Research Article

Distance to Spaces of Semicontinuous and Continuous Functions

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Received 22 March 2019; Accepted 16 April 2019; Published 8 May 2019

Academic Editor: Miguel Martin

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Given a topological space X , we establish formulas to compute the distance from a function $f \in \mathbb{R}^X$ to the spaces of upper semicontinuous functions and lower semicontinuous functions. For this, we introduce an index of upper semioscillation and lower semioscillation. We also establish new formulas about distances to some subspaces of continuous functions that generalize some classical results.

1. Introduction

Several classical and modern results deal with distances, optimization, equalities, and inequalities. For this, several authors have studied how to compute distances to some spaces of functions, for example, spaces of continuous functions ([1]), spaces of Baire-one functions ([2, 3]), and spaces of measurable functions and integrable functions ([4]). This kind of results has been used in a big number of papers (for instance, [5–15]).

Our aim here is to establish analogous formulas to study distances to spaces of upper semicontinuous functions and lower semicontinuous functions and to study distances to some subspaces of continuous functions that generalize the mentioned result in [1].

Recall that if X is a topological space, a function $f \in \mathbb{R}^X$ is upper (resp., lower) semicontinuous if for all $x \in X$ and $\varepsilon > 0$ there is U , a neighborhood of x , such that $f(z) < f(x) + \varepsilon$ (resp., $f(z) > f(x) - \varepsilon$) for all $z \in U$. Observe that f is upper (resp., lower) semicontinuous if and only if $f^{-1}([t, +\infty))$ (resp., $f^{-1}((-\infty, t])$) is a closed set for all $t \in \mathbb{R}$.

Throughout the paper X is a topological space, $C(X)$ denotes the subspace of $\mathbb{R}^X$ made up of all continuous functions, $usC(X)$ denotes the subspace of upper semicontinuous functions, and $lsC(X)$ denotes the subspace of lower semicontinuous functions. For $f \in \mathbb{R}^X$ and $C \subset \mathbb{R}^X$ we denote

$$d(f, C) = \inf_{g \in C} d(f, g) \quad (1)$$

where

$$d(f, g) = \sup_{x \in X} |f(x) - g(x)| \quad (2)$$

is the supremum metric (that can take the value $+\infty$).

In Section 2 we introduce the indexes of upper and lower semioscillation and use them to study the distances of a function to the spaces of upper semicontinuous functions and lower semicontinuous functions; see Theorem 5.

In Section 3 we establish some relations between distances to spaces of semicontinuous functions and spaces of continuous functions.

It is known that if X is a normal space and $f \in \mathbb{R}^X$, then

$$d(f, C(X)) = \frac{1}{2} \text{osc}(f); \quad (3)$$

see Theorem 8. In Section 4 we study the distances to some subspaces of continuous functions. Theorem 12 shows that if $f \in \mathbb{R}^X$ and $A \subset X$ is a closed subset of the set of points of continuity of f , then we can extend $f|_A$ to a continuous function that is also a best approximation of f in $C(X)$. Theorem 15 studies the distance from f to the continuous functions that have fixed values in some points. Both theorems generalize (3).

2. Upper and Lower Semioscillation

The oscillation of a function $f \in \mathbb{R}^X$ at $x \in X$ is defined as

$$\text{osc}(f, x) = \inf_{U \in \mathcal{U}_x} \sup \text{diam}(f(U)) \quad (4)$$

where $\mathcal{U}_x$ denotes the set of neighborhoods of x . The oscillation of a function is used in [1] to study distances to spaces of continuous functions (see Theorem 8). Inspired by this idea, we define the indexes of upper semioscillation and lower semioscillation.

Definition 1. Let $f \in \mathbb{R}^X$ and $x \in X$. We define the index of upper semioscillation of f in x as

$$\text{uosc}(f, x) = \inf_{U \in \mathcal{U}_x} \sup_{z \in U} (f(z) - f(x)) \quad (5)$$

where $\mathcal{U}_x$ denotes the set of neighborhoods of x , and the index of upper semioscillation of f as

$$\text{uosc}(f) = \sup_{x \in X} \text{uosc}(f, x). \quad (6)$$

Analogously we define the index of lower semioscillation of f in x as

$$\text{losc}(f, x) = \inf_{U \in \mathcal{U}_x} \sup_{z \in U} (f(x) - f(z)) \quad (7)$$

and the index of lower semioscillation of f as

$$\text{losc}(f) = \sup_{x \in X} \text{losc}(f, x). \quad (8)$$

Observe that $\text{uosc}(f, x) < \varepsilon$ (resp., $\text{losc}(f, x) < \varepsilon$) if and only if there exist $0 < \varepsilon' < \varepsilon$ and a neighborhood U of x such that $f(z) < f(x) + \varepsilon'$ (resp., $f(z) > f(x) - \varepsilon'$) for all $z \in U$. In particular f is upper (resp., lower) semicontinuous if and only if $\text{uosc}(f) = 0$ (resp., $\text{losc}(f) = 0$).

It is very clear that the index of oscillation and the indexes of upper semioscillation and lower semioscillation are related.

Proposition 2. If X is a topological space and $f \in \mathbb{R}^X$ is a function, then for all $x \in X$ we have that

$$\text{osc}(f, x) = \text{uosc}(f, x) + \text{losc}(f, x), \quad (9)$$

so

$$\text{osc}(f) \leq \text{uosc}(f) + \text{losc}(f). \quad (10)$$

We introduce the following known functions.

Definition 3. For $f \in \mathbb{R}^X$ we denote

$$\begin{aligned} \overline{f}(x) &= \inf_{U \in \mathcal{U}_x} \sup_{z \in U} f(z), \\ \underline{f}(x) &= \sup_{U \in \mathcal{U}_x} \inf_{z \in U} f(z). \end{aligned} \quad (11)$$

Proposition 4. Let $f \in \mathbb{R}^X$. Then $\overline{f}$ is an upper semicontinuous function, $\underline{f}$ is a lower semicontinuous function, and we have that

$$\begin{aligned} \overline{f}(x) &= f(x) + \text{uosc}(f, x), \\ \underline{f}(x) &= f(x) - \text{losc}(f, x). \end{aligned} \quad (12)$$

Proof. It is known and very easy to check that $\overline{f}$ is upper semicontinuous and $\underline{f}$ is lower semicontinuous. We also have that

$$\begin{aligned} \overline{f}(x) &= \inf_{U \in \mathcal{U}_x} \sup_{z \in U} f(z) \\ &= f(x) + \inf_{U \in \mathcal{U}_x} \sup_{z \in U} (f(z) - f(x)) \\ &= f(x) + \text{uosc}(f, x) \end{aligned} \quad (13)$$

and analogously $\underline{f}(x) = f(x) - \text{losc}(f, x)$. $\square$

The following theorem is the main result of this section.

Theorem 5. Let X be a topological space and $f \in \mathbb{R}^X$ a function. Then

$$\begin{aligned} d(f, \text{usC}(X)) &= \frac{1}{2} \text{uosc}(f), \\ d(f, \text{lsC}(X)) &= \frac{1}{2} \text{losc}(f). \end{aligned} \quad (14)$$

In fact there are $g \in \text{usC}(X)$ and $h \in \text{lsC}(X)$ such that $d(f, g) = d(f, \text{usC}(X))$ and $d(f, h) = d(f, \text{lsC}(X))$.

Proof. We prove the theorem in the $\text{usC}(X)$ case, and the other one can be done analogously or we can deduce it from the $\text{usC}(X)$ case applied to $-f$.

Suppose that $d(f, \text{usC}(X)) < d$. Choose $g \in \text{usC}(X)$ such that $d(f, g) < d$. Fix $x \in X$ and $\varepsilon > 0$. Since $g \in \text{usC}(X)$, there is a neighborhood U of x such that $g(z) < g(x) + \varepsilon$ for all $z \in U$ and then

$$f(z) - f(x) < (g(z) + d) - (g(x) - d) < \varepsilon + 2d. \quad (15)$$

Therefore, $\text{uosc}(f, x) \leq 2d$ for all $x \in X$ and all $d > d(f, \text{usC}(X))$ and then $\text{uosc}(f) \leq 2d(f, \text{usC}(X))$.

We have to prove that if $\text{uosc}(f) = \delta < +\infty$, there is $g \in \text{usC}(X)$ such that $d(f, g) \leq \delta/2$. For this we use the upper semicontinuous function

$$\overline{f}(x) = \inf_{U \in \mathcal{U}_x} \sup_{z \in U} f(z). \quad (16)$$

By Proposition 4 we have that

$$f(x) \leq \overline{f}(x) = f(x) + \text{uosc}(f, x) \leq f(x) + \delta, \quad (17)$$

so the upper semicontinuous function $g = \overline{f} - \delta/2$ satisfies that

$$f(x) - \frac{\delta}{2} \leq g(x) \leq f(x) + \frac{\delta}{2} \quad (18)$$

and the proof is over. $\square$

If a sequence of upper (resp., lower) semicontinuous functions is locally uniformly convergent to a function f , then f is also upper (resp., lower) semicontinuous. Theorem 5 can be used to get a quantitative version of this result.

Proposition 6. *Let X be a topological space and $(f_n)_n$ a sequence in $\mathbb{R}^X$ locally uniformly convergent to a function f . Then*

$$\begin{aligned} d(f, usC(X)) &\leq \liminf_n d(f_n, usC(X)), \\ d(f, lsC(X)) &\leq \liminf_n d(f_n, lsC(X)). \end{aligned} \quad (19)$$

Proof. Fix $x \in X$ and $\varepsilon > 0$ and choose $n \in \mathbb{N}$ and U , a neighborhood of x , such that $|f_m(z) - f(z)| < \varepsilon$ for all $z \in U$ and $m > n$. Fix $m > n$ and choose $V \subset U$, a neighborhood of x , such that

$$f_m(z) - f_m(x) < uosc(f_m, x) + \varepsilon \quad (20)$$

for all $z \in V$. Then

$$\begin{aligned} f(z) - f(x) &= f(z) - f_m(z) + f_m(z) - f_m(x) \\ &\quad + f_m(x) - f(x) < uosc(f_m, x) + 3\varepsilon \end{aligned} \quad (21)$$

for all $z \in V$, so $uosc(f, x) \leq uosc(f_m, x) + 3\varepsilon \leq uosc(f_m) + 3\varepsilon$ and then $uosc(f, x) \leq \liminf_n uosc(f_n) + 3\varepsilon$. Since $x \in X$ and $\varepsilon > 0$ are arbitrary, we get by Theorem 5 that $d(f, usC(X)) \leq \liminf_n d(f_n, usC(X))$. Analogously $d(f, lsC(X)) \leq \liminf_n d(f_n, lsC(X))$. $\square$

Example 7. The inequalities of Proposition 6 are sharp because they become an equality when we consider constant sequences of non-semicontinuous functions (for example, $f_n = f = \chi_A \in \mathbb{R}^{\mathbb{R}}$ where χ_A is the characteristic function of A). However, in general, the equalities do not hold. Consider $f_n = \chi_{\{n\}} \in \mathbb{R}^{\mathbb{R}}$. The sequence $(f_n)_n$ is locally uniformly convergent to the null function $f = 0$ and $d(f_n, lsC(\mathbb{R})) = 1/2$, so

$$d(f, lsC(\mathbb{R})) = 0 < \frac{1}{2} = \liminf_n d(f_n, lsC(\mathbb{R})). \quad (22)$$

3. Relations between Distances to Spaces of Semicontinuous and Continuous Functions

Since $C(X) \subset usC(X) \cap lsC(X)$, we have that

$$\begin{aligned} d(f, usC(X)) &\leq d(f, C(X)), \\ d(f, lsC(X)) &\leq d(f, C(X)) \end{aligned} \quad (23)$$

for all $f \in \mathbb{R}^X$. From Theorem 8 we can obtain that in some cases the distance $d(f, C(X))$ can be bounded using the distances $d(f, usC(X))$ and $d(f, lsC(X))$. For this we also need the following result.

Theorem 8 (see [1, 16]). *Let X be a topological space. Then the following statements are equivalent:*

- (1) X is normal,

(2) for each $f \in \mathbb{R}^X$ there is $g \in C(X)$ such that $d(f, g) = (1/2)osc(f)$,

(3) $d(f, C(X)) = (1/2)osc(f)$ for each $f \in \mathbb{R}^X$.

The version of Theorem 8 that appears in [1] is less general. They prove that if X is a paracompact space, then the formula $d(f, C(X)) = (1/2)osc(f)$ holds for all bounded functions $f \in \mathbb{R}^X$ and also says that this result holds for normal spaces.

Corollary 9. *Let X be a normal topological space and $f \in \mathbb{R}^X$ a function. Then*

$$d(f, C(X)) \leq d(f, usC(X)) + d(f, lsC(X)). \quad (24)$$

If f is an upper semicontinuous function, then

$$d(f, C(X)) = d(f, lsC(X)). \quad (25)$$

If f is a lower semicontinuous function, then

$$d(f, C(X)) = d(f, usC(X)). \quad (26)$$

Proof. By Proposition 2 $osc(f) \leq uosc(f) + losc(f)$ and then by Theorems 8 and 5, we have that

$$\begin{aligned} d(f, C(X)) &= \frac{1}{2}osc(f, x) \leq \frac{1}{2}(uosc(f) + losc(f)) \\ &= d(f, usC(X)) + d(f, lsC(X)). \end{aligned} \quad (27)$$

Observe now that if f is upper (resp., lower) semicontinuous, then $osc(f) = losc(f)$ (resp., $osc(f) = uosc(f)$), so equalities (25) and (26) also follow from Theorems 8 and 5. $\square$

Remark 10. Considering equality (26), one can think that if f is lower (resp., upper) semicontinuous, then the best approximation of f by upper (resp., lower) semicontinuous functions that appear in the proof of Theorem 5 is continuous. However, it is easy to check that it is not true. If we consider $X = \mathbb{R}$ and $f = \chi_{(0, +\infty)}$, i.e., f is the function defined by

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0, \\ 1 & \text{if } x > 0, \end{cases} \quad (28)$$

then f is lower semicontinuous but

$$\overline{f}(x) = \inf_{U \in \mathcal{U}_x} \sup_{z \in U} f(z) = \chi_{[0, +\infty)} = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x \geq 0 \end{cases} \quad (29)$$

is not continuous.

4. Distances to Subspaces of Continuous Functions

In this section we study generalizations of Theorem 8. We prove that we can obtain a best approximation g of f in $C(X)$ that preserves the value of f in some sets of continuity points and we also can force g to have fixed values at some points. First of all, we need the following known lemma that can be found in [17, Theorem 12.16].

Lemma 11. Let X be a topological normal space. Then if $g_1 \in \mathbb{R}^X$ is an upper semicontinuous function and $g_2 \in \mathbb{R}^X$ is a lower semicontinuous function such that $g_1 \leq g_2$, there exists a continuous function $g \in C(X)$ such that $g_1 \leq g \leq g_2$.

Theorem 12. Let X be a normal space, $f \in \mathbb{R}^X$, and A be a closed subset of X such that f is continuous at x for all $x \in A$. Then there is $g \in C(X)$ a continuous function such that $f|_A = g|_A$ and

$$d(f, g) = \frac{1}{2} \text{osc}(f) = d(f, C(X)). \quad (30)$$

Proof. Suppose that $\text{osc}(f) = \delta < \infty$ and define $h_1(x) = \overline{f}(x) - \delta/2$ and $h_2(x) = \underline{f}(x) + \delta/2$. By Proposition 4, h_1 is upper semicontinuous and h_2 is lower semicontinuous and $\overline{f}(x) - \underline{f}(x) = \text{uosc}(f, x) + \text{losc}(f, x) = \text{osc}(f, x) \leq \text{osc}(f) = \delta$, so $h_1 \leq h_2$. Define

$$g_i(x) = \begin{cases} f(x) & \text{if } x \in A, \\ h_i(x) & \text{if } x \notin A, \end{cases} \quad (31)$$

for $i = 0, 1$. Clearly $g_1(x) \leq g_2(x)$. Observe that if f is continuous in x , then $\overline{f}(x) = \underline{f}(x) = f(x)$, so

$$h_1(x) < g_1(x) = f(x) = g_2(x) < h_2(x) \quad (32)$$

for $x \in A$.

Let us prove that g_1 is upper semicontinuous. If $x \notin A$, there is a neighborhood U of x such that $g_1|_U = h_1|_U$ and then $\text{uosc}(g_1, x) = \text{uosc}(h_1, x) = 0$. If $x \in A$, there is a neighborhood U of x such that $\text{diam}(f(U)) < \varepsilon$ (f is continuous at x) and $h_1(z) < h_1(x) + \varepsilon$ for all $z \in U$. If $z \in U \cap A$, then $g_1(z) = f(z) < f(x) + \varepsilon = g_1(x) + \varepsilon$, and if $z \in U \cap A^c$, then $g_1(z) = h_1(z) < h_1(x) + \varepsilon < g_1(x) + \varepsilon$, so $\text{uosc}(g_1) = 0$. Then we have that g_1 is upper semicontinuous. Analogously g_2 is lower semicontinuous, and since $g_1 \leq g_2$, by Lemma 11 we have that there exists a continuous function $g \in C(X)$ such that

$$g_1 \leq g \leq g_2. \quad (33)$$

Observe that if $x \in A$, since $g_1(x) = f(x) = g_2(x)$, then $g(x) = f(x)$, so

$$f|_A = g|_A. \quad (34)$$

If $x \notin A$, then $g_1(x) = h_1(x) = \overline{f}(x) - \delta/2 = f(x) + \text{uosc}(f, x) - (1/2)\text{osc}(f)$, so

$$\begin{aligned} |g_1(x) - f(x)| &= \left| \text{uosc}(f, x) - \frac{1}{2} \text{osc}(f) \right| \\ &\leq \frac{1}{2} \text{osc}(f). \end{aligned} \quad (35)$$

Analogously $|g_2(x) - f(x)| \leq (1/2)\text{osc}(f)$, so

$$\begin{aligned} f(x) - \frac{1}{2} \text{osc}(f) &\leq g_1(x) \leq g(x) \leq g_2(x) \\ &\leq f(x) + \frac{1}{2} \text{osc}(f) \end{aligned} \quad (36)$$

and then

$$d(f, g) \leq \frac{1}{2} \text{osc}(f). \quad (37)$$

To finish, we have to prove that $d(f, h) \geq (1/2)\text{osc}(f)$ for all $h \in C(X)$. This is true by Theorem 8, but we include the proof here to get a self-contained proof. Suppose that h is a continuous function and define $d = d(f, h)$. For $x \in X$, choose U , a neighborhood of x , such that $\text{diam}(h(U)) < \varepsilon$. Then

$$\begin{aligned} |f(z) - f(x)| &\leq |f(z) - h(z)| + |h(z) - h(x)| \\ &\quad + |h(x) - f(x)| < 2d + \varepsilon \end{aligned} \quad (38)$$

for all $z \in U$ so $\text{osc}(f) \leq 2d$ and then

$$d(f, h) \geq \frac{1}{2} \text{osc}(f) \quad (39)$$

for all $h \in C(X)$. $\square$

Remark 13. Observe that the final part of the proof of Theorem 12 does not need the normal hypothesis, so if X is a topological space and $f \in \mathbb{R}^X$, then

$$d(f, C(X)) \geq \frac{1}{2} \text{osc}(f). \quad (40)$$

Example 14. The set A in Theorem 12 needs to be closed. Indeed consider the function $f = \chi_A \in \mathbb{R}^{\mathbb{R}}$ where $A = (0, +\infty)$. If $g \in C(\mathbb{R})$ is a continuous function such that $f|_A = g|_A$, then $g(0) = 1$, so

$$d(f, g) \geq |f(0) - g(0)| = 1 > \frac{1}{2} \text{osc}(f) = \frac{1}{2}. \quad (41)$$

If we change the hypothesis $A \subset \{x \in X : f \text{ is continuous at } x\}$ by $f|_A$ continuous, the theorem also fails. Consider now $f = \chi_{\{0\}} \in \mathbb{R}^{\mathbb{R}}$ and $A = \{0\}$. If $g \in C(\mathbb{R})$ is a continuous function such that $g(0) = f(0) = 1$, then clearly $d(f, g) \geq 1$ and $\text{osc}(f) = 1$, so $d(f, g) > (1/2)\text{osc}(f)$.

Keeping the last example in mind, we can ask how far is $f \in \mathbb{R}^X$ from the set of continuous functions that takes a fixed value $z \in \mathbb{R}$ in a point $x \in X$. The following theorem says that the distance that we obtain depends on $M = \max\{|z - \overline{f}(x)|, |z - \underline{f}(x)|\}$. Since $\overline{f}(x) = f(x) + \text{uosc}(f, x)$, $\underline{f}(x) = f(x) - \text{losc}(f, x)$, and $\underline{f} \leq \overline{f}$, we get that we can write M as

$$\begin{aligned} M &= \max\{z - f(x) + \text{losc}(f, x), f(x) + \text{uosc}(f, x) \\ &\quad - z\}. \end{aligned} \quad (42)$$

Theorem 15. Let X be a normal space, $f \in \mathbb{R}^X$, and $A \subset X$ be a discrete closed subset. For each $x \in A$, fix $z_x \in \mathbb{R}^X$. Then there is $g \in C(X)$ such that $g(x) = z_x$ for all $x \in A$ and

$$d(f, g) = \max\left\{\frac{1}{2} \text{osc}(f), M\right\} \quad (43)$$

where

$$\begin{aligned} M &= \sup_{x \in A} \max \{z_x - f(x) + \text{losc}(f, x), f(x) \\ &\quad + \text{uosc}(f, x) - z_x\} \\ &= \sup_{x \in A} \max \{|z_x - \overline{f}(x)|, |z_x - \underline{f}(x)|\}. \end{aligned} \quad (44)$$

Proof. Take $\delta = \max\{(1/2)\text{osc}(f), M\}$ and define $h_1(x) = \overline{f}(x) - \delta$ and $h_2(x) = \underline{f}(x) + \delta$. By Proposition 4 h_1 is upper semicontinuous, h_2 is lower semicontinuous, and $\overline{f}(x) - \underline{f}(x) = \text{uosc}(f, x) + \text{losc}(f, x) = \text{osc}(f, x) \leq \text{osc}(f) \leq 2\delta$, so $h_1 \leq h_2$. We have that

$$h_1(x) \leq z_x \leq h_2(x) \quad \text{for } x \in A. \quad (45)$$

Indeed, if $z_x \in A$, then $\overline{f}(x) - z_x = f(x) + \text{uosc}(f, x) - z_x \leq \delta$, so $h_1(x) = \overline{f}(x) - \delta \leq z_x$ and analogously $z_x \leq h_2(x)$. Define

$$g_i(x) = \begin{cases} z_x & \text{if } x \in A, \\ h_i(x) & \text{if } x \notin A, \end{cases} \quad (46)$$

for $i = 1, 2$. Let us see that g_1 is upper semicontinuous. If $x \notin A$, then there is U , a neighborhood of x , such that $g_1|_U = h_1|_U$, so $\text{uosc}(g_1, x) = \text{uosc}(h_1, x) = 0$. If $x \in A$, fix $\varepsilon > 0$ and pick U , a neighborhood of x , such that $A \cap U = \{x\}$ and $h_1(z) < h_1(x) + \varepsilon$ for all $z \in U$. Then

$$g_1(z) = h_1(z) < h_1(x) + \varepsilon \leq z_x + \varepsilon = g_1(x) + \varepsilon \quad (47)$$

for all $z \in U \setminus \{x\}$, so g_1 is upper semicontinuous. Analogously g_2 is lower semicontinuous, and since $g_1 \leq g_2$, by Lemma 11 there is a continuous function $g \in C(X)$ such that

$$g_1 \leq g \leq g_2. \quad (48)$$

Observe that if $x \in A$, then $z_x = g_1(x) \leq g(x) \leq g_2(x) = z_x$, so $g(x) = z_x$ for all $x \in A$. Let us see that $|f(x) - g_1(x)| \leq \delta$.

(i) Suppose that $x \in A$. Then

$$\begin{aligned} |f(x) - g_1(x)| &= |f(x) - z_x| \leq \max \{z_x - f(x) \\ &\quad + \text{losc}(f, x), f(x) + \text{uosc}(f, x) - z_x\} \leq \delta. \end{aligned} \quad (49)$$

(ii) Suppose that $x \notin A$ and $f(x) \leq g_1(x)$. Then

$$\begin{aligned} g_1(x) - f(x) &= \overline{f}(x) - \delta - f(x) \\ &= f(x) + \text{uosc}(f, x) - \delta - f(x) \\ &= \text{uosc}(f, x) - \delta \\ &\leq \text{uosc}(f, x) - \frac{1}{2}\text{osc}(f) \leq \frac{1}{2}\text{osc}(f) \\ &\leq \delta. \end{aligned} \quad (50)$$

(iii) Suppose that $x \notin A$ and $f(x) \geq g_1(x)$. Then

$$\begin{aligned} f(x) - g_1(x) &\leq \overline{f}(x) - g_1(x) = \overline{f}(x) - (\overline{f}(x) - \delta) \\ &= \delta. \end{aligned} \quad (51)$$

Thus, $|f(x) - g_1(x)| \leq \delta$ and analogously $|f(x) - g_2(x)| \leq \delta$. We have that

$$f(x) - \delta \leq g_1(x) \leq g(x) \leq g_2(x) \leq f(x) + \delta \quad (52)$$

and then

$$d(f, g) \leq \delta. \quad (53)$$

We have to prove that $d(f, g) \geq \delta$. Since $d(f, C(X)) = (1/2)\text{osc}(f)$, then $d(f, g) \geq (1/2)\text{osc}(f)$. Accordingly, the following claim finish the proof.

Claim. If h is a continuous function such that $h(x) = z_x$ for some fixed $x \in A$, then

$$\begin{aligned} d(f, h) &\geq L := \max \{z_x - f(x) + \text{losc}(f, x), f(x) \\ &\quad + \text{uosc}(f, x) - z_x\}. \end{aligned} \quad (54)$$

Suppose without loss of generality that $L = z_x - f(x) + \text{losc}(f, x)$. Fix $\varepsilon > 0$; then there is U , a neighborhood of x , such that $|z_x - h(y)| < \varepsilon$ for all $y \in U$. Choose $y \in U \setminus A$ such that $f(y) < f(x) - \text{losc}(f, x) + \varepsilon$. Then

$$\begin{aligned} h(y) - f(y) &\geq z_x - \varepsilon - (f(x) - \text{losc}(f, x) + \varepsilon) \\ &= z_x - f(x) + \text{losc}(f, x) - 2\varepsilon \\ &= L - 2\varepsilon. \end{aligned} \quad (55)$$

Since $\varepsilon > 0$ is arbitrary, we get that $d(f, h) \geq L$ and the proof of the claim is over. $\square$

Corollary 16. Let X be a normal space, $f \in \mathbb{R}^X$, and $A \subset X$ be a discrete closed subset. For each $x \in A$ fix $z_x \in \mathbb{R}^X$ and consider the affine subspace

$$Z = \{h \in C(X) : h(x) = z_x \text{ for all } x \in A\}. \quad (56)$$

Then

$$d(f, Z) = \max \left\{ \frac{1}{2}\text{osc}(f), M \right\} \quad (57)$$

where

$$\begin{aligned} M &= \sup_{x \in A} \max \{z_x - f(x) + \text{losc}(f, x), f(x) \\ &\quad + \text{uosc}(f, x) - z_x\} \\ &= \sup_{x \in A} \max \{|z_x - \overline{f}(x)|, |z_x - \underline{f}(x)|\}. \end{aligned} \quad (58)$$

Proof. By Theorem 15 we have that $d(f, Z) \leq \max\{(1/2)\text{osc}(f), M\}$. The other inequality is true by the claim of the proof of Theorem 15. $\square$

Corollary 17. Let X be a normal space, $f \in \mathbb{R}^X$, and $A \subset X$ be a discrete closed subset. For each $x \in A$ fix

$$z_x \in \left[\overline{f}(x) - \frac{1}{2} \text{osc}(f), \underline{f}(x) + \frac{1}{2} \text{osc}(f) \right]. \quad (59)$$

Then there is $g \in C(X)$ such that $g(x) = z_x$ for all $x \in A$ and

$$d(f, g) = \frac{1}{2} \text{osc}(f). \quad (60)$$

Proof. If $x \in A$ and $z_x \in [\overline{f}(x) - (1/2)\text{osc}(f), \underline{f}(x) + (1/2)\text{osc}(f)]$, we have that

$$\max \left\{ |z_x - \overline{f}(x)|, |z_x - \underline{f}(x)| \right\} \leq \frac{1}{2} \text{osc}(f) \quad (61)$$

so by Theorem 15 there is a function $g \in C(X)$ such that $g(x) = z_x$ and

$$d(f, g) = \max \left\{ \frac{1}{2} \text{osc}(f), M \right\} = \frac{1}{2} \text{osc}(f). \quad (62)$$

□

The proofs of Theorems 12 and 15 are very similar. In fact we can combine both theorems.

Theorem 18. Let X be a normal space, $f \in \mathbb{R}^X$, $A \subset X$ be a discrete closed subset, and B be a closed subset of X such that f is continuous at x for all $x \in B$. For each $x \in A$ fix $z_x \in \mathbb{R}^X$. Then there is $g \in C(X)$ such that $g(x) = z_x$ for all $x \in A$, $g(x) = f(x)$ for all $x \in B \setminus A$, and

$$d(f, g) = \max \left\{ \frac{1}{2} \text{osc}(f), M \right\} \quad (63)$$

where

$$\begin{aligned} M &= \sup_{x \in A} \max \{ z_x - f(x) + \text{losc}(f, x), f(x) \\ &\quad + \text{uosc}(f, x) - z_x \} \\ &= \sup_{x \in A} \max \left\{ |z_x - \overline{f}(x)|, |z_x - \underline{f}(x)| \right\}. \end{aligned} \quad (64)$$

We omit the proof because it is very similar to the proofs of Theorems 12 and 15. If we read the proof of Theorem 15, we have to change the definition of g_i by

$$g_i(x) = \begin{cases} z_x & \text{if } x \in A, \\ f(x) & \text{if } x \in B \setminus A \\ h_i(x) & \text{if } x \notin A \cup B, \end{cases} \quad (65)$$

and then we get by Lemma 11 $g \in C(X)$ that $g_1 \leq g \leq g_2$, and then we have to check that g is the desired function.

Data Availability

All our findings are theoretical ones and are included within the article.

Conflicts of Interest

The author declares no conflicts of interest.

Acknowledgments

The author is supported by the Spanish grants MTM2017-83262-C2-2-P of the Spanish Ministry of Economy and 20906/PI/18 of Fundación Séneca.

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