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Research Article

Conditional Fourier-Feynman Transforms with Drift on a Function Space

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In this paper we derive change of scale formulas for conditional analytic Fourier-Feynman transforms and conditional convolution products of the functions which are the products of generalized cylinder functions and the functions in a Banach algebra which is the space of generalized Fourier transforms of the complex Borel measures on $L_2[0,T]$ using two simple formulas for conditional expectations with a drift on an analogue of Wiener space. Then we prove that the conditional transform of the conditional convolution product can be expressed by the product of the conditional transforms of each function. Finally we establish various changes of scale formulas for the conditional transforms and the conditional convolution products.

1. Introduction

Let $C_0[0,T]$ denote the Wiener space, the space of real-valued continuous functions x on the interval $[0,T]$ with $x(0) = 0$. It has long been known that Wiener measure and Wiener measurability behave badly under the change of scale transformation and under translations on $C_0[0,T]$ [1, 2]. Yoo and his coauthors [3] presented a change of scale formula for Wiener integrals of functions on the abstract Wiener space $\mathbb{B}$ [4] which generalizes $C_0[0,T]$. We note that the functions used in [3] are the products of generalized cylinder functions on $\mathbb{B}$ and the functions on the Fresnel class [5] which is the space of Fourier-Stieltjes transforms of measures on a separable real Hilbert space densely embedded in $\mathbb{B}$, and note that they need not be bounded or continuous.

Let $C[0,T]$ denote an analogue of Wiener space which is the space of real-valued continuous functions on the interval $[0,T]$ [6]. Two kinds of integral transforms, which are known as an analytic conditional Fourier-Feynman transform and a conditional convolution product on $C[0,T]$, were introduced by the author and his coauthors [7–9] using some conditioning functions. In fact, the authors [9] investigated the conditional Fourier-Feynman transforms and the conditional convolution products of the cylinder functions on $C[0,T]$ and established various relationships that occur among them using the conditioning function $X_{n+1} : C[0,T] \longrightarrow \mathbb{R}^{n+2}$ given by $X_{n+1}(x) = (x(t_0), x(t_1), \ldots, x(t_n), x(t_{n+1}))$, where $0 = t_0 < t_1 < \cdots < t_n < t_{n+1} = T$. Moreover they derived several changes of scale formulas for the conditional transforms and the conditional convolution products, which simplify the evaluations of the conditional expectations, because the probability measure used on $C[0,T]$ may not be scale-invariant. We note that X_{n+1} has no drift and contains the present position $x(t_{n+1})$ of the path $x \in C[0,T]$. In [7], Cho extended the results in [9] using a more generalized stochastic process $Z : C[0,T] \times [0,T] \longrightarrow \mathbb{R}$ given by $Z(x,t) = (h\chi_{[0,t]}, x) + x(0) + a(t)$ for $x \in C[0,T]$ and $t \in [0,T]$, where $(\cdot,\cdot)$ denotes the Paley-Wiener-Zygmund stochastic integral [6], and $h(\neq 0$ a.e.) and a are of bounded variation and absolutely continuous, respectively, on $[0,T]$. Here he used the conditioning function $Z_{n+1}(x) = (Z(x,t_0), Z(x,t_1), \ldots, Z(x,t_{n+1}))$ for $x \in C[0,T]$ which generalizes X_{n+1}, and the functions a and $(h\chi_{[0,t]}, x)$ have the effect that the normal density defining the premeasure of cylinder sets on $C[0,T]$ contains the mean function $a(t)$ and the variance function $\int_0^t [h(u)]^2 du$, respectively. Using the conditioning function

$Z_n(x) = (Z(x,t_0), Z(x,t_1), \ldots, Z(x,t_n))$ for $x \in C[0,T]$, Cho [8] also derived similar results in [7]. We note that Z_{n+1} contains the present positions of generalized Wiener paths and Z_n does not; that is, the conditional expectation at $Z_{n+1}(x) = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$ generalizes the probability of the Wiener paths x which pass through ξ_j at time t_j for $j = 0, 1, \ldots, n, n+1$ while Z_n describes the probability of the Wiener paths which pass through ξ_j at time t_j for $j = 0, 1, \ldots, n$. In other words, Z_{n+1} describes the conditional expectation of Wiener paths which pass through ξ_j at each past time t_j $(j = 0, 1, 2, \ldots, n)$ and ξ_{n+1} at the present time $t_{n+1} = T$, while Z_n only describes the conditional expectation of Wiener paths which pass through ξ_j at each past time t_j $(j = 0, 1, 2, \ldots, n)$.

In this paper, using two simple formulas for conditional expectations over paths [10, 11], we evaluate conditional expectations of the products of generalized cylinder functions and the functions in a Banach algebra which plays significant roles in Feynman integration theories and quantum mechanics. Then we investigate their relationships. In particular, we establish change of scale formulas for the conditional transforms and the conditional convolution products. In these evaluation formulas and changes of scale formulas, we use multivariate normal distributions so that the evaluation formulas are unaffected by the present positions of paths, despite the existences of the present positions of the paths in the conditioning functions. We also note that the change scale formulas in [7–9] are expressed by finite sums, while the formulas in this paper are expressed by a limit with a complete orthonormal set of $L_2[0,T]$. Moreover, the conditional Fourier-Feynman transforms and convolution products of cylinder functions in [7–9] are still cylinder functions while the transforms and the convolutions of the functions in this paper are not cylinder functions. In fact, with the conditioning functions Z_n and Z_{n+1}, we evaluate conditional expectations, namely, the conditional Fourier-Feynman transforms and the conditional convolution products of the functions given by

$$f\left(\left(v_1, Z(x,\cdot)\right), \ldots, \left(v_r, Z(x,\cdot)\right)\right) \cdot \int_{L_2[0,T]} \exp\left\{i\left(v, Z(x,\cdot)\right)\right\} d\sigma(v), \tag{1}$$

where σ is a complex Borel measure of bounded variation on $L_2[0,T]$, $f \in L_p(\mathbb{R}^r)$ with $1 \le p \le \infty$, and $\{v_1, \ldots, v_r\}$ is an orthonormal subset of $L_2[0,T]$. Then we show that the conditional Fourier-Feynman transform $T_q^{(2)}[[(\Psi_1 * \Psi_2)_q \mid Z_n](\cdot, \vec{\xi}_n) \mid Z_n]$ of the conditional convolution product $[(\Psi_1 * \Psi_2)_q \mid Z_n](\cdot, \vec{\xi}_n)$ of the functions Ψ_i $(i = 1, 2)$ defined by

$$\Psi_i(x) = f_i\left(\left(v_1, Z(x,\cdot)\right), \ldots, \left(v_r, Z(x,\cdot)\right)\right) \cdot \int_{L_2[0,T]} \exp\left\{i\left(v, Z(x,\cdot)\right)\right\} d\sigma_i(v), \tag{2}$$

where σ_i is a complex Borel measure of bounded variation on $L_2[0,T]$, $f_i \in L_p(\mathbb{R}^r)$, can be expressed by the formula

$$\begin{aligned} &T_q^{(2)}\left[\left[(\Psi_1 * \Psi_2)_q \mid Z_n\right]\left(\cdot, \vec{\xi}_n\right) \mid Z_n\right]\left(y, \vec{\eta}_n\right) \\ &\quad = \left[T_q^{(1)}\left[\Psi_1 \mid Z_n\right]\left(\frac{1}{\sqrt{2}} y \right.\right. \\ &\qquad \left.\left. + \left(\sqrt{2}-1\right) a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n + \vec{\xi}_n\right) - \left(\sqrt{2}-1\right)\vec{a}_n\right)\right] \\ &\qquad \times \left[T_q^{(2)}\left[\Psi_2 \mid Z_n\right]\left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n - \vec{\xi}_n\right)\right.\right. \\ &\qquad \left.\left. + \vec{a}_n\right)\right] \end{aligned} \tag{3}$$

for a nonzero real q, almost surely $y \in C[0,T]$, and P_{Z_n} almost surely $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, where $\vec{a}_n = (a(t_0), a(t_1), \ldots, a(t_n))$ and P_{Z_n} is the probability distribution of Z_n on the Borel class of $\mathbb{R}^{n+1}$. Moreover, replacing a_n by a, we can recover the same formula with Z_{n+1} and the effects of drift a will be investigated on the polygonal function of a so that the results do not depend on a particular choice of the initial distribution of the paths. We also note that the functions in (1) extend the initial state of the Schrödinger equation [12].

2. A Function Space and Preliminary Results

For a positive real T, let $C[0,T]$ denote the space of real-valued continuous functions on the time interval $[0,T]$ with the supremum norm and let w_φ be the analogue of Wiener measure according to the probability measure φ on the Borel class of $\mathbb{R}$ [6].

Let $\mathbb{C}$ and $\mathbb{C}_+$ denote the sets of complex numbers and complex numbers with positive real parts, respectively. Let $F : C[0,T] \longrightarrow \mathbb{C}$ be integrable and let X_τ be a random vector on $C[0,T]$ assuming that the value space of X_τ is a normed space with the Borel σ-algebra. We will adopt the conditional expectation $E[F \mid X_\tau]$ of F given X_τ, which is described in [8, 13, 14].

For an extended real number p with $1 < p \le \infty$, suppose that p and p' are related by $1/p + 1/p' = 1$ (possibly $p' = 1$ if $p = \infty$). Let $q \in \mathbb{R} - \{0\}$. For $\lambda \in \mathbb{C}_+$ let F_λ be a measurable function on $C[0,T]$ such that $\lim_{\lambda \longrightarrow -iq} \|F_\lambda - F\|_{p'} = 0$. Then we write $\lim_{\lambda \longrightarrow -iq}(w^{p'})(F_\lambda) = F$.

Define a stochastic process $X : C[0,T] \times [0,T] \longrightarrow \mathbb{R}$ by $X(x,t) = (h\chi_{[0,t]}, x)$ for $x \in C[0,T]$ and $t \in [0,T]$, where χ denotes an indicator function. For $t \in [0,T]$, let $b(t) = \int_0^t [h(s)]^2 ds$ and, for any function f on $[0,T]$, define the polygonal function $P_{b,n+1}(f)$ of f by

$$\begin{aligned} P_{b,n+1}(f)(t) = \sum_{j=1}^{n+1} &\left[\frac{b(t_j) - b(t)}{b(t_j) - b(t_{j-1})} f(t_{j-1}) \right. \\ &\left. + \frac{b(t) - b(t_{j-1})}{b(t_j) - b(t_{j-1})} f(t_j)\right] \times \chi_{(t_{j-1}, t_j]}(t) + f(0) \\ &\cdot \chi_{\{0\}}(t) \end{aligned} \tag{4}$$

for $t \in [0,T]$. For $\vec{\xi}_{n+1} = (\xi_0, \xi_1, \ldots, \xi_n, \xi_{n+1}) \in \mathbb{R}^{n+2}$, define the polygonal function $P_{b,n+1}(\vec{\xi}_{n+1})$ of $\vec{\xi}_{n+1}$ by the right-hand side of (4), where $f(t_j)$ is replaced by ξ_j. If $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$, $P_{b,n+1}(\vec{\xi}_n)$ is interpreted as $P_{b,n+1}(\vec{\xi}_{n+1})$ with $\xi_{n+1} = \xi_n$. For $x \in C[0,T]$ and $t \in [0,T]$ let

$$\begin{aligned} A(t) &= a(t) - P_{b,n+1}(a)(t), \\ X_{b,n+1}(x,t) &= X(x,t) - P_{b,n+1}(X(x,\cdot))(t), \\ Z_{b,n+1}(x,t) &= Z(x,t) - P_{b,n+1}(Z(x,\cdot))(t). \end{aligned} \tag{5}$$

For $\vec{a}, \vec{u} \in \mathbb{R}^r$, $\lambda \in \mathbb{C}$, and any nonsingular positive $r \times r$ matrix A_r on $\mathbb{R}$, let

$$\begin{aligned} \Psi_r\left(\lambda, \vec{a}, A_r, \vec{u}\right) &= \left[\frac{\lambda^r}{(2\pi)^r \det(A_r)}\right]^{1/2} \\ &\cdot \exp\left\{-\frac{\lambda}{2}\left\langle A_r^{-1}\left(\vec{u} - \vec{a}\right), \vec{u} - \vec{a}\right\rangle_{\mathbb{R}}\right\}, \end{aligned} \tag{6}$$

where $\langle \cdot, \cdot \rangle_{\mathbb{R}}$ denotes the dot product on $\mathbb{R}^r$.

For a function $F : C[0,T] \longrightarrow \mathbb{C}$, let $F_Z^\lambda(x,y) = F(Z(\lambda^{-1/2}x, \cdot) + y)$, let $Z_{n+1}^\lambda(x) = Z_{n+1}(\lambda^{-1/2}x)$, and let $Z_n^\lambda(x) = Z_n(\lambda^{-1/2}x)$ for $\lambda > 0$ and $x, y \in C[0,T]$. Suppose that $E[F_Z^\lambda(\cdot, y)]$ exists, where the expectation is taken over the first variable. By the same method as in Theorem 2.5 of [10] we have

$$\begin{aligned} &E\left[F_Z^\lambda(\cdot, y) \mid Z_{n+1}^\lambda\right]\left(\vec{\xi}_{n+1}\right) \\ &= E\left[F\left(y + \lambda^{-1/2}X_{b,n+1}(x,\cdot) + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right] \end{aligned} \tag{7}$$

for $P_{Z_{n+1}^\lambda}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, and for $P_{Z_n^\lambda}$ a.e. $\vec{\xi}_n = (\xi_0, \xi_1, \ldots, \xi_n) \in \mathbb{R}^{n+1}$

$$\begin{aligned} &E\left[F_Z^\lambda(\cdot, y) \mid Z_n^\lambda\right]\left(\vec{\xi}_n\right) = \int_{\mathbb{R}} \Psi_1\left(\lambda, a(T) - a(t_n),\right. \\ &\left. b(T) - b(t_n), \xi_{n+1} - \xi_n\right) E\left[F\left(y + \lambda^{-1/2}X_{b,n+1}(x,\cdot)\right.\right. \\ &\left.\left. + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right] d\xi_{n+1}, \end{aligned} \tag{8}$$

where $\vec{\xi}_{n+1} = (\xi_0, \ldots, \xi_n, \xi_{n+1})$, the expectation is taken over the variable x and $P_{Z_{n+1}^\lambda}$, $P_{Z_n^\lambda}$ are the probability distributions of Z_{n+1}^λ, Z_n^λ on $\mathscr{B}(\mathbb{R}^{n+2})$, $\mathscr{B}(\mathbb{R}^{n+1})$, respectively. Let $I_F^\lambda(y, \vec{\xi}_{n+1})$ and $I_F^\lambda(y, \vec{\xi}_n)$ be the right-hand sides of (7) and (8), respectively. If, for w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, $I_F^\lambda(y, \vec{\xi}_{n+1})$ has an analytic extension $J_\lambda(F)(y, \vec{\xi}_{n+1})$ on $\mathbb{C}_+$, then it is called a generalized analytic conditional Fourier-Wiener transform of F given Z_{n+1} with the parameter λ and is denoted by

$$T_\lambda\left[F \mid Z_{n+1}\right]\left(y, \vec{\xi}_{n+1}\right) = J_\lambda(F)\left(y, \vec{\xi}_{n+1}\right). \tag{9}$$

Moreover if $T_\lambda[F \mid Z_{n+1}](y, \vec{\xi}_{n+1})$ has a point-wise limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is called a generalized L_1-analytic conditional Fourier-Feynman transform of F given Z_{n+1} with the parameter q and is denoted by

$$\begin{aligned} &T_q^{(1)}\left[F \mid Z_{n+1}\right]\left(y, \vec{\xi}_{n+1}\right) \\ &= \lim_{\lambda \longrightarrow -iq} T_\lambda\left[F \mid Z_{n+1}\right]\left(y, \vec{\xi}_{n+1}\right). \end{aligned} \tag{10}$$

For $1 < p \leq \infty$, we define a generalized L_p-analytic conditional Fourier-Feynman transform $T_q^{(p)}[F \mid Z_{n+1}]$ of F given Z_{n+1} by the formula

$$\begin{aligned} &T_q^{(p)}\left[F \mid Z_{n+1}\right]\left(\cdot, \vec{\xi}_{n+1}\right) \\ &= \lim_{\lambda \longrightarrow -iq}\left(w^{p'}\right)\left(T_\lambda\left[F \mid Z_{n+1}\right]\left(\cdot, \vec{\xi}_{n+1}\right)\right) \\ &\qquad \text{(if exists)}. \end{aligned} \tag{11}$$

$T_\lambda[F \mid Z_n](y, \vec{\xi}_n)$ and $T_q^{(p)}[F \mid Z_n](y, \vec{\xi}_n)$ are similarly defined with $I_F^\lambda(y, \vec{\xi}_n)$.

Let G be defined on $C[0,T]$. For $\lambda > 0$ and $y \in C[0,T]$, suppose that $E[F_{Z/\sqrt{2}}^\lambda(\cdot, y/\sqrt{2}) G_{-Z/\sqrt{2}}^\lambda(\cdot, y/\sqrt{2})]$ exists. By the same method as in Theorem 2.5 of [10] we have

$$\begin{aligned} &E\left[F_{Z/\sqrt{2}}^\lambda\left(\cdot, \frac{y}{\sqrt{2}}\right) G_{-Z/\sqrt{2}}^\lambda\left(\cdot, \frac{y}{\sqrt{2}}\right) \mid Z_{n+1}^\lambda\right]\left(\vec{\xi}_{n+1}\right) \\ &= E\left[F\left(\frac{1}{\sqrt{2}}\left[y + \lambda^{-1/2}X_{b,n+1}(x,\cdot)\right.\right.\right. \\ &\left.\left. + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right]\right) \times G\left(\frac{1}{\sqrt{2}}\left[y - \lambda^{-1/2}X_{b,n+1}(x,\cdot)\right.\right. \\ &\left.\left.\left. - A - P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right]\right)\right], \end{aligned} \tag{12}$$

$$\begin{aligned} &E\left[F_{Z/\sqrt{2}}^\lambda\left(\cdot, \frac{y}{\sqrt{2}}\right) G_{-Z/\sqrt{2}}^\lambda\left(\cdot, \frac{y}{\sqrt{2}}\right) \mid Z_n^\lambda\right]\left(\vec{\xi}_n\right) \\ &= \int_{\mathbb{R}} \Psi_1\left(\lambda, a(T) - a(t_n),\right. \\ &\left. b(T) - b(t_n), \xi_{n+1} - \xi_n\right) E\left[F\left(\frac{1}{\sqrt{2}}\left[y\right.\right.\right. \\ &\left.\left. + \lambda^{-1/2}X_{b,n+1}(x,\cdot) + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right]\right) G\left(\frac{1}{\sqrt{2}}\left[y\right.\right. \\ &\left.\left.\left. - \lambda^{-1/2} \times X_{b,n+1}(x,\cdot) - A\right.\right.\right. \\ &\left.\left.\left. - P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right]\right)\right] d\xi_{n+1}. \end{aligned} \tag{13}$$

Let $K_{F,G}^\lambda(y, \vec{\xi}_{n+1})$ and $K_{F,G}^\lambda(y, \vec{\xi}_n)$ be the right-hand sides of (12) and (13), respectively. If $K_{F,G}^\lambda(y, \vec{\xi}_{n+1})$ has an analytic

extension $J_\lambda(F,G)(y,\vec{\xi}_{n+1})$ on $\mathbb{C}_+$, then it is called a generalized conditional convolution product of F and G given Z_{n+1} with the parameter λ and denoted by

$$[(F*G)_\lambda \mid Z_{n+1}]\left(y,\vec{\xi}_{n+1}\right) = J_\lambda(F,G)\left(y,\vec{\xi}_{n+1}\right). \quad (14)$$

Moreover if $[(F*G)_\lambda \mid Z_{n+1}](y,\vec{\xi}_{n+1})$ has a point-wise limit as λ approaches $-iq$ through $\mathbb{C}_+$, then it is denoted by

$$\begin{aligned}&\left[(F*G)_q \mid Z_{n+1}\right]\left(y,\vec{\xi}_{n+1}\right)\\ &\quad = \lim_{\lambda\longrightarrow -iq}\left[(F*G)_\lambda \mid Z_{n+1}\right]\left(y,\vec{\xi}_{n+1}\right).\end{aligned} \quad (15)$$

$[(F*G)_\lambda \mid Z_n](y,\vec{\xi}_n)$ and $[(F*G)_q \mid Z_n](y,\vec{\xi}_n)$ are similarly understood with $K^\lambda_{F,G}(y,\vec{\xi}_n)$.

For $j = 1,\ldots,n,n+1$, let $\alpha_j = (h/[b(t_j) - b(t_{j-1})]^{1/2})\chi_{(t_{j-1},t_j]}$, let V be the subspace of $L_2[0,T]$ generated by $\{\alpha_1,\ldots,\alpha_n,\alpha_{n+1}\}$, and let $V^\perp$ be the orthogonal complement of V. Let $\mathscr{P} : L_2[0,T] \longrightarrow V$ be the projection given by $\mathscr{P}v = \sum_{j=1}^{n+1}\langle v,\alpha_j\rangle\alpha_j$ and let $\mathscr{P}^\perp : L_2[0,T] \longrightarrow V^\perp$ be the orthogonal projection.

The following lemmas are essential to prove the results in the next sections [8].

Lemma 1. *Let $v \in L_2[0,T]$. Then for w_φ a.e. $x \in C[0,T]$ we have*

$$\begin{aligned}(v, X(x,\cdot)) &= (M_h v, x),\\ (v, P_{b,n+1}(X(x,\cdot))) &= (\mathscr{P}M_h v, x),\end{aligned} \quad (16)$$

where $M_h : L_2[0,T] \longrightarrow L_2[0,T]$ is the multiplication operator defined by $M_h u = hu$ for $u \in L_2[0,T]$.

Lemma 2. *Let $v \in L_2[0,T]$ and $\vec{\xi}_{n+1} = (\xi_0,\xi_1,\ldots,\xi_n,\xi_{n+1}) \in \mathbb{R}^{n+2}$. Then we have*

$$\begin{aligned}\left(v, P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right) &= \left(v, P_{b,n+1}\left(\vec{\xi}_n\right)\right)\\ &\quad + \langle v\alpha_{n+1},\alpha_{n+1}\rangle(\xi_{n+1}-\xi_n),\end{aligned} \quad (17)$$

where $\vec{\xi}_n = (\xi_0,\xi_1,\ldots,\xi_n)$ and $\langle\cdot,\cdot\rangle$ denotes the inner product on $L_2[0,T]$.

Lemma 3. *Let $\{v_1,\ldots,v_r\}$ be a subset of $L_2[0,T]$ such that $\{M_h v_1,\ldots,M_h v_r\}$ is an independent set. Then the random vector $(\vec{v},Z(x,\cdot)) \equiv ((v_1,Z(x,\cdot)),\ldots,(v_r,Z(x,\cdot)))$ has the multivariate normal distribution with mean vector $(\vec{v},a) \equiv ((v_1,a),\ldots,(v_r,a))$ and covariance matrix $\Sigma_{M_h} = [\langle M_h v_i, M_h v_j\rangle]_{r\times r}$. Moreover, for any Borel measurable function $f : \mathbb{R}^r \longrightarrow \mathbb{C}$ we have*

$$\begin{aligned}&\int_{C[0,T]} f\left(\left(\vec{v},Z(x,\cdot)\right)\right)dw_\varphi(x)\\ &\quad \overset{*}{=} \int_{\mathbb{R}^r} f\left(\vec{u}\right)\Psi_r\left(1,\left(\vec{v},a\right),\Sigma_{M_h},\vec{u}\right)d\vec{u},\end{aligned} \quad (18)$$

where $\overset{*}{=}$ means that if either side exists, then both sides exist and they are equal.

3. Generalized Conditional Fourier-Feynman Transforms

In this section we derive the conditional Fourier-Feynman transforms of functions on $C[0,T]$ and investigate the effects of the drift a.

Let $1 \le p \le \infty$, let r be any fixed positive integer, and let $\{v_1,v_2,\ldots,v_r\}$ be an orthonormal subset of $L_2[0,T]$ such that both $\{M_h v_1,\ldots,M_h v_r\}$ and $\{\mathscr{P}^\perp M_h v_1,\ldots,\mathscr{P}^\perp M_h v_r\}$ are independent sets. Let $\mathscr{A}^{(p)}$ be the space of cylinder functions F_r having the form

$$F_r(x) = f\left(\left(\vec{v},x\right)\right) \quad (19)$$

for w_φ a.e. $x \in C[0,T]$, where $f \in L_p(\mathbb{R}^r)$ and $(\vec{v},x) = ((v_1,x),\ldots,(v_r,x))$. Without loss of generality we can take f to be Borel measurable.

Let $\mathscr{M}(L_2[0,T])$ be the class of all $\mathbb{C}$-valued Borel measures of bounded variation over $L_2[0,T]$ and let $\mathcal{S}_{w_\varphi}$ be the space of all functions F which for $\sigma \in \mathscr{M}(L_2[0,T])$ have the form

$$F(x) = \int_{L_2[0,T]} \exp\{i(v,x)\}\,d\sigma(v) \quad (20)$$

for w_φ a.e. $x \in C[0,T]$. Note that $\mathcal{S}_{w_\varphi}$ is a Banach algebra [6].

Let $\mathscr{C} = \{e_1,\ldots,e_r\}$ be the orthonormal set obtained from $\mathscr{C}' \equiv \{\mathscr{P}^\perp M_h v_1,\ldots,\mathscr{P}^\perp M_h v_r\}$ by the Gram-Schmidt orthonormalization process. Keeping the orders of the elements in $\mathscr{C}$ and $\mathscr{C}'$, respectively, let B be the matrix of change of coordinates from $\mathscr{C}'$ to $\mathscr{C}$. For $v \in L_2[0,T]$, $\lambda \in \mathbb{C}$, and $\vec{u} \in \mathbb{R}^r$, let $c_j(v) = \langle v,e_j\rangle$ for $j = 1,\ldots,r$, let $\vec{c}(\mathscr{P}^\perp M_h v) = (c_1(\mathscr{P}^\perp M_h v),\ldots,c_r(\mathscr{P}^\perp M_h v))$, let $A_1(\lambda,v) = \exp\{-(1/2\lambda)[\|\mathscr{P}^\perp M_h v\|^2 - \|\vec{c}(\mathscr{P}^\perp M_h v)\|^2_{\mathbb{R}}]\}$, and let $A_2(v,\vec{u}) = \exp\{i\langle\vec{c}(\mathscr{P}^\perp M_h v),\vec{u}\rangle_{\mathbb{R}}\}$.

Theorem 4. *Let $\Psi(x) = F(x)F_r(x)$ for w_φ a.e. $x \in C[0,T]$, where $F_r(\in \mathscr{A}^{(p)}(1 \le p \le \infty))$ and $F(\in \mathcal{S}_{w_\varphi})$ are given by (19) and (20), respectively. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned}T_\lambda[\Psi \mid Z_{n+1}]\left(y,\vec{\xi}_{n+1}\right) &= \int_{L_2[0,T]} B_1\left(y,v,0,A,\vec{\xi}_{n+1}\right)\\ &\quad\cdot A_1(\lambda,v)\\ &\quad\cdot \int_{\mathbb{R}^r} f\left(\vec{u}B + \left(\vec{v},y+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)\\ &\quad\cdot A_2\left(v,\vec{u}\right)\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right)d\vec{u}\,d\sigma(v),\end{aligned} \quad (21)$$

where I_r is the $r\times r$ identity matrix and, for $w_1,w_2 \in L_2[0,T]$, and for $x \in C[0,T]$

$$B_1\left(y, w_1, w_2, x, \vec{\xi}_{n+1}\right) = \exp\left\{i\left[\left(w_1 + w_2, y\right) + \left(w_1 - w_2, x + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right]\right\}. \tag{22}$$

Furthermore $T_\lambda[\Psi \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ *for* $0 < p_1 \leq p$. *If* $p = 1$, *then, for a nonzero real* q, $T_q^{(1)}[\Psi \mid Z_{n+1}](y, \vec{\xi}_{n+1})$ *is given by the right-hand side of (21) with replacing* λ *by* $-iq$. *In this case,* $T_q^{(1)}[\Psi \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ *for* $0 < p_1 \leq \infty$.

Proof. For $\lambda > 0$, w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have

$$\begin{aligned} I_\Psi^\lambda\left(y, \vec{\xi}_{n+1}\right) &= E\left[\Psi\left(y + \lambda^{-1/2} X_{b,n+1}(x,\cdot) + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right] = \int_{L_2[0,T]} B_1\left(y, v, 0, A, \vec{\xi}_{n+1}\right) \\ &\cdot \int_{C[0,T]} \exp\left\{i\lambda^{-1/2}\left(\mathscr{P}^\perp M_h v, x\right)\right\} \\ &\times f\left(\lambda^{-1/2}\left(\mathscr{P}^\perp M_h \vec{v}, x\right) + \left(\vec{v}, y + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right) dw_\varphi(x)\, d\sigma(v) \end{aligned} \tag{23}$$

by Lemma 1, where $(\mathscr{P}^\perp M_h \vec{v}, x) = ((\mathscr{P}^\perp M_h v_1, x), \ldots, (\mathscr{P}^\perp M_h v_r, x))$. For $v \in L_2[0,T]$, let $e_{r+1} = (1/c_{r+1}(\mathscr{P}^\perp M_h v))[\mathscr{P}^\perp M_h v - \sum_{j=1}^r c_j(\mathscr{P}^\perp M_h v) e_j]$ if $c_{r+1}(\mathscr{P}^\perp M_h v) \neq 0$, where $c_{r+1}(\mathscr{P}^\perp M_h v) = (\|\mathscr{P}^\perp M_h v\|^2 - \|\vec{c}(\mathscr{P}^\perp M_h v)\|_{\mathbb{R}}^2)^{1/2}$. By Lemma 3 we have

$$\begin{aligned} &I_\Psi^\lambda\left(y, \vec{\xi}_{n+1}\right) \\ &= \int_{L_2[0,T]} B_1\left(y, v, 0, A, \vec{\xi}_{n+1}\right) \int_{C[0,T]} \exp\left\{i\lambda^{-1/2}\sum_{j=1}^{r+1} c_j\left(\mathscr{P}^\perp M_h v\right) \cdot \left(e_j, x\right)\right\} \times f\left(\lambda^{-1/2}\left(\vec{e}, x\right) B + \left(\vec{v}, y + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right) dw_\varphi(x)\, d\sigma(v) \\ &= \left(\frac{1}{2\pi}\right)^{(r+1)/2} \int_{L_2[0,T]} B_1\left(y, v, 0, A, \vec{\xi}_{n+1}\right) \int_{\mathbb{R}^{r+1}} \exp\left\{i\lambda^{-1/2}\sum_{j=1}^{r+1} c_j\left(\mathscr{P}^\perp M_h v\right) \cdot u_j - \frac{1}{2}\sum_{j=1}^{r+1} u_j^2\right\} f\left(\lambda^{-1/2}\vec{u} B + \left(\vec{v}, y + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right) du_{r+1} d\vec{u}\, d\sigma(v), \end{aligned} \tag{24}$$

where $\vec{e} = (e_1, \ldots, e_r)$ and $\vec{u} = (u_1, \ldots, u_r)$. By simple calculations we have

$$\begin{aligned} I_\Psi^\lambda\left(y, \vec{\xi}_{n+1}\right) &= \int_{L_2[0,T]} B_1\left(y, v, 0, A, \vec{\xi}_{n+1}\right) A_1(\lambda, v) \\ &\cdot \int_{\mathbb{R}^r} A_2\left(v, \lambda^{-1/2}\vec{u}\right) \\ &\cdot f\left(\lambda^{-1/2}\vec{u} B + \left(\vec{v}, y + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right) \\ &\cdot \Psi_r\left(1, \vec{0}, I_r, \vec{u}\right) d\vec{u}\, d\sigma(v) \\ &= \int_{L_2[0,T]} B_1\left(y, v, 0, A, \vec{\xi}_{n+1}\right) A_1(\lambda, v) \\ &\cdot \int_{\mathbb{R}^r} f\left(\vec{u} B + \left(\vec{v}, y + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right) \\ &\cdot A_2\left(v, \vec{u}\right) \Psi_r\left(\lambda, \vec{0}, I_r, \vec{u}\right) d\vec{u}\, d\sigma(v). \end{aligned} \tag{25}$$

Note that if $1 \leq p < \infty$, then we have by the change of variable theorem

$$\begin{aligned} &\left\|f\left(\cdot B + \left(\vec{v}, y + A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)\right\|_p^p \\ &\quad = \left|\det\left(B^{-1}\right)\right| \|f\|_p^p < \infty, \end{aligned} \tag{26}$$

and for $\lambda \in \mathbb{C}_+$ we have $|A_1(\lambda, v) A_2(v, \vec{u})| \leq 1$ by the Bessel inequality. By the Morera theorem with aid of the Hölder inequality and the dominated convergence theorem, $T_\lambda[\Psi \mid Z_{n+1}](y, \vec{\xi}_{n+1})$ exists for $\lambda \in \mathbb{C}_+$. For $\lambda \in \mathbb{C}_+$ we have by Lemma 3 and the Young inequality [15]

$$\begin{aligned} &\left\|T_\lambda\left[\Psi \mid Z_{n+1}\right]\left(\cdot, \vec{\xi}_{n+1}\right)\right\|_p^p \\ &\quad \leq |\det(B)| \|\sigma\|^p \int_{\mathbb{R}^r} \left[\int_{\mathbb{R}^r} \left|f\left(\left(\vec{z} - \vec{u}\right) B + \left(\vec{v}, A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)\right| \times \left|\Psi_r\left(\lambda, \vec{0}, I_r, \vec{u}\right)\right| d\vec{u}\right]^p d\vec{z} \\ &\quad = |\det(B)| \|\sigma\|^p \left\|\left|f\left(\cdot B + \left(\vec{v}, A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)\right| * \left|\Psi_r\left(\lambda, \vec{0}, I_r, \cdot\right)\right|\right\|_p^p < \infty \end{aligned} \tag{27}$$

so that $T_\lambda[\Psi \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ because $w_\varphi(C[0,T]) = 1 < \infty$. If $p = 1$, then the final results follow from the dominated convergence theorem. □

Remark 5. For $v \in L_2[0,T]$ let $d_j(v) = \langle v, \mathscr{P}^\perp M_h v_j \rangle$ for $j = 1, \ldots, r$. Let $B^{-1} = [\beta_{kl}]_{r \times r}$ be the matrix of change of coordinates from $\mathscr{C}$ to $\mathscr{C}'$. Now for $l =$

$1,\ldots,r$, $e_l = \sum_{k=1}^{r}\beta_{kl}\mathscr{P}^{\perp}M_h v_k$ so that for $\overrightarrow{d}(\mathscr{P}^{\perp}M_h v) = (d_1(\mathscr{P}^{\perp}M_h v),\ldots,d_r(\mathscr{P}^{\perp}M_h v))$ and $\overrightarrow{u} = (u_1,\ldots,u_r)$ we have

$$\begin{aligned}\left\|\overrightarrow{c}\,(\mathscr{P}^{\perp}M_h v)\right\|_{\mathbb{R}}^2 &= \sum_{l=1}^{r}\left[\sum_{k=1}^{r}\beta_{kl}d_k\left(\mathscr{P}^{\perp}M_h v\right)\right]^2 \\ &= \left\|\overrightarrow{d}\,(\mathscr{P}^{\perp}M_h v)\,B^{-1}\right\|_{\mathbb{R}}^2\end{aligned} \tag{28}$$

and $\langle\overrightarrow{c}(\mathscr{P}^{\perp}M_h v),\overrightarrow{u}\rangle_{\mathbb{R}} = \langle\overrightarrow{d}(\mathscr{P}^{\perp}M_h v)B^{-1},\overrightarrow{u}\rangle_{\mathbb{R}}$. Now the expression $\overrightarrow{c}(\mathscr{P}^{\perp}M_h v)$ in A_1 and A_2 can be replaced by $\overrightarrow{d}(\mathscr{P}^{\perp}M_h v)B^{-1}$.

By Lemma 1.1 in [16] and (27) with minor modifications, we have the following theorem.

Theorem 6. *Let Ψ be as given in Theorem 4 with $1 < p \le 2$, and let $1/p + 1/p' = 1$. Then for $\lambda \in \mathbb{C}_+$ and $P_{Z_{n+1}}$ a.e. $\overrightarrow{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have $T_\lambda[\Psi \mid Z_{n+1}](\cdot,\overrightarrow{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le p'$.*

Theorem 7. *Let the assumptions be as given in Theorem 6. Suppose that σ is concentrated on the set $M_h^{-1}(V)$. Then $T_q^{(p)}[\Psi \mid Z_{n+1}](y,\overrightarrow{\xi}_{n+1})$ is given by the right-hand side of (21), where λ is replaced by $-iq$ and $A_1(-iq,v) = A_2(v,\overrightarrow{u}) = 1$ for σ a.e. $v \in L_2[0,T]$ and $\overrightarrow{u} \in \mathbb{R}^r$. In this case, $T_q^{(p)}[\Psi \mid Z_{n+1}](\cdot,\overrightarrow{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le p'$.*

Proof. For σ a.e. $v \in L_2[0,T]$, $\overrightarrow{u} \in \mathbb{R}^r$, and $\lambda \in \mathbb{C}$, we have $A_1(\lambda,v) = A_2(v,\overrightarrow{u}) = 1$ so that it is not difficult to show

$$\begin{aligned}&T_\lambda\left[\Psi \mid Z_{n+1}\right]\left(y,\overrightarrow{\xi}_{n+1}\right) \\ &\quad= \int_{L_2[0,T]} B_1\left(y,v,0,A,\overrightarrow{\xi}_{n+1}\right) \\ &\quad\cdot\left(f\left(\cdot B + \left(\overrightarrow{v},A + P_{b,n+1}\left(\overrightarrow{\xi}_{n+1}\right)\right)\right)\right. \\ &\quad\left.*\,\Psi_r\left(\lambda,\overrightarrow{0},I_r,\cdot\right)\right)\left(B^{-1}\left(\overrightarrow{v},y\right)\right)d\sigma(v).\end{aligned} \tag{29}$$

Now $\|T_q^{(p)}[\Psi \mid Z_{n+1}](\cdot,\overrightarrow{\xi}_{n+1})\|_{p'}^{p'}$ is bounded by

$$\begin{aligned}&|\det(B)|\,\|\sigma\|^{p'}\left\|f\left(\cdot B + \left(\overrightarrow{v},A + P_{b,n+1}\left(\overrightarrow{\xi}_{n+1}\right)\right)\right)\right. \\ &\quad\left.*\,\Psi_r\left(-iq,\overrightarrow{0},I_r,\cdot\right)\right\|_{p'}^{p'}\end{aligned} \tag{30}$$

which is finite from Lemma 1.1 in [16]. By Lemma 3, Theorem 4, and the change of variable theorem, we have for $\overrightarrow{\xi}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}&\left\|T_\lambda\left[\Psi \mid Z_{n+1}\right]\left(\cdot,\overrightarrow{\xi}_{n+1}\right)\right. \\ &\quad\left.- T_q^{(p)}\left[\Psi \mid Z_{n+1}\right]\left(\cdot,\overrightarrow{\xi}_{n+1}\right)\right\|_{p'}^{p'} \le |\det(B)|\,\|\sigma\|^{p'} \\ &\quad\cdot\left\|f\left(\cdot B + \left(\overrightarrow{v},A + P_{b,n+1}\left(\overrightarrow{\xi}_{n+1}\right)\right)\right)\right. \\ &\quad*\,\Psi_r\left(\lambda,\overrightarrow{0},I_r,\cdot\right) \\ &\quad- f\left(\cdot B + \left(\overrightarrow{v},A + P_{b,n+1}\left(\overrightarrow{\xi}_{n+1}\right)\right)\right) \\ &\quad\left.*\,\Psi_r\left(-iq,\overrightarrow{0},I_r,\cdot\right)\right\|_{p'}^{p'}\end{aligned} \tag{31}$$

which converges to 0 as λ approaches $-iq$ through $\mathbb{C}_+$ by Lemma 1.2 in [16]. □

For $\lambda \in \mathbb{C}$ let $\Gamma_\lambda = \lambda/(b(T) - b(t_n))$, let $\langle\overrightarrow{v}\alpha_{n+1},\alpha_{n+1}\rangle = (\langle v_1\alpha_{n+1},\alpha_{n+1}\rangle,\ldots,\langle v_r\alpha_{n+1},\alpha_{n+1}\rangle)$, and let $\Lambda = 1 + [b(T) - b(t_n)]\|\langle\overrightarrow{v}\alpha_{n+1},\alpha_{n+1}\rangle B^{-1}\|_{\mathbb{R}}^2$. For $v \in L_2[0,T]$ and $\overrightarrow{u} \in \mathbb{R}^r$, let

$$\begin{aligned}&\Phi_r\left(\lambda,v,\overrightarrow{u}\right) \\ &\quad= \left(\frac{1}{\Lambda}\right)^{1/2}\exp\left\{\frac{1}{2\Gamma_\lambda\Lambda}\left[\lambda\left\langle\overrightarrow{u},\left\langle\overrightarrow{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}} + i\left[\left\langle v\alpha_{n+1},\alpha_{n+1}\right\rangle\right.\right.\right. \\ &\quad\left.\left.\left.- \left\langle\overrightarrow{c}\,(\mathscr{P}^{\perp}M_h v),\left\langle\overrightarrow{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}\right]\right]^2\right\}.\end{aligned} \tag{32}$$

Theorem 8. *Suppose that the assumptions and notations are as given in Theorem 4. Let $\overrightarrow{a}_n = (a(t_0),\ldots,a(t_n))$. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\overrightarrow{\xi}_n \in \mathbb{R}^{n+1}$*

$$\begin{aligned}T_\lambda\left[\Psi \mid Z_n\right]\left(y,\overrightarrow{\xi}_n\right) &= \int_{L_2[0,T]} B_1\left(y,v,0,a,\overrightarrow{\xi}_n\right. \\ &\quad\left.- \overrightarrow{a}_n\right)A_1(\lambda,v) \\ &\quad\cdot\int_{\mathbb{R}^r} f\left(\overrightarrow{u}B + \left(\overrightarrow{v},y + a + P_{b,n+1}\left(\overrightarrow{\xi}_n - \overrightarrow{a}_n\right)\right)\right) \\ &\quad\cdot A_2\left(v,\overrightarrow{u}\right)\Psi_r\left(\lambda,\overrightarrow{0},I_r,\overrightarrow{u}\right) \\ &\quad\cdot\Phi_r\left(\lambda,v,\overrightarrow{u}\right)d\overrightarrow{u}\,d\sigma(v),\end{aligned} \tag{33}$$

where B_1 and Φ_r are given by (22) and (32), respectively. Furthermore $T_\lambda[\Psi \mid Z_n](\cdot,\overrightarrow{\xi}_n) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le p$. If $p = 1$, then, for a nonzero real q, $T_q^{(1)}[\Psi \mid Z_n](y,\overrightarrow{\xi}_n)$ is given by the right-hand side of (33) with replacing λ by $-iq$. In this case, $T_q^{(1)}[\Psi \mid Z_n](\cdot,\overrightarrow{\xi}_n) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le \infty$.

Proof. For $\overrightarrow{\xi}_n = (\xi_0,\ldots,\xi_n) \in \mathbb{R}^{n+1}$ let $\overrightarrow{\xi}_{n+1} = (\xi_0,\ldots,\xi_n,\xi_{n+1})$ with $\xi_{n+1} \in \mathbb{R}$. For $\lambda > 0$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\overrightarrow{\xi}_n \in \mathbb{R}^{n+1}$ we have by Lemma 2

$$
\begin{aligned}
I_{\Psi}^{\lambda}\left(y,\vec{\xi}_n\right) &= \int_{\mathbb{R}} \Psi_1\left(\lambda, a(T)-a(t_n), b(T)-b(t_n), \xi_{n+1}-\xi_n\right) E\left[\Psi\left(y+\lambda^{-1/2}\times X_{b,n+1}(x,\cdot)+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right] d\xi_{n+1} \\
&= \left(\frac{\Gamma_\lambda}{2\pi}\right)^{1/2} \int_{L_2[0,T]} B_1\left(y,v,0,A,\vec{\xi}_n\right) A_1(\lambda,v) \int_{\mathbb{R}^{r+1}} \exp\left\{i\left\langle v\alpha_{n+1},\alpha_{n+1}\right\rangle\left[a(T)-a(t_n)\right]\right\} f\left(\vec{u}B+\left(\vec{v}, y+A+P_{b,n+1}\left(\vec{\xi}_n\right)\right.\right. \\
&\quad \left.\left.+\left(a(T)-a(t_n)+\xi\right)\left\langle \vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle\right)\right) A_2(v,\vec{u})\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right)\exp\left\{-\frac{\Gamma_\lambda}{2}\xi^2+i\left\langle v\alpha_{n+1},\alpha_{n+1}\right\rangle\xi\right\} d\vec{u}\,d\xi\,d\sigma(v) \\
&= \left(\frac{\Gamma_\lambda}{2\pi}\right)^{1/2}\left(\frac{\lambda}{2\pi}\right)^{r/2} \int_{L_2[0,T]} B_1\left(y,v,0,a,\vec{\xi}_n-\vec{a}_n\right) A_1(\lambda,v)\int_{\mathbb{R}^{r+1}} f\left(\vec{u}B+\left(\vec{v}, y+a+P_{b,n+1}\left(\vec{\xi}_n-\vec{a}_n\right)\right)\right) \\
&\quad \cdot \exp\left\{-\frac{\Gamma_\lambda}{2}\xi^2+i\left\langle v\alpha_{n+1},\alpha_{n+1}\right\rangle\xi-\frac{\lambda}{2}\left\|\xi\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}-\vec{u}\right\|_{\mathbb{R}}^2+i\left\langle\vec{c}\left(\mathscr{P}^{\perp}M_h v\right),\vec{u}-\xi\times\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}\right\} d\vec{u}\,d\xi\,d\sigma(v) \\
&= \left(\frac{\Gamma_\lambda}{2\pi}\right)^{1/2}\int_{L_2[0,T]} B_1\left(y,v,0,a,\vec{\xi}_n-\vec{a}_n\right) A_1(\lambda,v)\int_{\mathbb{R}^{r+1}} f\left(\vec{u}B+\left(\vec{v}, y+a+P_{b,n+1}\left(\vec{\xi}_n-\vec{a}_n\right)\right)\right) A_2(v,\vec{u})\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right) \\
&\quad \cdot \exp\left\{-\frac{1}{2}\left(\Gamma_\lambda+\lambda\left\|\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\|_{\mathbb{R}}^2\right)\xi^2\right. \\
&\quad \left.+\left[\lambda\left\langle\vec{u},\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}+i\left[\left\langle v\times\alpha_{n+1},\alpha_{n+1}\right\rangle-\left\langle\vec{c}\left(\mathscr{P}^{\perp}M_h v\right),\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}\right]\right]\xi\right\} d\vec{u}\,d\xi\,d\sigma(v) \\
&= \left(\frac{\Gamma_\lambda}{2\pi}\right)^{1/2}\int_{L_2[0,T]} B_1\left(y,v,0,a,\vec{\xi}_n-\vec{a}_n\right) A_1(\lambda,v)\int_{\mathbb{R}^{r+1}} f\left(\vec{u}B+\left(\vec{v}, y+a+P_{b,n+1}\left(\vec{\xi}_n-\vec{a}_n\right)\right)\right) A_2(v,\vec{u})\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right) \\
&\quad \cdot \exp\left\{-\frac{\Gamma_\lambda\Lambda}{2}\xi^2+\left[\lambda\left\langle\vec{u},\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}+i\left[\left\langle v\alpha_{n+1},\alpha_{n+1}\right\rangle-\left\langle\vec{c}\left(\mathscr{P}^{\perp}M_h v\right),\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}\right]\right] \\
&\quad \cdot\xi\right\} d\xi\,d\vec{u}\,d\sigma(v) = \int_{L_2[0,T]} B_1\left(y,v,0,a,\vec{\xi}_n-\vec{a}_n\right) A_1(\lambda,v)\int_{\mathbb{R}^r} f\left(\vec{u}B\right. \\
&\quad \left.+\left(\vec{v}, y+a+P_{b,n+1}\left(\vec{\xi}_n-\vec{a}_n\right)\right)\right) A_2(v,\vec{u})\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right)\Phi_r\left(\lambda,v,\vec{u}\right) d\vec{u}\,d\sigma(v).
\end{aligned}
\tag{34}
$$

For $\lambda \in \mathbb{C}_+$ we have $|A_1(\lambda,v)A_2(v,\vec{u})| \le 1$ by the Bessel inequality and

$$
\begin{aligned}
&\left|\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right)\Phi_r\left(\lambda,v,\vec{u}\right)\right| \\
&\quad \le \left(\frac{|\lambda|}{2\pi}\right)^{r/2}\exp\left\{-\frac{\operatorname{Re}\lambda}{2}\left\|\vec{u}\right\|_{\mathbb{R}}^2+\frac{\operatorname{Re}\lambda}{2\Lambda}\left[b(T)-b(t_n)\right]\right. \\
&\quad \left.\cdot\left\langle\vec{u},\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}^2\right\} \\
&\quad = \left(\frac{|\lambda|}{2\pi}\right)^{r/2}\exp\left\{-\frac{\operatorname{Re}\lambda}{2\Lambda}\left[\left\|\vec{u}\right\|_{\mathbb{R}}^2+\left[b(T)\right.\right.\right. \\
&\quad \left.-b(t_n)\right]\left[\left\|\vec{u}\right\|_{\mathbb{R}}^2\left\|\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\|_{\mathbb{R}}^2\right. \\
&\quad \left.\left.\left.-\left\langle\vec{u},\left\langle\vec{v}\alpha_{n+1},\alpha_{n+1}\right\rangle B^{-1}\right\rangle_{\mathbb{R}}^2\right]\right]\right\} \\
&\quad \le \left(\frac{|\lambda|}{2\pi}\right)^{r/2}\exp\left\{-\frac{\operatorname{Re}\lambda}{2\Lambda}\left\|\vec{u}\right\|_{\mathbb{R}}^2\right\}
\end{aligned}
\tag{35}
$$

by the Schwarz inequality. By the Morera theorem we establish the first part of the theorem. If $p = 1$, then the final results follow from the dominated convergence theorem. □

Theorem 9. *Let Ψ be as given in Theorem 4 with $1 < p \le 2$, and let $1/p + 1/p' = 1$. Then for $\lambda \in \mathbb{C}_+$ and P_{Z_n} a.e. $\vec{\xi}_n \in \mathbb{R}^{n+1}$, we have $T_\lambda[\Psi \mid Z_n](y,\vec{\xi}_n) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le p'$.*

Proof. For $\lambda \in \mathbb{C}_+$, $\|T_\lambda[\Psi \mid Z_n](\cdot,\vec{\xi}_n)\|_{p'}^{p'}$ is bounded by

$$
\begin{aligned}
&|\det(B)|\,\|\sigma\|^{p'}\left[\frac{|\lambda|\Lambda}{(\operatorname{Re}\lambda)}\right]^{p'r/2} \\
&\quad \times\left\|\left|f\left(\cdot B+\left(\vec{v}, a+P_{b,n+1}\left(\vec{\xi}_n-\vec{a}_n\right)\right)\right)\right|\right. \\
&\quad \left.*\,\Psi_r\left(\frac{(\operatorname{Re}\lambda)}{\Lambda},\vec{0},I_r,\cdot\right)\right\|_{p'}^{p'}
\end{aligned}
\tag{36}
$$

from (35), which is finite by Lemma 1.1 in [16]. □

Because $\langle v_j\alpha_{n+1},\alpha_{n+1}\rangle = 0$ if and only if $\langle M_h v_j,\alpha_{n+1}\rangle = 0$, we can prove the following theorem using the same method as used in the proof of Theorem 7.

Theorem 10. *Let the assumptions be as given in Theorem 7. Suppose that $\langle M_h v_j,\alpha_{n+1}\rangle = 0$ for $j = 1,\ldots,r$. Then for a nonzero real q and w_φ a.e. $y \in C[0,T]$, $T_q^{(p)}[\Psi \mid Z_n](y,\vec{\xi}_n)$ is given by the right-hand side of (33), where λ is replaced by $-iq$ and $A_1(-iq,v) = A_2(v,\vec{u}) = 1$, $\Phi_r(-iq,v,\vec{u}) = \exp\{[b(T) - b(t_n)]\langle v\alpha_{n+1},\alpha_{n+1}\rangle^2/(2qi)\}$ for σ a.e. $v \in L_2[0,T]$ and $\vec{u} \in \mathbb{R}^r$. In this case, $T_q^{(p)}[\Psi \mid Z_n](\cdot,\vec{\xi}_n) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le p'$.*

4. Generalized Conditional Convolution Products

In this section we derive the conditional convolution products of the functions in the previous sections and investigate the effects of the drift a.

Theorem 11. *Let F_r, G_r and f, g be related by (19), respectively, where $F_r \in \mathscr{A}^{(p_1)}$ and $G_r \in \mathscr{A}^{(p_2)}$ with $1 \le p_1, p_2 \le \infty$. Let $F, G(\in \mathcal{S}_{w_\varphi})$ and $\sigma_F, \sigma_G(\in \mathscr{M}(L_2[0,T]))$ be related by (20), respectively. Furthermore let $\Psi(x) = F(x)F_r(x)$, $H(x) = G(x)G_r(x)$ for w_φ a.e. $x \in C[0,T]$ and let $1/p_1 + 1/p_1' = 1$, $1/p_2 + 1/p_2' = 1$. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$
\begin{aligned}
&[(\Psi * H)_\lambda \mid Z_{n+1}]\left(y,\vec{\xi}_{n+1}\right)\\
&= \int_{(L_2[0,T])^2} B_1\left(y,\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}},A,\vec{\xi}_{n+1}\right)\\
&\cdot A_1\left(\lambda,\frac{1}{\sqrt{2}}(w_1-w_2)\right)\\
&\cdot \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v},y+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)+\vec{u}B\right]\right)\\
&\cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v},y-A-P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)-\vec{u}B\right]\right)\\
&\cdot A_2\left(\frac{1}{\sqrt{2}}(w_1-w_2),\vec{u}\right)\\
&\cdot \Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right)d\vec{u}\,d(\sigma_F\times\sigma_G)(w_1,w_2),
\end{aligned}
\tag{37}
$$

where B_1 is given by (22). Moreover $[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot,\vec{\xi}_{n+1}) \in L_{q_1}(C[0,T])$, where $0 < q_1 \le 1$ if either $p_2 \le p_1'$ or $p_1 \le p_2'$, $0 < q_1 \le p_2$ if $p_2 \ge p_1'$ and $0 < q_1 \le p_1$ if $p_1 \ge p_2'$.

Proof. For $\lambda > 0$, w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have

$$
\begin{aligned}
K_{\Psi,H}^{\lambda}\left(y,\vec{\xi}_{n+1}\right) &= E\left[\Psi\left(\frac{1}{\sqrt{2}}\left[y+\lambda^{-1/2}X_{b,n+1}(x,\cdot)+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right]\right)H\left(\frac{1}{\sqrt{2}}\left[y-\lambda^{-1/2}X_{b,n+1}(x,\cdot)-A-P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right]\right)\right]\\
&= \int_{(L_2[0,T])^2}\int_{C[0,T]}\exp\left\{\frac{i}{\sqrt{2}}\left[\left(w_1,y+\lambda^{-1/2}X_{b,n+1}(x,\cdot)+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)+\left(w_2,y-\lambda^{-1/2}X_{b,n+1}(x,\cdot)-A-P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right]\right\}\\
&\times f\left(\frac{1}{\sqrt{2}}\left(\vec{v},y+\lambda^{-1/2}X_{b,n+1}(x,\cdot)+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)\\
&\cdot g\left(\frac{1}{\sqrt{2}}\left(\vec{v},y-\lambda^{-1/2}X_{b,n+1}(x,\cdot)-A-P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)dw_\varphi(x)\,d(\sigma_F\times\sigma_G)(w_1,w_2)\\
&= \int_{(L_2[0,T])^2}B_1\left(y,\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}},A,\vec{\xi}_{n+1}\right)\int_{C[0,T]}\exp\left\{\frac{i}{\sqrt{2}}\left(w_1-w_2,\lambda^{-1/2}X_{b,n+1}(x,\cdot)\right)\right\}\\
&\cdot f\left(\frac{1}{\sqrt{2}}\left(\vec{v},y+\lambda^{-1/2}X_{b,n+1}(x,\cdot)+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)\\
&\cdot g\left(\frac{1}{\sqrt{2}}\times\left(\vec{v},y-\lambda^{-1/2}X_{b,n+1}(x,\cdot)-A-P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right)dw_\varphi(x)\,d(\sigma_F\times\sigma_G)(w_1,w_2).
\end{aligned}
\tag{38}
$$

Using the same method as used in the proof of Theorem 4, we have

$$
\begin{aligned}
K_{\Psi,H}^{\lambda}\left(y,\vec{\xi}_{n+1}\right) &= \int_{(L_2[0,T])^2}B_1\left(y,\frac{w_1}{\sqrt{2}},\frac{w_2}{\sqrt{2}},A,\vec{\xi}_{n+1}\right)\\
&\cdot A_1\left(\lambda,\frac{1}{\sqrt{2}}(w_1-w_2)\right)\\
&\cdot \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v},y+A+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)+\vec{u}B\right]\right)g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v},y-A-P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)-\vec{u}B\right]\right)\\
&\cdot A_2\left(\frac{1}{\sqrt{2}}(w_1-w_2),\vec{u}\right)\Psi_r\left(\lambda,\vec{0},I_r,\vec{u}\right)d\vec{u}\,d(\sigma_F\times\sigma_G)(w_1,w_2)
\end{aligned}
\tag{39}
$$

with replacing v by $(1/\sqrt{2})(w_1 - w_2)$ in A_1 and A_2. Now for $\lambda \in \mathbb{C}_+$ we have formally

$$\begin{aligned}
&\left\| [(\Psi * H)_\lambda \mid Z_{n+1}] \left(\cdot, \vec{\xi}_{n+1}\right) \right\|_1 \\
&\quad \le |\det(B)| \, \|\sigma_F\| \, \|\sigma_G\| \int_{\mathbb{R}^{2r}} \left| f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{z} + \vec{u}\right) B \right.\right.\right. \\
&\quad \left. + \left(\vec{v}, A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right]\right) g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{z} - \vec{u}\right) B \right.\right. \\
&\quad \left.\left.\left. - \left(\vec{v}, A + P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)\right]\right)\right| \times \left|\Psi_r\left(\lambda, \vec{0}, I_r, \right.\right. \\
&\quad \left.\left. \vec{u}\right)\right| d\left(\vec{u}, \vec{z}\right).
\end{aligned} \tag{40}$$

By the same method as used in the proof of Theorem 3.2 in [9], we have $[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_1(C[0,T])$ with the existence of $[(\Psi * H)_\lambda \mid Z_{n+1}](y, \vec{\xi}_{n+1})$ for $\lambda \in \mathbb{C}_+$ if either $p_2 \le p_1'$ or $p_1 \le p_2'$. Using the same methods as used in the proof of Theorem 3.2 in [9] again, we have the remainder part of the theorem. □

Theorem 12. *Let the assumptions be as given in Theorem 11. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e $\vec{\xi}_n \in \mathbb{R}^{n+1}$,*

$$\begin{aligned}
&[(\Psi * H)_\lambda \mid Z_n]\left(y, \vec{\xi}_n\right) = \int_{(L_2[0,T])^2} B_1\left(y, \frac{w_1}{\sqrt{2}}, \frac{w_2}{\sqrt{2}}, a, \vec{\xi}_n \right. \\
&\quad \left. - \vec{a}_n\right) A_1\left(\lambda, \frac{1}{\sqrt{2}}(w_1 - w_2)\right) \\
&\quad \cdot \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + a + P_{b,n+1}\left(\vec{\xi}_n - \vec{a}_n\right)\right) + \vec{u} B\right]\right) \\
&\quad \cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y - a - P_{b,n+1}\left(\vec{\xi}_n - \vec{a}_n\right)\right) - \vec{u} B\right]\right) \\
&\quad \cdot A_2\left(\frac{1}{\sqrt{2}}(w_1 - w_2), \vec{u}\right) \Phi_r\left(\lambda, \frac{1}{\sqrt{2}}(w_1 - w_2), \vec{u}\right) \\
&\quad \cdot \Psi_r\left(\lambda, \vec{0}, I_r, \right. \\
&\quad \left. \vec{u}\right) d\vec{u} \, d(\sigma_F \times \sigma_G)(w_1, w_2),
\end{aligned} \tag{41}$$

where B_1 and Φ_r are given by (22) and (32), respectively. Moreover $[(\Psi * H)_\lambda \mid Z_n](\cdot, \vec{\xi}_n) \in L_{q_1}(C[0,T])$, where $0 < q_1 \le 1$ if either $p_2 \le p_1'$ or $p_1 \le p_2'$, $0 < q_1 \le p_2$ if $p_2 \ge p_1'$ and $0 < q_1 \le p_1$ if $p_1 \ge p_2'$.

Proof. Replacing v by $(1/\sqrt{2})(w_1 - w_2)$ in the proof of Theorem 8 with Theorem 11, we have for $\vec{\xi}_n \in \mathbb{R}^{n+1}$, $\lambda > 0$, and $y \in C[0,T]$

$$\begin{aligned}
&K_{\Psi,H}^\lambda\left(y, \vec{\xi}_n\right) = \int_{(L_2[0,T])^2} B_1\left(y, \frac{w_1}{\sqrt{2}}, \frac{w_2}{\sqrt{2}}, a, \vec{\xi}_n - \vec{a}_n\right) \\
&\quad \cdot A_1\left(\lambda, \frac{1}{\sqrt{2}}(w_1 - w_2)\right) \\
&\quad \cdot \int_{\mathbb{R}^r} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + a + P_{b,n+1}\left(\vec{\xi}_n - \vec{a}_n\right)\right) + \vec{u} B\right]\right) \\
&\quad \cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y - a - P_{b,n+1}\left(\vec{\xi}_n - \vec{a}_n\right)\right) - \vec{u} B\right]\right) \\
&\quad \cdot A_2\left(\frac{1}{\sqrt{2}}(w_1 - w_2), \vec{u}\right) \Phi_r\left(\lambda, \frac{1}{\sqrt{2}}(w_1 - w_2), \vec{u}\right) \\
&\quad \cdot \Psi_r\left(\lambda, \vec{0}, I_r, \right. \\
&\quad \left. \vec{u}\right) d\vec{u} \, d(\sigma_F \times \sigma_G)(w_1, w_2).
\end{aligned} \tag{42}$$

Now for $\lambda \in \mathbb{C}_+$, the formal expression $\|[(\Psi * H)_\lambda \mid Z_n](\cdot, \vec{\xi}_n)\|_1$ is bounded by

$$\begin{aligned}
&|\det(B)| \, \|\sigma_F\| \, \|\sigma_G\| \left(\frac{|\lambda| \Lambda}{\operatorname{Re} \lambda}\right)^{r/2} \\
&\quad \cdot \int_{\mathbb{R}^{2r}} \left| f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{z} + \vec{u}\right) B \right.\right.\right. \\
&\quad \left. + \left(\vec{v}, a + P_{b,n+1}\left(\vec{\xi}_n - \vec{a}_n\right)\right)\right]\right) \\
&\quad \cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{z} - \vec{u}\right) B \right.\right. \\
&\quad \left.\left.\left. - \left(\vec{v}, a + P_{b,n+1}\left(\vec{\xi}_n - \vec{a}_n\right)\right)\right]\right)\right| \\
&\quad \times \Psi_r\left(\frac{\operatorname{Re} \lambda}{\Lambda}, \vec{0}, I_r, \vec{u}\right) d\left(\vec{u}, \vec{z}\right).
\end{aligned} \tag{43}$$

By the same process as used in the proof of Theorem 11, the proof is completed. □

Using the fact $w_\varphi(C[0,T]) < \infty$ and the same method as used in the proof of Theorem 3.3 in [9], we can prove the following two theorems.

Theorem 13. *Let Ψ and H be as given in Theorem 11. Let q be a nonzero real number. Then for $\lambda \in \mathbb{C}_+$ or $\lambda = -iq$, and $\vec{\xi}_{n+1} \in \mathbb{R}^{n+2}$, we have the following:*

(1) *if $F_r, G_r \in \mathscr{A}^{(1)}$, then $[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le 1$*

(2) *if $F_r \in \mathscr{A}^{(1)}$ and $G_r \in \mathscr{A}^{(2)}$, then $[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le 2$*

(3) *if $F_r \in \mathscr{A}^{(1)}$ and $G_r \in \mathscr{A}^{(\infty)}$, then $[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le \infty$*

(4) *if $F_r, G_r \in \mathscr{A}^{(2)}$, then $[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \in L_{p_1}(C[0,T])$ for $0 < p_1 \le \infty$*

Theorem 14. *If we replace Z_{n+1} by Z_n in Theorem 13, then the conclusions of the theorem are still true, where $\vec{\xi}_{n+1}$ is replaced by $\vec{\xi}_n \in \mathbb{R}^{n+1}$.*

Remark 15. The formal expressions $F \in L_{p_1}(C[0,T])(0 < p_1 \leq p)$ in each theorem of this section and of the previous sections include the fact that there exists a cylinder function $G \in \mathscr{A}^{(p)}$ with $|F(x)| \leq |G(x)|$ for w_φ a.e. $x \in C[0,T]$.

5. Relationships between Transforms and Convolution Products

In this section, we investigate the inverse transforms of the conditional Fourier-Feynman transforms of the functions as given in the previous sections. We also show that the analytic conditional Fourier-Feynman transforms of the conditional convolution products for the functions can be expressed as the products of the analytic conditional Fourier-Feynman transforms of each function.

Theorem 16. *Let q be a nonzero real number. Then, under the assumptions as given in Theorem 7, we have for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1}, \vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned}&\int_{C[0,T]} \Big| T_{\overline{\lambda}}\Big[T_\lambda[\Psi \mid Z_{n+1}]\Big(\cdot, \vec{\xi}_{n+1}\Big) \mid Z_{n+1}\Big]\Big(y, \vec{\eta}_{n+1}\Big) \\ &\quad - \Psi\Big(y + 2A \\ &\quad + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)\Big|^p dw_\varphi(y) \\ &\quad \longrightarrow 0\end{aligned} \tag{44}$$

for $1 \leq p < \infty$, and for $1 \leq p \leq \infty$

$$\begin{aligned}&T_{\overline{\lambda}}\Big[T_\lambda[\Psi \mid Z_{n+1}]\Big(\cdot, \vec{\xi}_{n+1}\Big) \mid Z_{n+1}\Big]\Big(y, \vec{\eta}_{n+1}\Big) \\ &\quad \longrightarrow \Psi\Big(y + 2A + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)\end{aligned} \tag{45}$$

for w_φ a.e. $y \in C[0,T]$, as λ approaches $-iq$ through $\mathbb{C}_+$.

Proof. For $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1}$, $\vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$, we have by the change of variable theorem, Theorems 4 and 7

$$\begin{aligned}&T_{\overline{\lambda}}\Big[T_\lambda[\Psi \mid Z_{n+1}]\Big(\cdot, \vec{\xi}_{n+1}\Big) \mid Z_{n+1}\Big]\Big(y, \vec{\eta}_{n+1}\Big) \\ &\quad = \int_{L_2[0,T]} B_1\Big(y, v, 0, 2A, \vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big) \\ &\quad \cdot \int_{\mathbb{R}^{2r}} f\Big(\Big(\vec{u} + \vec{w}\Big)B \\ &\quad + \Big(\vec{v}, y + 2A + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)\Big)\Psi_r\Big(\lambda, \vec{0}, \\ &\quad I_r, \vec{u}\Big)\Psi_r\Big(\overline{\lambda}, \vec{0}, I_r, \vec{w}\Big) d\vec{w}\, d\vec{u}\, d\sigma(v) \\ &\quad = \int_{L_2[0,T]} B_1\Big(y, v, 0, 2A, \vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big) \int_{\mathbb{R}^r} f\Big(\vec{z}B \\ &\quad + \Big(\vec{v}, y + 2A + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)\Big) \\ &\quad \cdot \Psi_r\Big(\frac{|\lambda|^2}{2\mathrm{Re}\,\lambda}, \vec{0}, I_r, \vec{z}\Big) d\vec{z}\, d\sigma(v) \\ &\quad = \epsilon^{-r}\Big(f(\cdot B) * \Psi_r\Big(1, \vec{0}, I_r, \cdot/\epsilon\Big)\Big)\Big(\Big(\vec{v}, y + 2A \\ &\quad + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)B^{-1}\Big) \times \int_{L_2[0,T]} B_1\Big(y, v, \\ &\quad 0, 2A, \vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big) d\sigma(v),\end{aligned} \tag{46}$$

where $\epsilon = (2\mathrm{Re}\,\lambda/|\lambda|^2)^{1/2} > 0$. Let $1 \leq p < \infty$. Then we have by Lemma 3 and the change of variable theorem

$$\begin{aligned}&\int_{C[0,T]} \Big| T_{\overline{\lambda}}\Big[T_\lambda[\Psi \mid Z_{n+1}]\Big(\cdot, \vec{\xi}_{n+1}\Big) \mid Z_{n+1}\Big] \\ &\quad \cdot \Big(y, \vec{\eta}_{n+1}\Big) - \Psi\Big(y + 2A \\ &\quad + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)\Big|^p dw_\varphi(y) \\ &\quad = \int_{C[0,T]} \Big| \Big[\epsilon^{-r}\Big(f(\cdot B) * \Psi_r\Big(1, \vec{0}, I_r, \cdot/\epsilon\Big)\Big) \\ &\quad \cdot \Big(\Big(\vec{v}, y + 2A + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)B^{-1}\Big) \\ &\quad - f\Big(\Big(\vec{v}, y + 2A + P_{b,n+1}\Big(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\Big)\Big)\Big)\Big] \\ &\quad \cdot \int_{L_2[0,T]} B_1\Big(y, v, 0, 2A, \vec{\xi}_{n+1} \\ &\quad + \vec{\eta}_{n+1}\Big) d\sigma(v)\Big|^p dw_\varphi(y) \leq |\det(B)|\, \|\sigma\|^p \\ &\quad \cdot \Big\|\epsilon^{-r}\Big(f(\cdot B) * \Psi_r\Big(1, \vec{0}, I_r, \cdot/\epsilon\Big)\Big) - f(\cdot B)\Big\|_p^p.\end{aligned} \tag{47}$$

Letting $\lambda \longrightarrow -iq$ through $\mathbb{C}_+$ which satisfies $\epsilon \longrightarrow 0$, we have the first part of the theorem by Theorem 1.18 in [17]. If $1 \leq p \leq \infty$, then the remainder part of the theorem follows from Theorem 1.25 in [17]. □

Theorem 17. *Let q be a nonzero real number. Then, under the assumptions as given in Theorem 10, we have for P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$*

$$\int_{C[0,T]} \left| T_{\overline{\lambda}} \left[T_\lambda [\Psi \mid Z_n] \left(\cdot, \vec{\xi}_n \right) \mid Z_n \right] \left(y, \vec{\eta}_n \right) - \Psi \left(y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) \right|^p dw_\varphi (y) \longrightarrow 0 \tag{48}$$

for $1 \le p < \infty$, *and for* $1 \le p \le \infty$

$$T_{\overline{\lambda}} \left[T_\lambda [\Psi \mid Z_n] \left(\cdot, \vec{\xi}_n \right) \mid Z_n \right] \left(y, \vec{\eta}_n \right) \longrightarrow \Psi \left(y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) \tag{49}$$

for w_φ *a.e.* $y \in C[0,T]$, *as* λ *approaches* $-iq$ *through* $\mathbb{C}_+$.

Proof. For $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$, we have by the change of variable theorem and Theorems 8 and 10

$$\begin{aligned} T_{\overline{\lambda}} \left[T_\lambda [\Psi \mid Z_n] \left(\cdot, \vec{\xi}_n \right) \mid Z_n \right] \left(y, \vec{\eta}_n \right) &= \epsilon^{-r} \left(f(\cdot B) * \Psi_r \left(1, \vec{0}, I_r, \cdot/\epsilon \right) \right) \\ &\cdot \left(\left(\vec{v}, y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) B^{-1} \right) \\ &\times \int_{L_2[0,T]} B_1 \left(y, v, 0, 2a, \vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \\ &\cdot A_3(\epsilon, v)\, d\sigma(v), \end{aligned} \tag{50}$$

where $\epsilon = (2\mathrm{Re}\,\lambda/|\lambda|^2)^{1/2} > 0$ and $A_3(\epsilon, v) = \{-\epsilon^2[b(T) - b(t_n)]\langle v\alpha_{n+1}, \alpha_{n+1}\rangle^2/2\}$. Let $1 \le p < \infty$. Then we have by Lemma 3, the Minkowski inequality, and the change of variable theorem

$$\begin{aligned} &\left[\int_{C[0,T]} \left| T_{\overline{\lambda}} \left[T_\lambda [\Psi \mid Z_n] \left(\cdot, \vec{\xi}_n \right) \mid Z_n \right] \left(y, \vec{\eta}_n \right) - \Psi \left(y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) \right|^p dw_\varphi (y) \right]^{1/p} \\ &\le \left[\int_{C[0,T]} \left| \left[\epsilon^{-r} \left(f(\cdot B) * \Psi_r \left(1, \vec{0}, I_r, \cdot/\epsilon \right) \right) \cdot \left(\left(\vec{v}, y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) B^{-1} \right) - f \left(\left(\vec{v}, y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) \right) \right] \right. \right. \\ &\times \int_{L_2[0,T]} B_1 \left(y, v, 0, 2a, \vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) A_3(\epsilon, v)\, d\sigma(v) \Big|^p dw_\varphi (y) \Big]^{1/p} \\ &+ \left[\int_{C[0,T]} \left| f \left(\left(\vec{v}, y + 2a + P_{b,n+1} \left(\vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \right) \right) \cdot \int_{L_2[0,T]} B_1 \left(y, v, 0, 2a, \vec{\xi}_n + \vec{\eta}_n - 2\vec{a}_n \right) \cdot \left[A_3(\epsilon, v) - 1 \right] d\sigma(v) \right|^p dw_\varphi (y) \right]^{1/p} \\ &\le |\det(B)|^{1/p} \|\sigma\| \cdot \left\| \epsilon^{-r} \left(f(\cdot B) * \Psi_r \left(1, \vec{0}, I_r, \cdot/\epsilon \right) \right) - f(\cdot B) \right\|_p \\ &+ \|f\|_p \times \int_{L_2[0,T]} \left| A_3(\epsilon, v) - 1 \right| d|\sigma|(v). \end{aligned} \tag{51}$$

Letting $\lambda \longrightarrow -iq$ through $\mathbb{C}_+$ which satisfies $\epsilon \longrightarrow 0$ and $A_3(\epsilon, v) \longrightarrow 1$, we have the first part of the theorem by Theorem 1.18 of [17]. If $1 \le p \le \infty$, then the remainder part of the theorem follows from Theorem 1.25 of [17]. □

Theorem 18. *Under the assumptions as given in Theorem 11, we have for* $\lambda \in \mathbb{C}_+$, w_φ *a.e.* $y \in C[0,T]$ *and* $P_{Z_{n+1}}$ *a.e.* $\vec{\xi}_{n+1}, \vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned} T_\lambda \left[[(\Psi * H)_\lambda \mid Z_{n+1}] \left(\cdot, \vec{\xi}_{n+1} \right) \mid Z_{n+1} \right] \left(y, \vec{\eta}_{n+1} \right) &= \left[T_\lambda [\Psi \mid Z_{n+1}] \left(\frac{1}{\sqrt{2}} y + \left(\sqrt{2} - 1 \right) a, \frac{1}{\sqrt{2}} \left(\vec{\eta}_{n+1} + \vec{\xi}_{n+1} \right) - \left(\sqrt{2} - 1 \right) a \right) \right] \\ &\left[T_\lambda [H \mid Z_{n+1}] \left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}} \left(\vec{\eta}_{n+1} - \vec{\xi}_{n+1} \right) + a \right) \right]. \end{aligned} \tag{52}$$

Proof. For $\lambda \in \mathbb{C}_+$, the existence of $T_\lambda[[(\Psi * H)_\lambda \mid Z_{n+1}](\cdot, \vec{\xi}_{n+1}) \mid Z_{n+1}](y, \vec{\eta}_{n+1})$ follows from Theorems

4 and 11. We have for w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1}, \vec{\eta}_{n+1} \in \mathbb{R}^{n+2}$

$$\begin{aligned}
&T_\lambda\left[\left[(\Psi * H)_\lambda \mid Z_{n+1}\right]\left(\cdot, \vec{\xi}_{n+1}\right) \mid Z_{n+1}\right]\left(y, \vec{\eta}_{n+1}\right) = \int_{(L_2[0,T])^2} B_1\left(y + A + P_{b,n+1}\left(\vec{\eta}_{n+1}\right), \frac{w_1}{\sqrt{2}}, \frac{w_2}{\sqrt{2}}, A, \vec{\xi}_{n+1}\right) A_1\left(\lambda,\right.\\
&\left.\frac{1}{\sqrt{2}}(w_1 - w_2)\right) A_1\left(\lambda, \frac{1}{\sqrt{2}}(w_1 + w_2)\right) \int_{\mathbb{R}^{2r}} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + 2A + P_{b,n+1}\left(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\right)\right) + \left(\vec{w} + \vec{u}\right) B\right]\right)\\
&\cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + P_{b,n+1}\left(\vec{\eta}_{n+1} - \vec{\xi}_{n+1}\right)\right) + \left(\vec{w} - \vec{u}\right) B\right]\right) A_2\left(\frac{1}{\sqrt{2}}(w_1 - w_2), \vec{u}\right) A_2\left(\frac{1}{\sqrt{2}}(w_1 + w_2), \vec{w}\right)\\
&\cdot \Psi_r\left(\lambda, \vec{0}, I_r, \vec{u}\right) \Psi_r\left(\lambda, \vec{0}, I_r, \vec{w}\right) d\left(\vec{u}, \vec{w}\right) d\left(\sigma_F \times \sigma_G\right)(w_1, w_2).
\end{aligned} \tag{53}$$

Let $\vec{\alpha} = (1/\sqrt{2})(\vec{w} + \vec{u})$ and $\vec{\beta} = (1/\sqrt{2})(\vec{w} - \vec{u})$. Then we have

$$\begin{aligned}
&T_\lambda\left[\left[(\Psi * H)_\lambda \mid Z_{n+1}\right]\left(\cdot, \vec{\xi}_{n+1}\right) \mid Z_{n+1}\right]\left(y, \vec{\eta}_{n+1}\right)\\
&= \int_{(L_2[0,T])^2} B_1\left(\frac{1}{\sqrt{2}} y, w_1, 0, \sqrt{2}A, \frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1} + \vec{\xi}_{n+1}\right)\right)\\
&\cdot B_1\left(\frac{1}{\sqrt{2}} y, w_2, 0, 0, \frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1} - \vec{\xi}_{n+1}\right)\right) A_1\left(\lambda,\right.\\
&\left.\frac{1}{\sqrt{2}}(w_1 - w_2)\right) A_1\left(\lambda, \frac{1}{\sqrt{2}}(w_1 + w_2)\right) \int_{\mathbb{R}^{2r}} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y\right.\right.\right.\\
&\left.\left.\left. + 2A + P_{b,n+1}\left(\vec{\xi}_{n+1} + \vec{\eta}_{n+1}\right)\right)\right] + \vec{\alpha} B\right) \times g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y\right.\right.\right.\\
&\left.\left.\left. + P_{b,n+1}\left(\vec{\eta}_{n+1} - \vec{\xi}_{n+1}\right)\right)\right] + \vec{\beta} B\right) A_2\left(\frac{1}{\sqrt{2}}(w_1\right.\\
&\left. - w_2), \frac{1}{\sqrt{2}}\left(\vec{\alpha} - \vec{\beta}\right)\right) A_2\left(\frac{1}{\sqrt{2}}(w_1 + w_2), \frac{1}{\sqrt{2}}\left(\vec{\alpha}\right.\right.\\
&\left.\left. + \vec{\beta}\right)\right) \Psi_r\left(\lambda, \vec{0}, I_r, \frac{1}{\sqrt{2}}\left(\vec{\alpha} - \vec{\beta}\right)\right) \Psi_r\left(\lambda, \vec{0}, I_r,\right.\\
&\left.\frac{1}{\sqrt{2}}\left(\vec{\alpha}\right.\right.\\
&\left.\left. + \vec{\beta}\right)\right) d\left(\vec{\alpha}, \vec{\beta}\right) d\left(\sigma_F \times \sigma_G\right)(w_1, w_2)
\end{aligned} \tag{54}$$

by the change of variable theorem. Now we have

$$\begin{aligned}
A_1(\lambda, w_1) A_1(\lambda, w_2) &= A_1\left(\lambda, \frac{(w_1 - w_2)}{\sqrt{2}}\right)\\
&\cdot A_1\left(\lambda, \frac{(w_1 + w_2)}{\sqrt{2}}\right),\\
A_2\left(w_1, \vec{\alpha}\right) A_2\left(w_2, \vec{\beta}\right)
&= A_2\left(\frac{(w_1 - w_2)}{\sqrt{2}}, \frac{\left(\vec{\alpha} - \vec{\beta}\right)}{\sqrt{2}}\right)\\
&\times A_2\left(\frac{(w_1 + w_2)}{\sqrt{2}}, \frac{\left(\vec{\alpha} + \vec{\beta}\right)}{\sqrt{2}}\right),\\
\Psi_r\left(\lambda, \vec{0}, I_r, \vec{\alpha}\right) \Psi_r\left(\lambda, \vec{0}, I_r, \vec{\beta}\right)
&= \Psi_r\left(\lambda, \vec{0}, I_r, \frac{\left(\vec{\alpha} - \vec{\beta}\right)}{\sqrt{2}}\right)\\
&\cdot \Psi_r\left(\lambda, \vec{0}, I_r, \frac{\left(\vec{\alpha} + \vec{\beta}\right)}{\sqrt{2}}\right)
\end{aligned} \tag{55}$$

so that we have by Theorem 4

$$\begin{aligned}
&T_\lambda\left[\left[(\Psi * H)_\lambda \mid Z_{n+1}\right]\left(\cdot, \vec{\xi}_{n+1}\right) \mid Z_{n+1}\right]\left(y, \vec{\eta}_{n+1}\right)\\
&= \left[T_\lambda\left[\Psi \mid Z_{n+1}\right]\left(\frac{1}{\sqrt{2}} y\right.\right.\\
&\left. + \left(\sqrt{2} - 1\right) a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1} + \vec{\xi}_{n+1}\right)\right.\\
&\left.\left. - \left(\sqrt{2} - 1\right) a\right)\right]\left[T_\lambda\left[H \mid Z_{n+1}\right]\left(\frac{1}{\sqrt{2}} y\right.\right.\\
&\left.\left. - a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1} - \vec{\xi}_{n+1}\right) + a\right)\right]
\end{aligned} \tag{56}$$

which is the desired result. □

Theorem 19. *Under the assumptions as given in Theorem 12, we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$*

$$\begin{aligned} &T_\lambda\left[[(\Psi * H)_\lambda \mid Z_n]\left(\cdot, \vec{\xi}_n\right) \mid Z_n\right]\left(y, \vec{\eta}_n\right) \\ &\quad = \left[T_\lambda[\Psi \mid Z_n]\left(\frac{1}{\sqrt{2}} y\right.\right. \\ &\quad \left.\left. + \left(\sqrt{2}-1\right) a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n + \vec{\xi}_n\right) - \left(\sqrt{2}-1\right) \vec{a}_n\right)\right] \\ &\quad \times \left[T_\lambda[H \mid Z_n]\left(\frac{1}{\sqrt{2}} y - a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n - \vec{\xi}_n\right)\right.\right. \\ &\quad \left.\left. + \vec{a}_n\right)\right]. \end{aligned} \tag{57}$$

Proof. For $\lambda \in \mathbb{C}_+$, the existence of $T_\lambda[[(\Psi * H)_\lambda \mid Z_n](\cdot, \vec{\xi}_n) \mid Z_n](y, \vec{\eta}_n)$ follows from Theorems 8 and 12. We have for w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\vec{\xi}_n, \vec{\eta}_n \in \mathbb{R}^{n+1}$

$$\begin{aligned} &T_\lambda\left[[(\Psi * H)_\lambda \mid Z_n]\left(\cdot, \vec{\xi}_n\right) \mid Z_n\right]\left(y, \vec{\eta}_n\right) = \int_{(L_2[0,T])^2} B_1\left(y + a + P_{b,n+1}\left(\vec{\eta}_n - \vec{a}_n\right), \frac{w_1}{\sqrt{2}}, \frac{w_2}{\sqrt{2}}, a, \vec{\xi}_n - \vec{a}_n\right) \times A_1\left(\lambda,\right. \\ &\left.\frac{1}{\sqrt{2}}\left(w_1 - w_2\right)\right) A_1\left(\lambda, \frac{1}{\sqrt{2}}\left(w_1 + w_2\right)\right) \int_{\mathbb{R}^{2r}} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + 2a + P_{b,n+1}\left(\vec{\eta}_n + \vec{\xi}_n - 2\vec{a}_n\right)\right) + \left(\vec{w} + \vec{u}\right) B\right]\right) \\ &\cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + P_{b,n+1}\left(\vec{\eta}_n - \vec{\xi}_n\right)\right) + \left(\vec{w} - \vec{u}\right) B\right]\right) A_2\left(\frac{1}{\sqrt{2}}\left(w_1 - w_2\right), \vec{u}\right) A_2\left(\frac{1}{\sqrt{2}} \times \left(w_1 + w_2\right), \vec{w}\right) \\ &\cdot \Phi_r\left(\lambda, \frac{1}{\sqrt{2}}\left(w_1 - w_2\right), \vec{u}\right) \Phi_r\left(\lambda, \frac{1}{\sqrt{2}}\left(w_1 + w_2\right), \vec{w}\right) \Psi_r\left(\lambda, \vec{0}, I_r, \vec{u}\right) \\ &\cdot \Psi_r\left(\lambda, \vec{0}, I_r, \vec{w}\right) d\left(\vec{u}, \vec{w}\right) d\left(\sigma_F \times \sigma_G\right)\left(w_1, w_2\right). \end{aligned} \tag{58}$$

Let $\vec{\alpha} = (1/\sqrt{2})(\vec{w} + \vec{u})$ and $\vec{\beta} = (1/\sqrt{2})(\vec{w} - \vec{u})$. By the change of variable theorem we have

$$\begin{aligned} &T_\lambda\left[[(\Psi * H)_\lambda \mid Z_n]\left(\cdot, \vec{\xi}_n\right) \mid Z_n\right]\left(y, \vec{\eta}_n\right) = \int_{(L_2[0,T])^2} B_1\left(\frac{1}{\sqrt{2}} y, w_1, 0, \sqrt{2} a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n + \vec{\xi}_n\right) - \sqrt{2}\vec{a}_n\right) B_1\left(\frac{1}{\sqrt{2}} y, w_2,\right. \\ &\left.0, 0, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n - \vec{\xi}_n\right)\right) A_1\left(\lambda, w_1\right) A_1\left(\lambda, w_2\right) \int_{\mathbb{R}^{2r}} f\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + 2a + P_{b,n+1}\left(\vec{\eta}_n + \vec{\xi}_n - 2\vec{a}_n\right)\right)\right] + \vec{\alpha} B\right) \\ &\cdot g\left(\frac{1}{\sqrt{2}}\left[\left(\vec{v}, y + P_{b,n+1}\left(\vec{\eta}_n - \vec{\xi}_n\right)\right)\right] + \vec{\beta} B\right) \Phi_r\left(\lambda, \frac{1}{\sqrt{2}}\left(w_1 - w_2\right), \frac{1}{\sqrt{2}}\left(\vec{\alpha} - \vec{\beta}\right)\right) \\ &\cdot \Phi_r\left(\lambda, \frac{1}{\sqrt{2}}\left(w_1 + w_2\right), \frac{1}{\sqrt{2}}\left(\vec{\alpha} + \vec{\beta}\right)\right) A_2\left(w_1, \vec{\alpha}\right) A_2\left(w_2, \vec{\beta}\right) \Psi_r\left(\lambda, \vec{0}, I_r, \vec{\alpha}\right) \\ &\cdot \Psi_r\left(\lambda, \vec{0}, I_r, \vec{\beta}\right) d\left(\vec{\alpha}, \vec{\beta}\right) d\left(\sigma_F \times \sigma_G\right)\left(w_1, w_2\right) \end{aligned} \tag{59}$$

using the same process as used in the proof of Theorem 18. From a long calculation it is not difficult to show

$$\begin{aligned} &\Phi_r\left(\lambda, w_1, \vec{\alpha}\right) \Phi_r\left(\lambda, w_2, \vec{\beta}\right) \\ &\quad = \Phi_r\left(\lambda, \frac{\left(w_1 - w_2\right)}{\sqrt{2}}, \frac{\left(\vec{\alpha} - \vec{\beta}\right)}{\sqrt{2}}\right) \\ &\quad \times \Phi_r\left(\lambda, \frac{\left(w_1 + w_2\right)}{\sqrt{2}}, \frac{\left(\vec{\alpha} + \vec{\beta}\right)}{\sqrt{2}}\right) \end{aligned} \tag{60}$$

so that we have by Theorem 8

$$\begin{aligned} &T_\lambda\left[[(\Psi * H)_\lambda \mid Z_n]\left(\cdot, \vec{\xi}_n\right) \mid Z_n\right]\left(y, \vec{\eta}_n\right) \\ &\quad = \left[T_\lambda[\Psi \mid Z_n]\left(\frac{1}{\sqrt{2}} y\right.\right. \\ &\quad \left.\left. + \left(\sqrt{2}-1\right) a, \frac{1}{\sqrt{2}}\left(\vec{\eta}_n + \vec{\xi}_n\right) - \left(\sqrt{2}-1\right) \vec{a}_n\right)\right] \end{aligned}$$

$$\times\left[T_\lambda\left[H\mid Z_n\right]\left(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_n-\vec{\xi}_n\right)+\vec{a}_n\right)\right] \tag{61}$$

which completes the proof. □

By Theorems 4, 7, 8, 10, 13, 14, 18, and 19, we have the following two theorems.

Theorem 20. *Let Ψ and H be as given in Theorem 11, where $F_r, G_r \in \mathscr{A}^{(1)}$. Let q be a nonzero real number. Then we have for w_φ a.e. $y \in C[0,T]$*

$$\begin{aligned}T_q^{(1)}&\left[\left[(\Psi*H)_q\mid Z_{n+1}\right]\left(\cdot,\vec{\xi}_{n+1}\right)\mid Z_{n+1}\right]\left(y,\vec{\eta}_{n+1}\right)\\&=\left[T_q^{(1)}\left[\Psi\mid Z_{n+1}\right]\left(\frac{1}{\sqrt{2}}y+\left(\sqrt{2}-1\right)a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1}+\vec{\xi}_{n+1}\right)-\left(\sqrt{2}-1\right)a\right)\right]\\&\quad\left[T_q^{(1)}\left[H\mid Z_{n+1}\right]\left(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1}-\vec{\xi}_{n+1}\right)+a\right)\right]\end{aligned} \tag{62}$$

for $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1},\vec{\eta}_{n+1}\in\mathbb{R}^{n+2}$, and for P_{Z_n} a.e. $\vec{\xi}_n,\vec{\eta}_n\in\mathbb{R}^{n+1}$

$$\begin{aligned}T_q^{(1)}&\left[\left[(\Psi*H)_q\mid Z_n\right]\left(\cdot,\vec{\xi}_n\right)\mid Z_n\right]\left(y,\vec{\eta}_n\right)\\&=\left[T_q^{(1)}\left[\Psi\mid Z_n\right]\left(\frac{1}{\sqrt{2}}y+\left(\sqrt{2}-1\right)a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_n+\vec{\xi}_n\right)-\left(\sqrt{2}-1\right)\vec{a}_n\right)\right]\\&\quad\times\left[T_q^{(1)}\left[H\mid Z_n\right]\left(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_n-\vec{\xi}_n\right)+\vec{a}_n\right)\right].\end{aligned} \tag{63}$$

Theorem 21. *Let the assumptions be as given in Theorem 20 except for $G_r\in\mathscr{A}^{(2)}$. Moreover suppose that σ_G is concentrated on $M_h^{-1}(V)$. Then we have*

$$\begin{aligned}T_q^{(2)}&\left[\left[(\Psi*H)_q\mid Z_{n+1}\right]\left(\cdot,\vec{\xi}_{n+1}\right)\mid Z_{n+1}\right]\left(y,\vec{\eta}_{n+1}\right)\\&=\left[T_q^{(1)}\left[\Psi\mid Z_{n+1}\right]\left(\frac{1}{\sqrt{2}}y+\left(\sqrt{2}-1\right)a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1}+\vec{\xi}_{n+1}\right)-\left(\sqrt{2}-1\right)a\right)\right]\\&\quad\left[T_q^{(2)}\left[H\mid Z_{n+1}\right]\left(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_{n+1}-\vec{\xi}_{n+1}\right)+a\right)\right].\end{aligned} \tag{64}$$

In addition, suppose that $\langle M_h\nu_j,\alpha_{n+1}\rangle=0$ for $j=1,\ldots,r$. Then we have

$$\begin{aligned}T_q^{(2)}&\left[\left[(\Psi*H)_q\mid Z_n\right]\left(\cdot,\vec{\xi}_n\right)\mid Z_n\right]\left(y,\vec{\eta}_n\right)\\&=\left[T_q^{(1)}\left[\Psi\mid Z_n\right]\left(\frac{1}{\sqrt{2}}y+\left(\sqrt{2}-1\right)a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_n+\vec{\xi}_n\right)-\left(\sqrt{2}-1\right)\vec{a}_n\right)\right]\\&\quad\times\left[T_q^{(2)}\left[H\mid Z_n\right]\left(\frac{1}{\sqrt{2}}y-a,\frac{1}{\sqrt{2}}\left(\vec{\eta}_n-\vec{\xi}_n\right)+\vec{a}_n\right)\right].\end{aligned} \tag{65}$$

6. Change of Scale Formulas for the Transforms and Convolutions

In this section we investigate change of scales for evaluating the conditional expectations of the functions as described in the previous sections.

Let $\{e_1,e_2,\ldots\}$ be a complete orthonormal basis of $L_2[0,T]$ containing $\mathscr{C}$. For $m\in\mathbb{N}$, $\lambda\in\mathbb{C}$, and $x\in C[0,T]$ let $H_m(\lambda,x)=\exp\{((1-\lambda)/2)\sum_{j=1}^m(e_j,x)^2\}$. Let q be a nonzero real number and let $\{\lambda_m\}_{m=1}^\infty$ be a sequence in $\mathbb{C}_+$ converging to $-iq$ as m approaches ∞.

Theorem 22. *Let $1\le p\le\infty$ and let Ψ be as given in Theorem 4. Then we have for $\lambda\in\mathbb{C}_+$, w_φ a.e. $y\in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$*

$$\begin{aligned}T_\lambda\left[\Psi\mid Z_{n+1}\right]\left(y,\vec{\xi}_{n+1}\right)&=\lim_{m\to\infty}\lambda^{m/2}\\&\quad\cdot\int_{C[0,T]}H_m(\lambda,x)\,\Psi\left(y+Z_{b,n+1}(x,\cdot)+P_{b,n+1}\left(\vec{\xi}_{n+1}\right)\right)dw_\varphi(x).\end{aligned} \tag{66}$$

Moreover if $p=1$, then $T_q^{(1)}[\Psi\mid Z_{n+1}](y,\vec{\xi}_{n+1})$ is given by the right-hand side of (66) with replacing λ by λ_m.

Proof. For $m>r$, $\lambda>0$ and a.e. $\vec{\xi}_{n+1}\in\mathbb{R}^{n+2}$ we have by the same process as used in the proof of Theorem 4

$$\begin{aligned}
&\int_{C[0,T]} H_m(\lambda, x)\, \Psi\Big(y + Z_{b,n+1}(x,\cdot) \\
&\quad + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big)\, dw_\varphi(x) \\
&\quad = \int_{L_2[0,T]} B_1\Big(y, v, 0, A, \overrightarrow{\xi}_{n+1}\Big) \\
&\quad \cdot \int_{C[0,T]} \exp\left\{\frac{1-\lambda}{2}\sum_{j=1}^{m}\left(e_j, x\right)^2 + i\left(\mathscr{P}^{\perp} M_h v, x\right)\right\} \\
&\quad \times f\Big(\Big(\mathscr{P}^{\perp} M_h \overrightarrow{v}, x\Big) + \Big(\overrightarrow{v}, y + A \\
&\quad + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big)\Big)\, dw_\varphi(x)\, d\sigma(v) \\
&\quad = \left(\frac{1}{2\pi}\right)^{(m+1)/2} \int_{L_2[0,T]} B_1\Big(y, v, 0, A, \overrightarrow{\xi}_{n+1}\Big) \\
&\quad \cdot \int_{\mathbb{R}^{m+1}} \exp\left\{-\frac{1}{2}u_{m+1}^2 - \frac{\lambda}{2}\sum_{j=1}^{m} u_j^2\right. \\
&\quad \left. + i\sum_{j=1}^{m+1} c_j\left(\mathscr{P}^{\perp} M_h v\right) u_j\right\} f\Big(\overrightarrow{u} B + \Big(\overrightarrow{v}, y + A \\
&\quad + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big)\Big)\, d\Big(\overrightarrow{u}, \left(u_{r+1}, \ldots, u_{m+1}\right)\Big)\, d\sigma(v),
\end{aligned} \tag{67}$$

where $\overrightarrow{u} = (u_1, \ldots, u_r)$ and B_1 is given by (22). By a simple calculation we have

$$\begin{aligned}
&\lambda^{m/2} \int_{C[0,T]} H_m(\lambda, x)\, \Psi\left(y + Z_{b,n+1}(x,\cdot)\right. \\
&\quad \left. + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\right) dw_\varphi(x) \\
&\quad = \int_{L_2[0,T]} B_1\Big(y, v, 0, A, \overrightarrow{\xi}_{n+1}\Big) \\
&\quad \cdot \int_{\mathbb{R}^r} \exp\left\{-\frac{1}{2}\left[\left\|\mathscr{P}^{\perp} M_h v\right\|^2\right.\right. \\
&\quad \left. - \sum_{j=1}^{m}\left[c_j\left(\mathscr{P}^{\perp} M_h v\right)\right]^2\right] - \frac{1}{2\lambda} \\
&\quad \left. \cdot \sum_{j=r+1}^{m}\left[c_j\left(\mathscr{P}^{\perp} M_h v\right)\right]^2\right\} f\Big(\overrightarrow{u} B \\
&\quad + \Big(\overrightarrow{v}, y + A + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big)\Big) A_2\left(v, \overrightarrow{u}\right) \\
&\quad \cdot \Psi_r\left(\lambda, \overrightarrow{0}, I_r, \overrightarrow{u}\right) d\overrightarrow{u}\, d\sigma(v)
\end{aligned} \tag{68}$$

which holds for $\lambda \in \mathbb{C}_+$ by an analytic continuation. Now, by the dominated convergence theorem and the Parseval identity, we have (66). The remainder part of the theorem follows from the dominated convergence theorem. □

Using a similar process as used in the proofs of Theorems 8, 22, and 23, we can prove the following theorems by the dominated convergence theorem.

Theorem 23. *Let $1 \le p \le \infty$ and let Ψ be as given in Theorem 4. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\overrightarrow{\xi}_n = (\xi_0, \ldots, \xi_n) \in \mathbb{R}^{n+1}$*

$$\begin{aligned}
&T_\lambda\left[\Psi \mid Z_n\right]\Big(y, \overrightarrow{\xi}_n\Big) = \lim_{m\to\infty} \lambda^{m/2} \\
&\quad \cdot \int_{\mathbb{R}} \int_{C[0,T]} \Psi_1\left(\lambda, a(T) - a(t_n), b(T) - b(t_n), \xi_{n+1}\right. \\
&\quad \left. - \xi_n\right) \times H_m(\lambda, x)\, \Psi\Big(y + Z_{b,n+1}(x,\cdot) \\
&\quad + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big)\, dw_\varphi(x)\, d\xi_{n+1},
\end{aligned} \tag{69}$$

where $\overrightarrow{\xi}_{n+1} = (\xi_0, \ldots, \xi_n, \xi_{n+1})$. Moreover if $p = 1$, then $T_q^{(1)}[\Psi \mid Z_n](y, \overrightarrow{\xi}_n)$ is given by the right-hand side of (69) with replacing λ by λ_m.

Theorem 24. *Let the assumptions be as given in Theorem 11. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and $P_{Z_{n+1}}$ a.e. $\overrightarrow{\xi}_{n+1} \in \mathbb{R}^{n+2}$*

$$\begin{aligned}
&\left[(\Psi * H)_\lambda \mid Z_{n+1}\right]\Big(y, \overrightarrow{\xi}_{n+1}\Big) = \lim_{m\to\infty} \lambda^{m/2} \\
&\quad \cdot \int_{C[0,T]} H_m(\lambda, x)\, \Psi\Big(\frac{1}{\sqrt{2}}\Big[y + Z_{b,n+1}(x,\cdot) \\
&\quad + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big]\Big) \times H\Big(\frac{1}{\sqrt{2}}\Big[y - Z_{b,n+1}(x,\cdot) \\
&\quad - P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big]\Big)\, dw_\varphi(x).
\end{aligned} \tag{70}$$

Moreover if $p_1 = p_2 = 1$, then $[(\Psi * H)_q \mid Z_{n+1}](y, \overrightarrow{\xi}_{n+1})$ is given by the right-hand side of (70) with replacing λ by λ_m.

Theorem 25. *Let the assumptions be as given in Theorem 11. Then we have for $\lambda \in \mathbb{C}_+$, w_φ a.e. $y \in C[0,T]$ and P_{Z_n} a.e. $\overrightarrow{\xi}_n = (\xi_0, \ldots, \xi_n) \in \mathbb{R}^{n+1}$*

$$\begin{aligned}
&\left[(\Psi * H)_\lambda \mid Z_n\right]\Big(y, \overrightarrow{\xi}_n\Big) \\
&\quad = \lim_{m\to\infty} \lambda^{m/2} \int_{\mathbb{R}} \int_{C[0,T]} \Psi_1\left(\lambda, a(T) - a(t_n),\right. \\
&\quad \left. b(T) - b(t_n), \xi_{n+1} - \xi_n\right) \times H_m(\lambda, x)\, \Psi\Big(\frac{1}{\sqrt{2}}\Big[y \\
&\quad + Z_{b,n+1}(x,\cdot) + P_{b,n+1}\Big(\overrightarrow{\xi}_{n+1}\Big)\Big]\Big) H\Big(\frac{1}{\sqrt{2}}\Big[y
\end{aligned}$$

$$- Z_{b,n+1}(x,\cdot)$$
$$- P_{b,n+1}\left(\overrightarrow{\xi}_{n+1}\right)\Big]\Big) dw_{\varphi}(x)\, d\xi_{n+1}, \tag{71}$$

where $\overrightarrow{\xi}_{n+1} = (\xi_0, \ldots, \xi_n, \xi_{n+1})$. *Moreover if* $p_1 = p_2 = 1$, *then* $[(\Psi * H)_q \mid Z_n](y, \overrightarrow{\xi}_n)$ *is given by the right-hand side of (71) with replacing* λ *by* λ_m.

Remark 26.

(1) In the evaluation of $T_\lambda[\Psi \mid Z_{n+1}]$ for $\lambda > 0$, the scale $\lambda^{-1/2}$ in Ψ is moved into H_m on (66). This is the reason why (66) is called a change of scale formula for the transform.

(2) All the results of this paper are independent for a particular choice of the initial distribution φ.

(3) The results of this paper generalize most of theorems in [8, 9, 13, 18].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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