\limsup_{x \to +\infty} \frac{h(\varphi(x)\log M(x))}{x} = \limsup_{x \to +\infty} \frac{h(\log M(x))}{x} = \nu, \tag{119}$$

as $\varrho = 0$.

4.1. The Proof of Theorem 28. By using the same argument as in the proof of Theorem 26, and combining with Lemma 34, we can easily prove Theorem 28.

4.2. The Proof of Theorem 29. Since $f(s_1, s_2)$ is of finite h-order (ρ_1^h, ρ_2^h), $0 < \rho_1^h, \rho_2^h < +\infty$, from Theorem 28, we have that for any $\varepsilon > 0$, and $\sigma_2 > 0$, there exists a real number $\sigma^{(1)} := \sigma^{(1)}(\varepsilon, \sigma_2)$ such that

$$\frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < \gamma\left\{\left(\rho_1^h + \varepsilon\right)\sigma_1\right\}, \quad \text{for } \sigma_1 \geq \sigma^{(1)}, \tag{120}$$

and for any $\sigma_1 > 0$, there exists a real number $\sigma^{(2)} := \sigma^{(2)}(\varepsilon, \sigma_2)$ such that

$$\frac{M^{(2)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < \gamma\left\{\left(\rho_2^h + \varepsilon\right)\sigma_2\right\}, \quad \text{for } \sigma_2 \geq \sigma^{(2)}. \tag{121}$$

Since $h(x)$ is strictly increasing and $\gamma(x)$ is the inverse function of $h(x)$, then $\gamma(x)$ is also strictly increasing. Thus, there exists a real number $\sigma = \sigma(\varepsilon) \geq \max\{\sigma^{(1)}, \sigma^{(2)}\}$ such that

$$\frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < \gamma\left\{\left(\rho_1^h + \varepsilon\right)\sigma_1 + \left(\rho_2^h + \varepsilon\right)\sigma_2\right\}, \tag{122}$$

and

$$\frac{M^{(2)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < \gamma\left\{\left(\rho_1^h + \varepsilon\right)\sigma_1 + \left(\rho_2^h + \varepsilon\right)\sigma_2\right\}, \tag{123}$$

for $\sigma_1, \sigma_2 > \sigma$. Thus, we have

$$\frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} + \frac{M^{(2)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} < 2\gamma\left\{\left(\rho_1^h + \varepsilon\right)\sigma_1 + \left(\rho_2^h + \varepsilon\right)\sigma_2\right\}, \tag{124}$$

that is,

$$h\left(\frac{1}{2}\left(\frac{M^{(1)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)} + \frac{M^{(2)}(\sigma_1, \sigma_2, f)}{M(\sigma_1, \sigma_2, f)}\right)\right) < \left(\rho_1^h + \varepsilon\right)\sigma_1 + \left(\rho_2^h + \varepsilon\right)\sigma_2 \tag{125}$$

for $\sigma_1, \sigma_2 > \sigma$. By Lemma 34, it follows

$$\limsup_{\sigma_1, \sigma_2 \to +\infty} \left\{\frac{h\left(\left[M^{(1)}(\sigma_1, \sigma_2, f) + M^{(2)}(\sigma_1, \sigma_2, f)\right]/M(\sigma_1, \sigma_2, f)\right)}{\rho_1^h\sigma_1 + \rho_2^h\sigma_2}\right\} := \eta \leq 1. \tag{126}$$

Here is to prove that $\eta = 1$. If $\eta < 1$, there exist two constants $\eta < \eta_1 < \eta_2 < 1$ such that $0 < \varepsilon < \rho_1^h(1-\eta_2)$ and

$$\frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} + \frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} < \exp\left\{\eta_1\left(\rho_1^h\sigma_1 + \rho_2^h\sigma_2\right)\right\}, \tag{127}$$

for $\sigma_1,\sigma_2 \geq \sigma$. Thus, from (127), we obtain that, for any $\sigma_2 > 0$, there exists a real number $\sigma^{(1)} := \sigma^{(1)}(\sigma_2)$ such that

$$\frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} < \frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} + \frac{M^{(1)}(\sigma_1,\sigma_2,f)}{M(\sigma_1,\sigma_2,f)} < \exp\left\{\eta_2\rho_1^h\sigma_1\right\}, \quad \text{for } \sigma_1 \geq \sigma^{(1)}. \tag{128}$$

Hence, since $0 < \varepsilon < \rho_1^h(1-\eta_2)$, from Theorem 28, we can easily obtain a contradiction with the assumption that $f(s_1,s_2)$ is of finite partial h-order ρ_1^h with respect to s_1.

Thus, $\eta = 1$; this completes the proof of Theorem 29.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest in the manuscript.

Authors' Contributions

Hong-Yan Xu contributed to conceptualization; Yong-Qin Cui and Hong-Yan Xu contributed to writing-original draft preparation; Hong-Yan Xu contributed to writing-review and editing; Yong-Qin Cui, Hong-Yan Xu, and Na Li contributed to funding acquisition. All authors read and approved the final manuscript.

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