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Research Article

Some Properties of Canonical Dual K-Bessel Sequences for Parseval K-Frames

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The concept of canonical dual K-Bessel sequences was recently introduced, a deep study of which is helpful in further developing and enriching the duality theory of K-frames. In this paper we pay attention to investigating the structure of the canonical dual K-Bessel sequence of a Parseval K-frame and some derived properties. We present the exact form of the canonical dual K-Bessel sequence of a Parseval K-frame, and a necessary and sufficient condition for a dual K-Bessel sequence of a given Parseval K-frame to be the canonical dual K-Bessel sequence is investigated. We also give a necessary and sufficient condition for a Parseval K-frame to have a unique dual K-Bessel sequence and equivalently characterize the condition under which the canonical dual K-Bessel sequence of a Parseval K-frame admits a unique dual K^*-Bessel sequence. Finally, we obtain a minimal norm property on expansion coefficients of elements in the range of K resulting from the canonical dual K-Bessel sequence of a Parseval K-frame.

1. Introduction

Throughout this paper, $\mathscr{H}$ and $\mathscr{K}$ are separable Hilbert spaces; $\mathbb{J}$ is a finite or countable index set. We denote by $B(\mathscr{H},\mathscr{K})$ the collection of all linear bounded operators from $\mathscr{H}$ to $\mathscr{K}$, and $B(\mathscr{H},\mathscr{H})$ is abbreviated as $B(\mathscr{H})$.

A sequence $\{f_j\}_{j\in\mathbb{J}}$ of elements in $\mathscr{H}$ is a *frame* if there exist constants $0 < A \le B < \infty$ such that

$$A\|f\|^2 \le \sum_{j\in\mathbb{J}} \left|\left\langle f, f_j\right\rangle\right|^2 \le B\|f\|^2, \quad \forall f \in \mathscr{H}. \tag{1}$$

The frame $\{f_j\}_{j\in\mathbb{J}}$ is a *Parseval frame* if $A = B = 1$. If only the right-hand inequality holds, then $\{f_j\}_{j\in\mathbb{J}}$ is called a *Bessel sequence* with Bessel bound B.

Associated with every Bessel sequence $\{x_j\}_{j\in\mathbb{J}}$ of $\mathscr{H}$ there is a linear bounded operator, called the *analysis operator* of $\{x_j\}_{j\in\mathbb{J}}$, defined by

$$\begin{gathered} U_X : \mathscr{H} \longrightarrow \ell^2(\mathbb{J}), \\ U_X(f) = \left\{\left\langle f, x_j\right\rangle\right\}_{j\in\mathbb{J}}. \end{gathered} \tag{2}$$

It is easy to check that the adjoint of U_X, U_X^*, is given by

$$\begin{gathered} U_X^* : \ell^2(\mathbb{J}) \longrightarrow \mathscr{H}, \\ U_X^*\left(\left\{c_j\right\}_{j\in\mathbb{J}}\right) = \sum_{j\in\mathbb{J}} c_j x_j. \end{gathered} \tag{3}$$

By composing U_X^* and U_X, we obtain the *frame operator* $S_X : \mathscr{H} \longrightarrow \mathscr{H}$:

$$S_X f = U_X^* U_X f = \sum_{j\in\mathbb{J}} \left\langle f, x_j\right\rangle x_j. \tag{4}$$

Note that S_X is a positive, self-adjoint operator, and it is invertible if and only if $\{x_j\}_{j\in\mathbb{J}}$ is a frame. Recall that a Bessel sequence $\{y_j\}_{j\in\mathbb{J}}$ in $\mathscr{H}$ is a *dual frame* of $\{x_j\}_{j\in\mathbb{J}}$ if

$$f = \sum_{j\in\mathbb{J}} \left\langle f, y_j\right\rangle x_j, \quad \forall f \in \mathscr{H}. \tag{5}$$

It is well-known that $\{S_X^{-1}x_j\}_{j\in\mathbb{J}}$ is a dual frame of $\{x_j\}_{j\in\mathbb{J}}$, which is called the *canonical dual frame*.

Frames were formally defined by Duffin and Schaeffer [1] in the early 1950s, when they were used to study some deep problems on nonharmonic Fourier series. Owing to the redundancy and flexibility, today they have served as an important tool in various fields; see [2–10] for more information on frame theory and its applications. Atomic systems for subspaces were first introduced by Feichtinger and Werther in [11] based on examples arising in sampling theory. When working on atomic systems for operators, Găvruţa [12] put forward the concept of K-frames for a given linear bounded operator K, which allows atomic decomposition of elements from the range of K and, in general, the range may not be closed. Moreover, it has been shown in [13–16] that in many ways K-frames behave completely differently from frames, although a K-frame is a generalization of a frame; see also [17, 18].

The classical canonical dual for a K-frame is absent since the frame operator may not be invertible, which has greatly contributed to the fact that there are few results on the duals of a K-frame. Recently, Guo in [15] proposed the concept of canonical dual K-Bessel sequences from the operator-theoretic point of view, a deep study of which is helpful in further developing and enriching the duality theory of K-frames. This paper is devoted to examining the structure of the canonical dual K-Bessel sequence of a Parseval K-frame and some derived properties. We present the exact form of the canonical dual K-Bessel sequence of a Parseval K-frame by means of the pseudo-inverse of K and a necessary and sufficient condition for a dual K-Bessel sequence of a given Parseval K-frame to be the canonical dual K-Bessel sequence. We also give a necessary and sufficient condition for a Parseval K-frame to have a unique dual K-Bessel sequence and equivalently characterize the condition for the canonical dual K-Bessel sequence to admit a unique dual K^*-Bessel sequence. We end the paper by showing that the canonical dual K-Bessel sequence of a Parseval K-frame gives rise to expansion coefficients of elements in the range of K with minimal norm.

We need to collect some definitions and basic properties for operators.

Definition 1. Suppose $K \in B(\mathscr{H})$. A sequence $\{f_j\}_{j\in\mathbb{J}}$ in $\mathscr{H}$ is said to be a K-frame, if there exist $0 < C \leq D < \infty$ such that

$$C\left\|K^* f\right\|^2 \leq \sum_{j\in\mathbb{J}}\left|\left\langle f, f_j\right\rangle\right|^2 \leq D\left\|f\right\|^2, \quad \forall f \in \mathscr{H}. \tag{6}$$

The constants C and D are called the lower and upper K-frame bounds.

Suppose $K \in B(\mathscr{H})$. A K-frame $\{f_j\}_{j\in\mathbb{J}}$ of $\mathscr{H}$ is said to be Parseval, if

$$\left\|K^* f\right\|^2 = \sum_{j\in\mathbb{J}}\left|\left\langle f, f_j\right\rangle\right|^2, \quad \forall f \in \mathscr{H}. \tag{7}$$

Definition 2. Let $\{f_j\}_{j\in\mathbb{J}}$ be a sequence in $\mathscr{H}$. For $\{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$, if $\sum_{j\in\mathbb{J}} c_j f_j = 0$ implies $c_j = 0$ for any $j \in \mathbb{J}$, then we say that $\{f_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-*linearly independent*.

The following results from operator theory will be used to prove our main results.

Lemma 3 (see [19]). *Suppose that $\Lambda \in B(\mathscr{H}, \mathscr{K})$ has closed range, then there exists a unique operator $\Lambda^\dagger \in B(\mathscr{K}, \mathscr{H})$, called the pseudo-inverse of Λ, satisfying*

$$\begin{aligned} \Lambda\Lambda^\dagger\Lambda &= \Lambda, \\ \Lambda^\dagger\Lambda\Lambda^\dagger &= \Lambda^\dagger, \\ \left(\Lambda\Lambda^\dagger\right)^* &= \Lambda\Lambda^\dagger, \\ \left(\Lambda^\dagger\Lambda\right)^* &= \Lambda^\dagger\Lambda, \\ \operatorname{Ker}\Lambda^\dagger &= \left(\mathit{Range}\left(\Lambda\right)\right)^\perp, \\ \mathit{Range}\left(\Lambda^\dagger\right) &= \left(\operatorname{Ker}\Lambda\right)^\perp. \end{aligned} \tag{8}$$

In the sequel, the notation $\Lambda^\dagger$ is reserved for the pseudo-inverse of Λ (if it exists).

Lemma 4 (see [20]). *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two Hilbert spaces. Also let $S \in B(\mathscr{H}_1, \mathscr{H})$ and $T \in B(\mathscr{H}_2, \mathscr{H})$. The following statements are equivalent.*

(1) *$Range(S) \subset Range(T)$*

(2) *There exists $\lambda > 0$ such that $SS^* \leq \lambda TT^*$*

(3) *There exists $\theta \in B(\mathscr{H}_1, \mathscr{H}_2)$ such that $S = T\theta$*

Moreover, if (1), (2), and (3) are valid, then there exists a unique operator θ such that

(a) $\|\theta\|^2 = \inf\{\mu : SS^* \leq \mu TT^*\}$

(b) $\operatorname{Ker} S = \operatorname{Ker}\theta$

(c) $Range(\theta) \subset \overline{Range}(T^*)$

Lemma 5. *Suppose that $K \in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathscr{H}$ with analysis operator U_F. Then $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame of $\mathscr{H}$ if and only if $Range(K) \subset Range(U_F^*)$.*

Proof. It is an immediate consequence of Lemma 4; we omit the details. □

2. Main Results

Suppose $K \in B(\mathscr{H})$ and $\{f_j\}_{j\in\mathbb{J}}$ is a K-frame of $\mathscr{H}$. From Theorem 3 in [12] we know that there always exists a Bessel sequence $\{g_j\}_{j\in\mathbb{J}}$ of $\mathscr{H}$ such that

$$Kf = \sum_{j\in\mathbb{J}}\left\langle f, g_j\right\rangle f_j, \quad \forall f \in \mathscr{H}, \tag{9}$$

which is called a *dual K-Bessel sequence* of $\{f_j\}_{j\in\mathbb{J}}$ (see Definition 2.5 in [15]). A direct calculation can show that a dual K-Bessel sequence is necessarily a K^*-frame.

In general, a K-frame may admit more than one dual K-Bessel sequence, as shown in the following example.

Example 6. Suppose that $\mathscr{H} = \mathbb{C}^3$, $\{g_j\}_{j=1}^3 = \{e_1, e_2, e_3\}$, where

$$e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}. \tag{10}$$

Define $K \in B(\mathscr{H})$ as follows:

$$\begin{aligned} K &: \mathscr{H} \longrightarrow \mathscr{H}, \\ Ke_1 &= e_1, \\ Ke_2 &= e_1, \\ Ke_3 &= e_2. \end{aligned} \tag{11}$$

Taking $f_j = Kg_j$ for $j = 1, 2, 3$, then

$$\begin{aligned} K^* f &= \sum_{j=1}^{3} \langle K^* f, e_j \rangle e_j = \sum_{j=1}^{3} \langle f, Ke_j \rangle e_j \\ &= \sum_{j=1}^{3} \langle f, Kg_j \rangle e_j = \sum_{j=1}^{3} \langle f, f_j \rangle e_j, \end{aligned} \tag{12}$$

for every $f \in \mathscr{H}$. Therefore $\|K^* f\|^2 = \sum_{j=1}^{3} |\langle f, f_j \rangle|^2$, showing that $\{f_j\}_{j=1}^3$ is a Parseval K-frame of $\mathscr{H}$. Since

$$\begin{aligned} Kf &= K\left(\sum_{j=1}^{3} \langle f, e_j \rangle e_j \right) = \sum_{j=1}^{3} \langle f, e_j \rangle Ke_j \\ &= \sum_{j=1}^{3} \langle f, g_j \rangle f_j, \end{aligned} \tag{13}$$

it follows that $\{g_j\}_{j=1}^3$ is a dual K-Bessel sequence of $\{f_j\}_{j=1}^3$. Let

$$h_1 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix}, \quad h_2 = \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix}, \quad h_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \tag{14}$$

and then it is easy to check that $\{h_j\}_{j=1}^3$ is Bessel sequence of $\mathscr{H}$. Now for any $f \in \mathscr{H}$ we have

$$\begin{aligned} \sum_{j=1}^{3} \langle f, h_j \rangle f_j &= \langle f, e_1 + 2e_2 \rangle e_1 + \langle f, -e_2 \rangle e_1 \\ &\quad + \langle f, e_3 \rangle e_2 \\ &= \langle f, e_1 + e_2 \rangle f_1 + \langle f, e_3 \rangle f_3 \\ &= \sum_{j=1}^{3} \langle f, g_j \rangle f_j = Kf. \end{aligned} \tag{15}$$

Hence $\{h_j\}_{j=1}^3$ is a dual K-Bessel sequence of $\{f_j\}_{j=1}^3$ and is different from $\{g_j\}_{j=1}^3$.

Guo in [15] proved that, among all dual K-Bessel sequences of a given K-frame, there is a unique dual K-Bessel sequence whose analysis operator obtains the minimal norm, which is called the *canonical dual K-Bessel sequence*. Motivated by the idea of [21], in the following we characterize the exact structure of the canonical dual K-Bessel sequence of a Parseval K-frame under the condition that K has closed range. We need the following two lemmas first. Since their proofs are similar to Theorems 2.7 and 2.8 in [21], respectively, we omit the details.

Lemma 7. *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$, then $\{K^\dagger f_j\}_{j\in\mathbb{J}}$ is a dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$.*

Lemma 8. *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$ with analysis operator U_F, then $\{g_j\}_{j\in\mathbb{J}}$ in $\mathscr{H}$ is a dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$ if and only if there exists $\varphi \in B(\mathscr{H}, \ell^2(\mathbb{J}))$ such that $U_F^*\varphi = 0$ and $\langle f, g_j - K^\dagger f_j \rangle = (\varphi f)_j$ for every $f \in \mathscr{H}$ and every $j \in \mathbb{J}$.*

By using Lemmas 7 and 8 we can obtain the following result which shows that the dual K-Bessel sequence $\{K^\dagger f_j\}_{j\in\mathbb{J}}$ of Parseval K-frame $\{f_j\}_{j\in\mathbb{J}}$ stated in Lemma 7 is exactly the canonical dual K-Bessel sequence. For details of the proof, the reader can check the proof for Theorem 2.10 in [21], step by step.

Theorem 9. *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$ with analysis operator U_F, then $\{K^\dagger f_j\}_{j\in\mathbb{J}}$ is the canonical dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$.*

Remark 10. (1) The canonical dual K-Bessel sequence of the Parseval K-frame $\{f_j\}_{j\in\mathbb{J}}$, which will be denoted by $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ later, is actually a Parseval frame on $(\operatorname{Ker} K)^\perp$ since

$$\begin{aligned} \sum_{j\in\mathbb{J}} \left|\langle f, \widetilde{f}_j \rangle\right|^2 &= \left\|K^* (K^\dagger)^* f\right\|^2 = \left\|(K^\dagger K)^* f\right\|^2 \\ &= \left\|K^\dagger K f\right\|^2 = \|f\|^2 \end{aligned} \tag{16}$$

for every $f \in (\operatorname{Ker} K)^\perp$, by Lemma 3.

(2) The canonical dual K-Bessel sequence of the Parseval K-frame $\{f_j\}_{j\in\mathbb{J}}$ is precisely a Parseval $K^{\dagger}K$-frame since

$$\begin{aligned}\sum_{j\in\mathbb{J}}\left|\left\langle f,\widetilde{f}_j\right\rangle\right|^2&=\sum_{j\in\mathbb{J}}\left|\left\langle f,K^{\dagger}f_j\right\rangle\right|^2=\left\|K^*\left(K^{\dagger}\right)^*f\right\|^2\\&=\left\|\left(K^{\dagger}K\right)^*f\right\|^2,\quad\forall f\in\mathscr{H}.\end{aligned}\tag{17}$$

(3) Although the canonical dual K-Bessel sequence of the Parseval K-frame $\{f_j\}_{j\in\mathbb{J}}$ is not a Parseval K-frame in general, it can naturally generate a new one in the form $\{K\widetilde{f}_j\}_{j\in\mathbb{J}}$. Indeed, by Lemma 3 we have

$$\begin{aligned}\sum_{j\in\mathbb{J}}\left|\left\langle f,K\widetilde{f}_j\right\rangle\right|^2&=\sum_{j\in\mathbb{J}}\left|\left\langle f,KK^{\dagger}f_j\right\rangle\right|^2\\&=\sum_{j\in\mathbb{J}}\left|\left\langle\left(KK^{\dagger}\right)^*f,f_j\right\rangle\right|^2\\&=\left\|K^*\left(KK^{\dagger}\right)^*f\right\|^2=\left\|\left(KK^{\dagger}K\right)^*f\right\|^2\\&=\left\|K^*f\right\|^2,\quad\forall f\in\mathscr{H}.\end{aligned}\tag{18}$$

We give a necessary and sufficient condition for a dual K-Bessel sequence of a Parseval K-frame to be the canonical dual K-Bessel sequence.

Theorem 11. *Suppose that $K\in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$ with a dual K-Bessel sequence $\{g_j\}_{j\in\mathbb{J}}$. Then $\{g_j\}_{j\in\mathbb{J}}$ is the canonical dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$ if and only if $U_G^*U_G=U_G^*U_H$ for any dual K-Bessel sequence $\{h_j\}_{j\in\mathbb{J}}$ of $\{f_j\}_{j\in\mathbb{J}}$, where U_G and U_H denote the analysis operators of $\{g_j\}_{j\in\mathbb{J}}$ and $\{h_j\}_{j\in\mathbb{J}}$, respectively.*

Proof. Let us first assume that $\{g_j\}_{j\in\mathbb{J}}=\{\widetilde{f}_j\}_{j\in\mathbb{J}}$. If we denote by U_F the analysis operator of $\{f_j\}_{j\in\mathbb{J}}$, then a direct calculation can show that $U_G=U_F(K^{\dagger})^*$. From this fact and taking into account the fact that

$$\begin{aligned}U_F^*\left(U_Gf-U_Hf\right)&=\sum_{j\in\mathbb{J}}\left\langle f,\widetilde{f}_j\right\rangle f_j-\sum_{j\in\mathbb{J}}\left\langle f,h_j\right\rangle f_j\\&=0,\quad\forall f\in\mathscr{H},\end{aligned}\tag{19}$$

we obtain

$$\begin{aligned}\left\langle\left(U_G-U_H\right)f,U_Gg\right\rangle&=\left\langle\left(U_G-U_H\right)f,U_F\left(K^{\dagger}\right)^*g\right\rangle\\&=\left\langle K^{\dagger}U_F^*\left(U_G-U_H\right)f,g\right\rangle=0\end{aligned}\tag{20}$$

for any $f,g\in\mathscr{H}$. Thus $U_G^*(U_G-U_H)f=0$; equivalently, $U_G^*U_G=U_G^*U_H$.

Conversely, let $U_G^*U_G=U_G^*U_H$ for any dual K-Bessel sequence $\{h_j\}_{j\in\mathbb{J}}$ of $\{f_j\}_{j\in\mathbb{J}}$. Then

$$\left\|U_G\right\|^2=\left\|U_G^*U_G\right\|=\left\|U_G^*U_H\right\|\le\left\|U_G\right\|\left\|U_H\right\|,\tag{21}$$

and it follows that $\|U_G\|\le\|U_H\|$, implying that $\{g_j\}_{j\in\mathbb{J}}$ is the canonical dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$. This completes the proof. □

A natural problem arises: under what condition will a Parseval K-frame admit a unique dual K-Bessel sequence? To this problem, we have the following.

Theorem 12. *Suppose that $K\in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$ with analysis operator U_F. Then $\{f_j\}_{j\in\mathbb{J}}$ has a unique dual K-Bessel sequence if and only if $Range(U_F)=\ell^2(\mathbb{J})$.*

Proof. Assume first that $Range(U_F)=\ell^2(\mathbb{J})$. Then U_F^* is injective. Let $\{g_j\}_{j\in\mathbb{J}}$ and $\{h_j\}_{j\in\mathbb{J}}$ be two dual K-Bessel sequences of $\{f_j\}_{j\in\mathbb{J}}$; then it is easy to check that $\{\langle f,g_j-h_j\rangle\}_{j\in\mathbb{J}}\in\ell^2(\mathbb{J})$ for each $f\in\mathscr{H}$ and that

$$\begin{aligned}0=Kf-Kf&=\sum_{j\in\mathbb{J}}\left\langle f,g_j\right\rangle f_j-\sum_{j\in\mathbb{J}}\left\langle f,h_j\right\rangle f_j\\&=\sum_{j\in\mathbb{J}}\left\langle f,g_j-h_j\right\rangle f_j=U_F^*\left(\left\{\left\langle f,g_j-h_j\right\}_{j\in\mathbb{J}}\right).\end{aligned}\tag{22}$$

Thus $\langle f,g_j-h_j\rangle=0$ for any $j\in\mathbb{J}$ and $f\in\mathscr{H}$ and, consequently, $g_j=h_j$ for any $j\in\mathbb{J}$.

For the opposite implication, assume contrarily that $Range(U_F)\neq\ell^2(\mathbb{J})$. Since $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame, it is easily seen that $KK^*=U_F^*U_F$. Thus $Range(U_F^*)=Range(K)$ by Lemma 4, and U_F has closed range as a consequence. Let $S\in B(\mathscr{H},\ell^2(\mathbb{J}))$ be an invertible operator and $0\neq\{a_j\}_{j\in\mathbb{J}}\in(Range(U_F))^{\perp}$. Taking $h=S^{-1}(\{a_j\}_{j\in\mathbb{J}})$ and $g_j=\overline{a}_jh$ for each $j\in\mathbb{J}$, then, for every $f\in\mathscr{H}$,

$$\begin{aligned}\sum_{j\in\mathbb{J}}\left|\left\langle f,g_j\right\rangle\right|^2&=\sum_{j\in\mathbb{J}}\left|\left\langle f,\overline{a}_jh\right\rangle\right|^2=\sum_{j\in\mathbb{J}}\left|\left\langle f,h\right\rangle\right|^2\left|a_j\right|^2\\&=\left|\left\langle f,h\right\rangle\right|^2\left\|\left\{a_j\right\}_{j\in\mathbb{J}}\right\|^2\\&\le\left\|\left\{a_j\right\}_{j\in\mathbb{J}}\right\|^2\|h\|^2\|f\|^2,\end{aligned}\tag{23}$$

meaning that $\{g_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathscr{H}$. Now let $h_j=\widetilde{f}_j+g_j$ for every $j\in\mathbb{J}$; then it is easily seen that $\{h_j\}_{j\in\mathbb{J}}$ is a Bessel sequence of $\mathscr{H}$. Since $\{a_j\}_{j\in\mathbb{J}}$ is orthogonal to $Range(U_F)$,

$$\begin{aligned}\left\langle\sum_{j\in\mathbb{J}}\left\langle f,g_j\right\rangle f_j,e\right\rangle&=\sum_{j\in\mathbb{J}}\left\langle f,\overline{a}_jh\right\rangle\left\langle f_j,e\right\rangle\\&=\sum_{j\in\mathbb{J}}\left\langle f,h\right\rangle a_j\left\langle f_j,e\right\rangle\\&=\left\langle f,h\right\rangle\left\langle\left\{a_j\right\}_{j\in\mathbb{J}},\left\{\left\langle e,f_j\right\rangle\right\}_{j\in\mathbb{J}}\right\rangle\\&=\left\langle f,h\right\rangle\left\langle\left\{a_j\right\}_{j\in\mathbb{J}},U_Fe\right\rangle=0\end{aligned}\tag{24}$$

for any $e, f \in \mathscr{H}$. Therefore $\sum_{j\in\mathbb{J}}\langle f, g_j\rangle f_j = 0$ for any $f \in \mathscr{H}$, which yields

$$\begin{aligned}\sum_{j\in\mathbb{J}}\left\langle f, h_j\right\rangle f_j &= \sum_{j\in\mathbb{J}}\left\langle f, \widetilde{f}_j\right\rangle f_j + \sum_{j\in\mathbb{J}}\left\langle f, g_j\right\rangle f_j \\ &= \sum_{j\in\mathbb{J}}\left\langle f, \widetilde{f}_j\right\rangle f_j = Kf.\end{aligned} \tag{25}$$

Since $\{a_j\}_{j\in\mathbb{J}} \neq 0$, there exists $j_0 \in \mathbb{J}$ such that $a_{j_0} \neq 0$ and, thus, $g_{j_0} \neq 0$, since a simple calculation gives $(a_{j_0}/|a_{j_0}|^2)S(g_{j_0}) = \{a_j\}_{j\in\mathbb{J}}$. Hence $\{h_j\}_{j\in\mathbb{J}}$ is a dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$ and is different from $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$, a contradiction. □

It is interesting that the $\ell^2(\mathbb{J})$-linear independence of a Parseval K-frame $\{f_j\}_{j\in\mathbb{J}}$ can immediately lead to the $\ell^2(\mathbb{J})$-linear independence of $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ and vice versa and that the uniqueness of the dual K-Bessel sequence of $\{f_j\}_{j\in\mathbb{J}}$ implies the uniqueness of the dual K^*-Bessel sequence of $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$.

Theorem 13. *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$. Then the following results hold:*

(1) $\{f_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-linearly independent if and only if $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-linearly independent.

(2) If $\{f_j\}_{j\in\mathbb{J}}$ admits a unique dual K-Bessel sequence, then $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ admits a unique dual K^*-Bessel sequence.

Proof. (1) We first prove the necessity. Let U_F be the analysis operator of $\{f_j\}_{j\in\mathbb{J}}$; then

$$\begin{aligned}\sum_{j\in\mathbb{J}}\left\langle f, f_j\right\rangle f_j &= U_F^*U_F f = KK^*f = \sum_{j\in\mathbb{J}}\left\langle K^*f, \widetilde{f}_j\right\rangle f_j \\ &= \sum_{j\in\mathbb{J}}\left\langle f, K\widetilde{f}_j\right\rangle f_j, \quad \forall f \in \mathscr{H}.\end{aligned} \tag{26}$$

Therefore,

$$\begin{aligned}0 &= \sum_{j\in\mathbb{J}}\left\langle f, f_j\right\rangle f_j - \sum_{j\in\mathbb{J}}\left\langle f, K\widetilde{f}_j\right\rangle f_j \\ &= \sum_{j\in\mathbb{J}}\left\langle f, f_j - K\widetilde{f}_j\right\rangle f_j.\end{aligned} \tag{27}$$

Since $\{f_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-linearly independent, it follows that $\langle f, f_j - K\widetilde{f}_j\rangle = 0$ for any $f \in \mathscr{H}$ and any $j \in \mathbb{J}$, and $f_j = K\widetilde{f}_j$ for any $j \in \mathbb{J}$ as a consequence. Suppose now that $\sum_{j\in\mathbb{J}} c_j\widetilde{f}_j = 0$ for some $\{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$, then

$$0 = K\sum_{j\in\mathbb{J}}c_j\widetilde{f}_j = \sum_{j\in\mathbb{J}}c_jK\widetilde{f}_j = \sum_{j\in\mathbb{J}}c_jf_j. \tag{28}$$

Again by the $\ell^2(\mathbb{J})$-linear independence of $\{f_j\}_{j\in\mathbb{J}}$ we obtain $c_j = 0$ for each $j \in \mathbb{J}$.

For the sufficiency, let $\sum_{j\in\mathbb{J}} c_jf_j = 0$ for $\{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$. Then

$$0 = K^\dagger\sum_{j\in\mathbb{J}}c_jf_j = \sum_{j\in\mathbb{J}}c_jK^\dagger f_j = \sum_{j\in\mathbb{J}}c_j\widetilde{f}_j. \tag{29}$$

Thus $c_j = 0$ for every $j \in \mathbb{J}$, and the conclusion follows.

(2) Since $\{f_j\}_{j\in\mathbb{J}}$ has a unique dual K-Bessel sequence, by Theorem 12 we know that its analysis operator U_F is surjective and, thus, U_F^* is injective, which implies that $\{f_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-linearly independent. Hence, by (1), $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ is also $\ell^2(\mathbb{J})$-linearly independent, from which we conclude that $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ has a unique dual K^*-Bessel sequence. □

We also present the following condition for the canonical dual K-Bessel sequence of a Parseval K-frame to have a unique dual K^*-Bessel sequence.

Theorem 14. *Suppose that $K \in B(\mathscr{H})$ has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$. If the remainder of $\{f_j\}_{j\in\mathbb{J}}$ fails to be a new K-frame whenever any one of its elements is deleted, then $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ has a unique dual K^*-Bessel sequence.*

Proof. To prove that $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ admits a unique dual K^*-Bessel sequence, it is sufficient to show that $\{\widetilde{f}_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-linearly independent, and it is enough to show that $\{f_j\}_{j\in\mathbb{J}}$ is $\ell^2(\mathbb{J})$-linearly independent by Theorem 13(1). Suppose on the contrary that there is $\{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$ with $c_{j_0} \neq 0$ for some $j_0 \in \mathbb{J}$ such that $\sum_{j\in\mathbb{J}} c_jf_j = 0$. Then $f_{j_0} = -\sum_{j\neq j_0}(c_j/c_{j_0})f_j$. Denote by U_F the analysis operator of $\{f_j\}_{j\in\mathbb{J}}$; then, by Lemma 5, $Range(K) \subset Range(U_F^*)$. Thus for each $f \in Range(K)$ there exists $\{a_j\}_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$ such that

$$\begin{aligned}f &= U_F^*\left\{a_j\right\}_{j\in\mathbb{J}} = \sum_{j\in\mathbb{J}}a_jf_j = \sum_{j\neq j_0}a_jf_j - a_{j_0}\sum_{j\neq j_0}\frac{c_j}{c_{j_0}}f_j \\ &= \sum_{j\neq j_0}\left(a_j - \frac{a_{j_0}}{c_{j_0}}c_j\right)f_j.\end{aligned} \tag{30}$$

Clearly, $\{f_j\}_{j\in\mathbb{J}\setminus\{j_0\}}$ is a Bessel sequence of $\mathscr{H}$ and $\{a_j - (a_{j_0}/c_{j_0})c_j\}_{j\in\mathbb{J}\setminus\{j_0\}} \in \ell^2(\mathbb{J}\setminus\{j_0\})$. Therefore, $f \in Range(U_{F_{j_0}}^*)$ and, consequently, $Range(K) \subset Range(U_{F_{j_0}}^*)$, where $U_{F_{j_0}}$ is the analysis operator of $\{f_j\}_{j\in\mathbb{J}\setminus\{j_0\}}$. Again by Lemma 5 it follows that $\{f_j\}_{j\in\mathbb{J}\setminus\{j_0\}}$ is a K-frame of $\mathscr{H}$, which leads to a contradiction. □

At the end of the paper, we show that the canonical K-dual Bessel sequences of Parseval K-frames give rise to expansion coefficients of elements in $Range(K)$ with minimal norm.

Theorem 15. *Suppose that K has closed range and $\{f_j\}_{j\in\mathbb{J}}$ is a Parseval K-frame of $\mathscr{H}$. Then for any $\{c_j\}_{j\in\mathbb{J}} \in \ell^2(\mathbb{J})$ satisfying $Kf = \sum_{j\in\mathbb{J}} c_jf_j$, we have*

$$\sum_{j\in\mathbb{J}}\left|c_j\right|^2 = \sum_{j\in\mathbb{J}}\left|c_j - \left\langle f, \widetilde{f}_j\right\rangle\right|^2 + \sum_{j\in\mathbb{J}}\left|\left\langle f, \widetilde{f}_j\right\rangle\right|^2. \tag{31}$$

Proof. It is easy to check that

$$\begin{aligned}
&\sum_{j\in\mathbb{J}}\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)\left\langle \tilde{f}_j,f\right\rangle \\
&\quad=\left\langle \sum_{j\in\mathbb{J}}\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)\tilde{f}_j,f\right\rangle \\
&\quad=\left\langle K^{\dagger}\sum_{j\in\mathbb{J}}\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)f_j,f\right\rangle \\
&\quad=\left\langle K^{\dagger}\left(Kf-Kf\right),f\right\rangle=0
\end{aligned}\tag{32}$$

for every $f\in\mathscr{H}$. Therefore,

$$\begin{aligned}
&\sum_{j\in\mathbb{J}}\left|c_j\right|^2=\sum_{j\in\mathbb{J}}c_j\bar{c}_j=\sum_{j\in\mathbb{J}}\left(\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)+\left\langle f,\tilde{f}_j\right\rangle\right) \\
&\quad\cdot\overline{\left(\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)+\left\langle f,\tilde{f}_j\right\rangle\right)} \\
&\quad=\sum_{j\in\mathbb{J}}\left(\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)\overline{\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)}\right. \\
&\quad+\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)\left\langle \tilde{f}_j,f\right\rangle+\left\langle f,\tilde{f}_j\right\rangle\overline{\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)} \\
&\quad\left.+\left\langle f,\tilde{f}_j\right\rangle\left\langle \tilde{f}_j,f\right\rangle\right)=\sum_{j\in\mathbb{J}}\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right) \\
&\quad\cdot\overline{\left(c_j-\left\langle f,\tilde{f}_j\right\rangle\right)}+\sum_{j\in\mathbb{J}}\left\langle f,\tilde{f}_j\right\rangle\left\langle \tilde{f}_j,f\right\rangle=\sum_{j\in\mathbb{J}}\left|c_j\right. \\
&\quad\left.-\left\langle f,\tilde{f}_j\right\rangle\right|^2+\sum_{j\in\mathbb{J}}\left|\left\langle f,\tilde{f}_j\right\rangle\right|^2.
\end{aligned}\tag{33}$$

□

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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