Research Article

Existence and Nonexistence of Positive Solutions for Fractional Three-Point Boundary Value Problems with a Parameter

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In this work, we investigate the existence and nonexistence of positive solutions for p-Laplacian fractional differential equations with a parameter. On the basis of the properties of Green’s function and Guo-Krasnosel’skii fixed point theorem on cones, the existence and nonexistence of positive solutions are obtained for the boundary value problems. We also give some examples to illustrate the effectiveness of our main results.

1. Introduction

Fractional calculus has played an important role in the modeling of different physical and natural phenomena, such as fluid mechanics, control system, and many other branches of engineering. In recent years, there are many papers concerning the existence of positive solutions for nonlinear fractional differential equations; see [1–18] and the references cited therein.

In [1], Han et al. studied the existence of positive solutions for the following problems with the generalized p-Laplacian operator:

\[
\begin{align*}
D_0^\beta \left( \phi \left( D_0^\alpha u (t) \right) \right) &= \lambda f \left( u (t) \right), \quad 0 < t < 1, \\
u (0) &= u' (0) = u' (1) = 0, \\
\phi \left( D_0^\alpha u (0) \right) &= (\phi \left( D_0^\alpha u (1) \right))^l = 0.
\end{align*}
\]

On the basis of the properties of Green’s function and Guo-Krasnosel’skii fixed point theorem on cones, some new existence results of at least one or two positive solutions are obtained by different eigenvalue interval for the aforementioned boundary value problems.

Xu et al. [2] showed the existence of multiple positive solutions to singular positone and semipositone m-point boundary value problems of nonlinear fractional differential equations

\[
\begin{align*}
D_0^\alpha u (t) + f \left( t, u (t) \right) &= 0, \quad 0 < t < 1, \\
u (0) &= 0, \\
\alpha u (1) &= \sum_{i=1}^{m-2} \beta_i u (\eta_i),
\end{align*}
\]

where \(1 < \alpha < 2\) and \(D_0^\alpha\) is the standard Riemann-Liouville fractional derivative. By means of the Leray-Schauder nonlinear alternative and a fixed point theorem on cones, they deduce multiple positive solutions to singular positive and semipositive m-point boundary value problems.

Shen et al. [3] investigate the following fractional thermostat model with a parameter:

\[
\begin{align*}
\mathcal{C} D^\alpha u (t) + \lambda f \left( t, u (t) \right) &= 0, \quad t \in (0, 1), \\
u' (0) &= 0, \\
\beta \mathcal{D}^{-1} u' (1) + u (\eta) &= 0,
\end{align*}
\]

where \(1 < \alpha \leq 2\), \(0 \leq \eta \leq 1\), \(\beta > 0\), and \(\mathcal{C} D^\alpha\) is the Caputo’s fractional derivatives. Using the fixed point theorem on cones, the existence and nonexistence results for positive solutions are discussed for the boundary value problems.
Motivated by the aforementioned works, this paper is concerned with the positive solutions for p-Laplacian fractional differential equation with a parameter

\[ D_0^\alpha \left( \phi_p \left( D_0^{\alpha_n}u(t) \right) \right) = \lambda f(t, u(t)), \quad t \in (0, 1), \quad (4) \]

\[ \phi_p \left( D_0^{\alpha_n}u(0) \right) = 0, \quad i = 0, 1, 2, \ldots, l - 2, \quad (5) \]

\[ \left| \phi_p \left( D_0^{\alpha_n}u(t) \right) \right|_{l=1}^{|t_1|} = b \left| \phi_p \left( D_0^{\alpha_n}u(t) \right) \right|_{l=1}^{|t|}, \quad (6) \]

\[ u^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, n - 2, \quad (7) \]

\[ u'(1) = au' \left( \xi \right), \quad (8) \]

where \( \lambda > 0, \phi_p(u) = |u|^{p-2}u, p > 1, D_0^\alpha \) and \( D_0^{\alpha_n} \) are the Riemann-Liouville fractional derivatives, \( 3 < n - 1 < \alpha \leq n, \ 3 < l - 1 < \beta \leq l, \) and \( l + n - 1 < \alpha + \beta \leq l + n \). Based on the properties of Green's function and Guo-Krasnosel'skii fixed point theorem on cones, the existence of positive solutions are obtained for problems (4)-(8).

We will always suppose the following conditions are satisfied:

1. \((H_1)\) \( a > 0, b > 0, 0 < \xi < 1 \) and \( 0 < \alpha \xi^{\alpha-2} < 1, 0 < b\xi^{\beta-2} < 1; \)
2. \((H_2)\) \( f(t, u) : [0, 1] \times (0, \infty) \rightarrow (0, \infty) \) is continuous.

2. Background and Definitions

For the convenience of the reader, we state some basic definitions and lemmas about fractional calculus theory, which can be found in [19, 20].

Definition 1. The fractional integral of order \( \alpha > 0 \) of a function \( y : (0, +\infty) \rightarrow \mathbb{R} \) is given by

\[ \Gamma(\alpha) = \int_0^{+\infty} e^{-x} x^{\alpha-1} dx, \quad (10) \]

\[ \Gamma(\alpha) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} y(s) ds, \quad (9) \]

provided that the right side is pointwise defined on \((0, +\infty)\), where

\[ G(t, s) = \frac{1}{A} \begin{cases} 
\frac{\xi^\alpha - (1 - \xi)^{\alpha-1}}{\alpha^\alpha - (1 - \alpha)^{\alpha-1} (\xi - \alpha)^{\alpha-2} - (1 - \alpha \xi^{\alpha-2}) (t - s)^{\alpha-1}}, & 0 \leq s \leq \min \{t, \xi\} \leq 1, \\
\frac{\xi^\alpha - (1 - \xi)^{\alpha-1}}{\alpha^\alpha - (1 - \alpha)^{\alpha-1} (\xi - \alpha)^{\alpha-2}}, & 0 \leq t \leq s \leq \xi \leq 1, \\
\frac{\xi^\alpha - (1 - s)^{\alpha-1}}{\alpha^\alpha - (1 - \alpha \xi^{\alpha-2}) (t - s)^{\alpha-1}}, & 0 \leq \xi \leq s \leq t \leq 1, \\
\frac{\alpha^\alpha - (1 - s)^{\alpha-1}}{\alpha^\alpha - (1 - s)^{\alpha-2}}, & 0 \leq \max \{t, \xi\} \leq s \leq 1,
\end{cases} \]

\[ H(s, \tau) = \frac{1}{B} \begin{cases} 
\frac{s^\beta - (1 - s)^{\beta-1}}{\beta^\beta - (1 - b \xi^{\beta-2}) (s - \tau)^{\beta-1}}, & 0 \leq s \leq \tau \leq \xi \leq 1, \\
\frac{s^\beta - (1 - \tau)^{\beta-1}}{\beta^\beta - (1 - b \xi^{\beta-2}) (s - \tau)^{\beta-1}}, & 0 \leq s \leq \tau \leq \xi \leq 1, \\
\frac{s^\beta - (1 - \tau)^{\beta-1}}{\beta^\beta - (1 - \tau)^{\beta-2} - (1 - b \xi^{\beta-2}) (s - \tau)^{\beta-2}}, & 0 \leq \xi \leq \tau \leq s \leq 1, \\
\frac{s^\beta - (1 - \tau)^{\beta-1}}{\beta^\beta - (1 - \tau)^{\beta-2}}, & 0 \leq \max \{s, \xi\} \leq \tau \leq 1,
\end{cases} \]

Definition 2. For a continuous function \( y : (0, +\infty) \rightarrow \mathbb{R} \), the Riemann-Liouville derivative of fractional order \( \alpha > 0 \) is defined as

\[ D_0^\alpha y(t) = \frac{1}{\Gamma(n - \alpha)} \int_0^t \left( \frac{d}{dt} \right)^{n-\alpha} y(s) ds, \quad (11) \]

where \( n = [\alpha] + 1 \), provided that the right side is pointwise defined on \((0, +\infty)\).

Lemma 3. Let \( \alpha > 0 \). Assume that \( u, D_0^{\alpha_n}u \in L^1(0, 1) \). Then

\[ I_0^\alpha D_0^{\alpha_n}u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2} + \cdots + c_N t^{\alpha-N}, \quad c_i \in \mathbb{R}, \ i = 1, 2, \ldots, N \]

as the unique solution, where \( N \) is the smallest integer greater than or equal to \( \alpha \).

In order to obtain our main results, we need the following Guo-Krasnosel'skii fixed point theorem in [21].

Theorem 4. Let \((E, \| \cdot \|)\) be a Banach space, and \( P \subset E \) be a cone in \( E \). Assume that \( \Omega_1 \) and \( \Omega_2 \) are open subsets of \( E \) such that \( 0 \in \Omega_1 \), \( \overline{\Omega_1} \setminus \Omega_2 \). If

\[ T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \rightarrow P \]

is a completely continuous operator such that either

(i) if \( \|Tu\| < \|u\|, u \in P \cap \partial\Omega_1 \), and \( \|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_2 \), or

(ii) if \( \|Tu\| \geq \|u\|, u \in P \cap \partial\Omega_1 \), and \( \|Tu\| \leq \|u\|, u \in P \cap \partial\Omega_2 \),

then \( T \) has a fixed point in \( P \cap (\overline{\Omega_2} \setminus \Omega_1) \).

3. Preliminary Lemmas

Lemma 5. The boundary value problems (4)-(8) are equivalent to the following equation:

\[ u(t) = \int_0^1 G(t, s) \phi_p \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds, \quad (14) \]

where
\( \phi_q(s) \) is the inverse function of \( \phi_p(s) \), a.e., \( \phi_q(s) = |s|^{q-2}s, 1/p + 1/q = 1 \) and \( A = (1 - a\kappa^{\alpha-2})\Gamma(\alpha), B = (1 - b\kappa^{\beta-2})\Gamma(\beta) \).

Proof. According to Lemma 3, (4) is equivalent to the following integral equation:

\[
\phi_p(D_0^\alpha u(t)) = \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau + c_1 t^{\beta-1} + c_2 t^{\beta-2} + \cdots + c_l t^{\beta-1}.
\]

(17)

Conditions (5) imply that \( c_2 = \cdots = c_l = 0 \), i.e.,

\[
\phi_p(D_0^\alpha u(t)) = \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau
\]

(18)

So,

\[\left[ \phi_p(D_0^\alpha u(t)) \right]' = \frac{\lambda}{\Gamma(\beta-1)} \int_0^t (t-\tau)^{\beta-2} f(\tau, u(\tau)) d\tau + c_1 (\beta - 1) t^{\beta-2}.\]

(19)

From (6), we have

\[c_1 = -\frac{\lambda}{B} \int_0^1 (1-\tau)^{\beta-2} f(\tau, u(\tau)) d\tau + \frac{\lambda b}{B} \int_0^\xi (\xi-\tau)^{\beta-2} f(\tau, u(\tau)) d\tau.\]

(20)

By use of (18) and (20), we get

\[
\phi_p(D_0^\alpha u(t)) = \frac{\lambda}{\Gamma(\beta)} \int_0^t (t-\tau)^{\beta-1} f(\tau, u(\tau)) d\tau - \frac{\lambda t^{\beta-1}}{B} \int_0^1 (1-\tau)^{\beta-2} f(\tau, u(\tau)) d\tau + \frac{\lambda b t^{\beta-1}}{B} \int_0^\xi (\xi-\tau)^{\beta-2} f(\tau, u(\tau)) d\tau
\]

(21)

\[= -\lambda \int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau.\]

Therefore,

\[
D_0^\alpha u(t) = -\phi_q \left( \lambda \int_0^1 H(t, \tau) f(\tau, u(\tau)) d\tau \right). \]

(22)

In view of Lemma 3, we have

\[u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds + d_1 t^{\alpha-1} + d_2 t^{\alpha-2} + \cdots + d_n t^{\alpha-n}.
\]

(23)

Conditions (7) imply that \( d_2 = \cdots = d_n = 0 \), i.e.,

\[u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds + d_1 t^{\alpha-1}.
\]

(24)

So,

\[u'(t) = -\frac{1}{\Gamma(\alpha-1)} \int_0^t (t-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds + d_1 (\alpha-1) t^{\alpha-2}.
\]

From (8), we get

\[d_1 = \frac{1}{A} \int_0^1 (1-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds - \frac{a}{A} \int_0^\xi (\xi-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds.
\]

(25)

By use of (24) and (26), we can obtain

\[u(t) = -\frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds + \frac{t^{\alpha-1}}{A} \cdot \int_0^t (t-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds + \frac{a t^{\alpha-1}}{A} \cdot \int_0^\xi (\xi-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) d\tau \right) ds.
\]

(27)

The proof is complete.

Lemma 6. Functions \( G(t, s) \) and \( H(s, \tau) \) defined by (15) and (16), respectively, are continuous on \([0, 1] \times [0, 1]\) and satisfy

(i) \( G(t, s) \geq 0, H(s, \tau) \geq 0 \), for \( t, s, \tau \in [0, 1] \);
(ii) \( G(t, s) \geq (1/\Gamma(a))t^{a-1}s(1-s)^{a-2}, H(s, \tau) \geq (1/\Gamma(\beta))\tau^{\beta-1}(1-\tau)^{\beta-2} \), for \( t, s, \tau \in [0, 1] \);

(iii) \( G(t, s) \leq k_1s(1-s)^{a-2}, H(s, \tau) \leq k_2\tau(1-\tau)^{\beta-2}, \) for \( t, s, \tau \in [0, 1] \), where

\[
k_1 = \frac{1}{A} \left[ (\alpha - 1) + (\alpha - 2)a\xi^{a-3} \right],
\]

\[
k_2 = \frac{1}{B} \left[ (\beta - 1) + (\beta - 2)b\xi^{\beta-3} \right].
\]

Proof. If \( 0 \leq s < \min\{t, \xi\} \leq 1 \), then

\[
AG(t, s) = t^{a-1} - (t - s)^{a-1} (\xi - s)^{a-2}
\]

\[
- (1 - a\xi^{a-2}) (t - s)^{a-1}
\]

\[
\geq t^{a-1} (1-s)^{a-2} - a(\xi - s)^{a-2}
\]

\[
- (1 - a\xi^{a-2}) (t - ts)^{a-1}
\]

\[
= t^{a-1} (1-s)^{a-2} - a\xi^{a-2} t^{a-1} (1-s)^{a-2}
\]

\[
- (1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2}
\]

\[
= (1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2}
\]

\[
(1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2} [1 - (1-s)]
\]

\[
= (1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2} \geq 0,
\]

\[
AG(t, s) = t^{a-1} (1-s)^{a-2} - a\xi^{a-2} (t - s)^{a-1}
\]

\[
= t^{a-1} (1-s)^{a-2} - a\xi^{a-2} (t - s)^{a-1}
\]

\[
- (t - s)^{a-1} + a\xi^{a-2} (t - s)^{a-1}
\]

\[
= t^{a-1} (1-s)^{a-2} - a\xi^{a-2} (t - s)^{a-1}
\]

\[
- at^{a-1} (1-s)^{a-2} + a\xi^{a-2} (t - s)^{a-1}
\]

\[
= t (t - ts)^{a-2} - t (s - t)^{a-2} + s (t - s)^{a-2}
\]

\[
- at (t - ts)^{a-2} + a (t - s)^{a-2}
\]

\[
\leq (\alpha - 2) (t - ts)^{a-2} (s - st) + s (t - s)^{a-2}
\]

\[
- at (t - ts)^{a-2} + at (t - s)^{a-2}
\]

\[
= (\alpha - 1) s (1-s)^{a-2}
\]

\[
+ at (t - s)^{a-2} (t - ts)^{a-2}.
\]

If \( 0 \leq s \leq t \leq \xi \leq 1 \), then \( at[(t\xi - s\xi)^{a-2} - (t\xi - ts)^{a-2}] \leq 0 \).

If \( 0 \leq s \leq \xi \leq t \leq 1 \), then

\[
ag[(t\xi - s\xi)^{a-2} - (t\xi - ts)^{a-2}]
\]

\[
\leq ag[(\alpha - 2) (t\xi - s\xi)^{a-2} (st - s\xi)]
\]

\[
\leq (\alpha - 2) a\xi^{a-3} (t - s)^{a-2} (t - \xi) s
\]

\[
\leq (\alpha - 2) a\xi^{a-3} (1-s)^{a-2}.
\]

So we get if \( 0 \leq s \leq \min\{t, \xi\} \leq 1 \), then

\[
\frac{1}{\Gamma(a)}t^{a-1} (1-s)^{a-2} \leq G(t, s) \leq k_1s (1-s)^{a-2},
\]

where \( k_1 = (1/A)[(\alpha - 1) + (\alpha - 2)a\xi^{a-3}] \). Obviously, \( Ak_1 > 1 \).

If \( 0 \leq t \leq s \leq \xi \leq 1 \), then

\[
AG(t, s) = t^{a-1} (1-s)^{a-2} - a\xi^{a-1} (\xi - s)^{a-2}
\]

\[
\geq t^{a-1} (1-s)^{a-2} - a\xi^{a-1} (\xi - s)^{a-2}
\]

\[
- (1 - a\xi^{a-2}) (t - ts)^{a-1}
\]

\[
= t^{a-1} (1-s)^{a-2} - a\xi^{a-2} t^{a-1} (1-s)^{a-2}
\]

\[
- (1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2}
\]

\[
= (1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2}
\]

\[
(1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2} [1 - (1-s)]
\]

\[
= (1 - a\xi^{a-2}) t^{a-1} (1-s)^{a-2} \geq 0,
\]

\[
AG(t, s) = t^{a-1} (1-s)^{a-2} - a\xi^{a-2} (t - s)^{a-1}
\]

\[
= t^{a-1} (1-s)^{a-2} - a\xi^{a-2} (t - s)^{a-1}
\]

\[
- (t - s)^{a-1} + a\xi^{a-2} (t - s)^{a-1}
\]

\[
= t^{a-1} (1-s)^{a-2} - a\xi^{a-2} (t - s)^{a-1}
\]

\[
- at^{a-1} (1-s)^{a-2} + a\xi^{a-2} (t - s)^{a-1}
\]

\[
= t (t - ts)^{a-2} - t (s - t)^{a-2} + s (t - s)^{a-2}
\]

\[
- at (t - ts)^{a-2} + a (t - s)^{a-2}
\]

\[
\leq (\alpha - 2) (t - ts)^{a-2} (s - st) + s (t - s)^{a-2}
\]

\[
- at (t - ts)^{a-2} + at (t - s)^{a-2}
\]

\[
= (\alpha - 1) s (1-s)^{a-2}
\]

\[
+ at (t - s)^{a-2} (t - ts)^{a-2}.
\]
\begin{align*}
&\leq (\alpha - 2)(t-ts)^{\alpha-3}(s-st) + s(1-s)^{\alpha-2} \\
&\quad + k_{1}s(1-s)^{\alpha-2} \\
&\leq (\alpha - 2)s(1-s)^{\alpha-2} + s(1-s)^{\alpha-2} \\
&\quad + k_{1}s(1-s)^{\alpha-2} \\
&= (\alpha - 1 + k_{1})s(1-s)^{\alpha-2} \\
&\leq Ak_{1}s(1-s)^{\alpha-2}.
\end{align*}

If $0 \leq \max\{t, \xi\} \leq s \leq 1$, we get

\[ AG(t, s) = t^{\alpha-1}(1-s)^{\alpha-2} \leq (1 - a \xi^{\alpha-2}) t^{\alpha-1}s (1-s)^{\alpha-2}, \]

\[ AG(t, s) = t^{\alpha-1}(1-s)^{\alpha-2} \leq Ak_{1}s(1-s)^{\alpha-2}. \]

Consequently, we get

\[ \frac{1}{\Gamma(\alpha)} t^{\alpha-1}s (1-s)^{\alpha-2} \leq G(t, s) \leq k_{1}s (1-s)^{\alpha-2} \]

for $t, s \in [0, 1]$. \hfill \Box

Similarly, we can get

\[ \frac{1}{\Gamma(\beta)} \tau^{\beta-1}(1-\tau)^{\beta-2} \leq H(s, \tau) \leq k_{2}\tau (1-\tau)^{\beta-2} \]

for $s, \tau \in [0, 1]$, where $k_{2} = (1/B)(\beta - 1) + (\beta - 2)b_{k}^{\beta-3}$.

The proof is complete.

Let $E$ be the real Banach space $C[0, 1]$ with the maximum norm and define the cone $P \subset E$ by

\[ P = \{ u | u \in E \text{ and } u(t) \geq k_{a}^{-1}\|u\|, t \in [0, 1]\}. \]

So we obtain

\[ \|Tu(t)\| \leq k_{1}B(2, \alpha - 1) (k_{2})^{q-1} \left( \lambda \int_{0}^{1} \tau (1-\tau)^{\beta-2} f(\tau, u(\tau)) \, d\tau \right)^{q-1}, \]

\[ Tu(t) = \int_{0}^{1} G(t, s) \phi_{q} \left( \lambda \int_{0}^{1} H(s, \tau) f(\tau, u(\tau)) \, d\tau \right) ds \]

\[ \geq \int_{0}^{1} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}s (1-s)^{\alpha-2} \phi_{q} \left( \lambda \int_{0}^{1} \frac{1}{\Gamma(\beta)} \tau^{\beta-1}(1-\tau)^{\beta-2} f(\tau, u(\tau)) \, d\tau \right) ds = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left( \frac{1}{\Gamma(\beta)} \right)^{q-1}. \]

\begin{align*}
&\int_{0}^{1} \frac{1}{\Gamma(\alpha)} t^{\alpha-1}s (1-s)^{\alpha-2} \phi_{q} \left( \lambda \int_{0}^{1} \frac{1}{\Gamma(\beta)} \tau^{\beta-1}(1-\tau)^{\beta-2} f(\tau, u(\tau)) \, d\tau \right) ds = \frac{t^{\alpha-1}}{\Gamma(\alpha)} \left( \frac{1}{\Gamma(\beta)} \right)^{q-1} \\
&\cdot \int_{0}^{1} \frac{1}{\Gamma(\beta)} \tau^{\beta-1}(1-\tau)^{\beta-2} f(\tau, u(\tau)) \, d\tau \cdot \int_{0}^{1} \tau (1-\tau)^{\beta-2} f(\tau, u(\tau)) \, d\tau \]

\[ = \frac{B((\beta - 1)(q - 1) + 2, \alpha - 1)}{\Gamma(\alpha) \Gamma(\beta)^{q-1}} \left( \lambda \int_{0}^{1} \tau (1-\tau)^{\beta-2} f(\tau, u(\tau)) \, d\tau \right)^{q-1} \geq k_{a}^{-1}\|Tu\|. \]
Therefore, $T: P \rightarrow P$. In view of continuity of $G, H$ and $f$, we have $T: P \rightarrow P$ is continuous.

Let $\Omega \subset P$ be bounded, i.e., there exists a positive constant $M > 0$ such that $\|u\| \leq M$, for all $u \in \Omega$, let $L = \max_{0 \leq s \leq 1, 0 \leq t \leq M, f(t, u)} \|f(t, u)\| + 1$, then, for $t \in [0, 1]$ and $u \in \Omega$, we get

\[
Tu(t) = \int_0^t \int_0^1 G(t, s) \phi_\lambda \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) \, d\tau \right) \, d\tau 
\]

\[
\leq \int_0^t k_1 s (1-s)^{\alpha-2} \phi_\lambda \left( \lambda \int_0^1 k_2 \tau (1-\tau)^{\beta-2} L \, d\tau \right) \, ds 
\]

\[
= k_1 B(2, \alpha - 1) (\lambda k_2 B(2, \beta - 1) L)^{\frac{\alpha - 1}{\alpha - 2}} < +\infty.
\]

Hence, $T(\Omega)$ is uniformly bounded. On the other hand, suppose (A2) holds, there is $\varepsilon > 0$, such that $|t_1 - t_2| < \delta$ imply

\[
|G(t_1, s) - G(t_2, s)| < \frac{\varepsilon}{[\lambda k_2 B(2, \beta - 1) L]^\frac{\alpha - 1}{\alpha - 2}} (\lambda k_2 B(2, \beta - 1) L)^{\frac{\alpha - 1}{\alpha - 2}}
\]

for any $s \in [0, 1]$.

Then, for all $u \in \Omega$,

\[
|Tu(t_1) - Tu(t_2)| \leq \int_0^t |G(t_1, s) - G(t_2, s)| \cdot \phi_\lambda \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) \, d\tau \right) \, ds
\]

\[
\leq \frac{\varepsilon}{[\lambda k_2 B(2, \beta - 1) L]^\frac{\alpha - 1}{\alpha - 2}} (\lambda k_2 B(2, \beta - 1) L)^{\frac{\alpha - 1}{\alpha - 2}} = \varepsilon.
\]

Hence, $T(\Omega)$ is equicontinuous. By Arzela-Ascoli theorem, we have $T: P \rightarrow P$ is completely continuous. 

\section{Main Results}

For convenience we introduce the following notations. Let

\begin{align*}
(A1) & \quad \limsup_{u \to 0^+} \frac{f(t, u)}{\phi_p(u)} = f^0, \\
(A2) & \quad \liminf_{u \to +\infty} \frac{f(t, u)}{\phi_p(u)} = f_\infty, \\
(A3) & \quad \liminf_{u \to -\infty} \frac{f(t, u)}{\phi_p(u)} = f_0.
\end{align*}

\[
\text{(A4)} \quad \limsup_{u \to +\infty} \frac{f(t, u)}{\phi_p(u)} = f^\infty.
\]

The following theorems are the main results in this paper.

\begin{theorem}
If (A1) and (A2) hold, and

\[
\left( \frac{k_1 B(2, \alpha - 1) \Gamma(\alpha)}{k B(\beta - 1) \left( (\alpha - 1) (p - 1) + 2, \beta - 1 \right) f_\infty} \right)^{p - 1} < \frac{B(\alpha - 1) (p - 1) + 2, \beta - 1 \, f_\infty}{k_1 B(2, \beta - 1) f_\infty},
\]

then for each

\[
\lambda \in \left( \frac{\Gamma(\alpha)}{k B(\beta - 1) \left( (\alpha - 1) (p - 1) + 2, \beta - 1 \right) f_\infty}, \frac{1}{k_1 B(2, \beta - 1) f_\infty} \right),
\]

the boundary value problems (4)–(8) have at least one positive solution. Here we impose $1/f_\infty = 0$ if $f_\infty = +\infty$ and $1/f_0 = +\infty$ if $f_0 = 0$.

\end{theorem}

\begin{proof}
Suppose (A1) holds, we may choose $H_1 = 0$, so that, for each $(t, u) \in [0, 1] \times [0, H_1]$, there are $f(t, u) \leq (f^0 + \varepsilon)\phi_p(u)$. Thus, if $u \in P$ and $\|u\| = H_1$, then by (40) and Lemma 6, we have

\[
Tu(t) = \int_0^1 \int_0^1 G(t, s) \cdot \phi_\lambda \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) \, d\tau \right) \, ds \leq \frac{k_1 s (1-s)^{\alpha - 2}}{[\lambda k_2 B(2, \beta - 1) L]^\frac{\alpha - 1}{\alpha - 2}} (k_2 B(2, \beta - 1) f_\infty)^{\frac{\alpha - 1}{\alpha - 2}} \leq k_1 B(2, \alpha - 1) \left( \lambda k_2 B(2, \beta - 1) (f^0 + \varepsilon) \phi_p(u) \right)^{\frac{\alpha - 1}{\alpha - 2}} H_1,
\]

\end{proof}

Let $\Omega_1 = \{ u \in E : \|u\| < H_1 \}$, then the previous inequality show that $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial \Omega_1$. On the other hand, suppose (A2) holds, there is $H_2 > 0$ such that $f(t, u) \geq (f_\infty - \varepsilon)\phi_p(u)$, for $(t, u) \in [0, 1] \times [H_2, +\infty)$. Thus, if $u \in P$ and $\|u\| = H_2 = \max\{2H_1, H_2\}$, by (40) and Lemma 6, we have
\[ \| Tu(t) \| = \left\| \int_0^1 G(t, s) \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) \, d\tau \right) \, ds \right\| \geq \left\| \int_0^1 \frac{1}{\Gamma(\alpha)} (s(1-s))^{\alpha-2} \cdot \phi_q \left( \lambda \int_0^1 \frac{1}{\Gamma(\beta)} \tau (1-\tau)^{\beta-2} (f(\tau, u(\tau)) \phi_p(\tau)) \, d\tau \right) \, ds \right\| = \left\| \frac{1}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-2} \right\| \]

\[ \geq B((\beta-1)(q-1)+2,\alpha-1) \left( \frac{\lambda}{\Gamma(\beta)} \int_0^1 \tau (1-\tau)^{\beta-2} (f(\tau, u(\tau)) \phi_p(\tau)) \, d\tau \right)^{q-1} \]

\[ \geq B((\beta-1)(q-1)+2,\alpha-1) \left( \frac{\lambda B((\alpha-1)(p-1)+2,\beta-1)(f(\tau, u(\tau)) \phi_p(\tau))}{\phi_p(\tau)} \right)^{q-1} kH_2 \geq H_2. \]

Let \( \Omega_2 = \{ u \in E : \| u \| < H_2 \} \), then the previous inequality show that \( \| Tu \| \geq \| u \| \) for \( u \in P \cap \partial \Omega_2 \).

Thus, from Theorem 4, we know that the operator \( T \) has a fixed point in \( P \cap (\Omega_2 \setminus \Omega_1) \). \( \square \)

**Theorem 9.** If (A3) and (A4) hold, and

\[ \left( \frac{k_1 B(2,\alpha-1) \Gamma(\alpha)}{k B((\beta-1)(q-1)+2,\alpha-1)} \right)^{p-1} \]

\[ < \frac{B((\alpha-1)(p-1)+2,\beta-1) f_0}{k_2 B(2,\beta-1) f^{\infty} \Gamma(\beta)}. \]

then for each

\[ \| Tu(t) \| = \left\| \int_0^1 G(t, s) \phi_q \left( \lambda \int_0^1 H(s, \tau) f(\tau, u(\tau)) \, d\tau \right) \, ds \right\| \geq \left\| \int_0^1 \frac{1}{\Gamma(\alpha)} (s(1-s))^{\alpha-2} \cdot \phi_q \left( \lambda \int_0^1 \frac{1}{\Gamma(\beta)} \tau (1-\tau)^{\beta-2} (f(\tau, u(\tau)) \phi_p(\tau)) \, d\tau \right) \, ds \right\| = \left\| \frac{1}{\Gamma(\alpha)} \int_0^1 (s(1-s))^{\alpha-2} \right\|

\[ \geq B((\beta-1)(q-1)+2,\alpha-1) \left( \frac{\lambda}{\Gamma(\beta)} \int_0^1 \tau (1-\tau)^{\beta-2} (f(\tau, u(\tau)) \phi_p(\tau)) \, d\tau \right)^{q-1} \]

\[ \geq B((\beta-1)(q-1)+2,\alpha-1) \left( \frac{\lambda B((\alpha-1)(p-1)+2,\beta-1)(f(\tau, u(\tau)) \phi_p(\tau))}{\phi_p(\tau)} \right)^{q-1} kH_2 \geq H_2. \]

Let \( \Omega_3 = \{ u \in E : \| u \| < H_3 \} \), then the previous inequality shows that \( \| Tu \| \geq \| u \| \) for \( u \in P \cap \partial \Omega_3 \).

Suppose (A4) holds, we consider two cases.

### Case 1. Suppose \( f \) is bounded, then there exists some \( l_1 > 0 \), such that \( f(t,u) \leq l_1 \), for \( u \in [0,1], u \in (0,\infty) \). Thus, if \( u \in P \) and if
by (40) and Lemma 6, we have

\[ \|Tu(t)\| = \left\| \int_0^1 G(t, s) \phi_q \left( \lambda \int_0^1 H(s, r) f(r, u(r)) \, dr \right) \, ds \right\| \]

\[ \leq \int_0^1 k_1 s (1-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 k_2 (1-r)^{\beta-2} \, dr \right) \, ds \]

\[ \leq k_1 B (2, \alpha-1) \left\| \lambda k_2 B (2, \beta-1) \right\|_{\mathbb{R}^2} \leq H_{41}. \]

\[ \|Tu(t)\| = \left\| \int_0^1 G(t, s) \phi_q \left( \lambda \int_0^1 H(s, r) f(r, u(r)) \, dr \right) \, ds \right\| \]

\[ \leq \int_0^1 k_1 s (1-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 k_2 (1-r)^{\beta-2} \, dr \right) \, ds \leq k_1 B (2, \alpha-1) \left\| \lambda k_2 B (2, \beta-1) \right\|_{\mathbb{R}^2} \leq H_{42}. \]

Case 2. Suppose \( f \) is unbounded; there is \( h_1 > 0 \) such that \( f(t, u) \leq (f^\infty + \varepsilon) \phi_p (u) \), for \( (t, u) \in [0, 1] \times [h_1, +\infty) \). Then there exists some \( H_{42} > \max\{2H_3, h_3\} \), such that \( f(t, u) \leq \max_{\omega_{[0,1]}} f(t, H_{42}) \), for \( u \in (0, H_{42}) \).

Thus, if \( u \in P \) and \( \|u\| = H_{42} \), by (40) and Lemma 6, we have

\[ \|Tu(t)\| = \left\| \int_0^1 G(t, s) \phi_q \left( \lambda \int_0^1 H(s, r) f(r, u(r)) \, dr \right) \, ds \right\| \]

\[ \leq \int_0^1 k_1 s (1-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 k_2 (1-r)^{\beta-2} \, dr \right) \, ds \leq k_1 B (2, \alpha-1) \left\| \lambda k_2 B (2, \beta-1) \right\|_{\mathbb{R}^2} \leq H_{42}. \]

Let \( \Omega_4 = \{ u \in E : \|u\| < H_{41} \} \), \( H_{41} = \max\{H_{41}, H_{42}\} \), then the previous inequality shows that \( \|Tu\| \leq \|u\| \) for \( u \in P \cap \partial \Omega_4 \). Thus, from Theorem 4, we know that the operator \( T \) has a fixed point in \( P \cap (\Omega_4 \setminus \Omega_3) \). \( \Box \)

**Theorem 10.** If \( f^0 < +\infty \) and \( f^\infty < +\infty \), then there exists \( \lambda_1 > 0 \) such that for all \( 0 < \lambda < \lambda_1 \), the boundary value problems (4)-(8) have no positive solution.

**Proof.** Since \( f^0 < +\infty \) and \( f^\infty < +\infty \), there exist positive constants \( M_1, M_2, r_1, r_2 \), such that \( r_1 < r_2 \) and

\[ f(t, u) \leq M_1 \phi_p (u) \quad \text{for} \quad t \in [0, 1], u \in [0, r_1]; \]

\[ f(t, u) \leq M_2 \phi_p (u) \quad \text{for} \quad t \in [0, 1], u \in [r_2, +\infty). \]

Let

\[ M_0 = \max \left\{ M_1, M_2, \sup_{u \in [r_1, r_2]} \frac{\max_{[0,1]} f(t, u)}{\phi_p (u)} \right\} > 0. \]

then we have \( f(t, u) \leq M_0 \phi_p (u) \) for \( t \in [0, 1], u \in [0, +\infty) \).

Assume \( w(t) \) is a positive solution of the boundary value problems (4)-(8); let

\[ \lambda_1 = \left( \frac{1}{k_1 B (2, \alpha-1)} \right)^{p-1} \left( k_2 B (2, \beta-1) M_0 \right) \]

then for all \( t \in [0, 1] \), we get

\[ \|w(t)\| = \|Tu(t)\| = \left\| \int_0^1 G(t, s) \phi_q \left( \lambda \int_0^1 H(s, r) f(r, w(r)) \, dr \right) \, ds \right\| \]

\[ \leq \int_0^1 k_1 s (1-s)^{\alpha-2} \phi_q \left( \lambda \int_0^1 k_2 (1-r)^{\beta-2} \, dr \right) \, ds \leq k_1 B (2, \alpha-1) \left\| \lambda k_2 B (2, \beta-1) \right\|_{\mathbb{R}^2} \leq H_{42}. \]

which is a contraction. Therefore, the boundary value problems (4)-(8) have no positive solution. \( \Box \)

**Theorem 11.** If \( f_0 > 0 \) and \( f_\infty > 0 \), then there exists \( \lambda_2 > 0 \) such that, for all \( \lambda > \lambda_2 \), the boundary value problems (4)-(8) have no positive solution.

**Proof.** Since \( f_0 > 0 \) and \( f_\infty > 0 \), there exist positive constants \( M_3, M_4, r_3, r_4 \), such that \( r_3 < r_4 \) and

\[ f(t, u) \geq M_3 \phi_p (u) \quad \text{for} \quad t \in [0, 1], u \in [0, r_3]; \]

\[ f(t, u) \geq M_4 \phi_p (u) \quad \text{for} \quad t \in [0, 1], u \in [r_4, +\infty). \]

Let

\[ M_5 = \min \left\{ M_3, M_4, \inf_{u \in [r_3, r_4]} \frac{\min_{[0,1]} f(t, u)}{\phi_p (u)} \right\} > 0. \]

then we have

\[ f(t, u) \geq M_5 \phi_p (u) \quad \text{for} \quad t \in [0, 1], u \in [0, +\infty). \]

Suppose \( w(t) \) is a positive solution of the problems (4)-(8); let

\[ \lambda_2 = \left( \frac{\Gamma (\alpha)}{k B ((\beta-1) (q-1) + 2, \alpha-1)} \right)^{p-1} \left( \frac{\Gamma (\beta)}{B ((\alpha-1) (p-1) + 2, \beta-1) M_5} \right) \]

then for all \( t \in [0, 1] \), we get

\[ f(t, u) \leq M_0 \phi_p (u) \quad \text{for} \quad t \in [0, 1], u \in [0, +\infty). \]

\[ \Box \]
\[ \|w(t)\| = \|Tw(t)\| = \left\| \int_0^1 G(t,s) \phi_\eta \left( \lambda \int_0^1 H(s,\tau) f(\tau, w(\tau)) \, d\tau \right) \, ds \right\| \]
\[ \geq \int_0^1 \frac{1}{\Gamma(\alpha)} \alpha^{-2}(1-s)^{\alpha-2} \phi_\eta \left( \lambda \int_0^1 \frac{1}{\Gamma(\beta)} \beta^{-2}(1-\tau)^{\beta-2} M_s \phi_p (w) \, d\tau \right) \, ds \]
\[ > \frac{B((\beta-1)(q-1)+2,\alpha-1)}{\Gamma(\alpha)} \left( \frac{\lambda_2 B(\alpha-1)(p-1)+2,\beta-1) M_s \phi_p (w)}{\Gamma(\beta)} \right)^{q-1} k \|w(t)\| = \|w(t)\|, \]

which is a contraction. Therefore, the boundary value problems (4)-(8) have no positive solution. \qed

5. Examples

In this section, we give some simple examples to explain our results.

Example 1. For the problems (4)-(8), let \( \alpha = 3.2, \beta = 3.1, a = 0.8, b = 0.9, \xi = 0.2, p = 1.5 \), then we get \( q = 3, k = 0.030716 \). Let

\[ f(t,u) = \frac{u^{0.5}(100u+3)}{u+3}, \] (65)

then \( f^0 = 1, f_{\infty} = 100 \). From a direct calculation, we get

\[ \left[ \frac{\Gamma(\alpha)}{k B((\beta-1)(q-1)+2,\alpha-1)} \right]^{p-1} \]
\[ \cdot \frac{\Gamma(\beta)}{B((\alpha-1)(p-1)+2,\beta-1) f_{\infty}} = 21.618258, \] (66)
\[ \left[ \frac{1}{k_1 B(2,\alpha-1)} \right]^{p-1} \]
\[ \cdot \frac{1}{k_2 B(2,\beta-1) f^0} \approx 30.516030. \]

In view of Theorem 8, we get that problems (4)-(8) have at least one positive solution when \( \lambda \in (108.091288, 152.580152) \).

Example 2. For the problems (4)-(8), let \( \alpha = 3.2, \beta = 3.1, a = 0.8, b = 0.9, \xi = 0.2, p = 1.5 \), then we get \( q = 3, k = 0.030716 \). Let

\[ f(t,u) = \frac{u^{0.5}(0.2u+20)}{u+1}, \] (67)

then \( f^0 = 20, f_{\infty} = 0.2 \). From a direct calculation, we get

\[ \left[ \frac{\Gamma(\alpha)}{k B((\beta-1)(q-1)+2,\alpha-1)} \right]^{p-1} \]
\[ \cdot \frac{\Gamma(\beta)}{B((\alpha-1)(p-1)+2,\beta-1) f^0} \approx 108.091288, \] (68)
\[ \left[ \frac{1}{k_1 B(2,\alpha-1)} \right]^{p-1} \]
\[ \cdot \frac{1}{k_2 B(2,\beta-1) f_{\infty}} \approx 152.580152. \]

In view of Theorem 8, we get that problems (4)-(8) have at least one positive solution when \( \lambda \in (108.091288, 152.580152) \).

6. Conclusions

The fixed point theorem on cones is used to solve the three-point boundary value problems of a kind of nonlinear fractional differential equation with a parameter. Under certain conditions of the nonlinearity, the existence and nonexistence of positive solutions are obtained for the boundary value problems when the parameter belongs to different intervals.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Authors’ Contributions

The authors contributed equally to this paper. All authors read and approved the final manuscript.

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References


