Boundedness of Multidimensional Dunkl-Hausdorff Operators

Radouan Daher and Faouaz Saadi

Department of Mathematics, Laboratory of Topology, Algebra, Geometry and Discrete Mathematics, Faculty of Sciences Ain Chock University Hassan II, Casablanca, Morocco

Correspondence should be addressed to Faouaz Saadi; sadfaouaz@gmail.com

Received 2 February 2020; Accepted 5 March 2020; Published 10 April 2020

1. Introduction

The Hausdorff operator is one of the most important operators in harmonic analysis, and it is used to solve certain classical problems in analysis. In the one-dimensional case, Hausdorff operators on the real line were introduced in [1] and studied on the Hardy space in [2]. The natural generalization in several dimensions was introduced and studied in [3–5]. The reader can see a recent survey article [6] by Liflyand which contains the main results on Hausdorff operators in various settings and bibliography until 2013.

Dunkl theory generalizes classical Fourier analysis on \( \mathbb{R}^n \). It started twenty years ago with Dunkl’s seminal work in [7] and was further developed by several mathematicians. In the frame of extending these results to the context of Dunkl theory, we have introduced and studied recently in [8] the Dunkl-Hausdorff operators in the one-dimensional case on the spaces \( L^p_\lambda (\mathbb{R}) \) and \( H^1_\lambda (\mathbb{R}) \). Now, we are going to study the multidimensional Dunkl-Hausdorff operators in the spirit of those in [3, 5]. We can only establish a complete result for the finite reflection group \( G = \mathbb{Z}^2_+ \) with the associated measure \( \mu_\kappa \) given for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \) by

\[
d\mu_\kappa (x) = h^2_\kappa (x)dx,
\]

with \( h_\kappa \) the \( \mathbb{Z}^2_+ \)-invariant function is defined by

\[
h_\kappa (x) = \prod_{j=1}^{n} |x_j|^{\kappa_j} = \prod_{j=1}^{n} h_\kappa (x_j),
\]

where \( \kappa_1, \ldots, \kappa_n \) are nonnegative real numbers.

The paper is organised as follows. In Section 2, we collect some definitions and results related to Dunkl’s analysis. In particular, we list the properties of the Dunkl transform and Hardy space associated with the Dunkl operators that will be relevant for the sequel. Section 3 is devoted to proving our main results.

2. Preliminaries

This section is devoted to the preliminaries and background. For further survey about this theory, the reader may refer to [9–11].

2.1. Notation. We denote by \( L^p_\kappa (\mathbb{R}^n) \) the space \( L^p (\mathbb{R}^n; d\mu_\kappa) \) and we use the shorter notation \( \| \cdot \|_{\kappa,p} \) instead of \( \| \cdot \|_{L^p_\kappa (\mathbb{R}^n)} \). For \( p \in [1, +\infty] \), the space \( L^p_\kappa (\mathbb{R}^n) \) is of course the space of measurable functions on \( \mathbb{R}^n \) such that

\[
\| f \|_{\kappa,p} = \left( \int_{\mathbb{R}^n} |f(y)|^p d\mu_\kappa (y) \right)^{1/p} < +\infty, \quad \text{if } 1 \leq p < +\infty,
\]

\[
\| f \|_{\kappa,\infty} = \operatorname{ess sup}_{y \in \mathbb{R}^n} |f(y)| < +\infty, \quad \text{otherwise}.
\]
2. Dunkl Transform. Let \( e_1, \ldots, e_n \) be the standard basis of \( \mathbb{R}^n \). We denote by \( \sigma_j \) (for each \( j \), from 1 to \( n \)) the reflection with respect to the hyperplane perpendicular to \( e_j \); that is, to say, for every \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \),

\[
\sigma_j(x) = x - 2\frac{\langle x, e_j \rangle}{\|e_j\|^2} e_j = (x_1, \ldots, x_{j-1}, -x_j, x_{j+1}, \ldots, x_n).
\]

(4)

Of course \( \langle \cdot, \cdot \rangle \) is the usual inner product on \( \mathbb{R}^n \times \mathbb{R}^n \) and \( \| \cdot \| \) is the associated norm. Let \( G \) be the finite reflection group generated by \( \{ \sigma_j : j = 1, \ldots, n \} \); so, \( G \) is isomorphic to \( S_2 \). Let \( \kappa_1, \kappa_2, \ldots, \kappa_n \) be nonnegative real numbers.

Associated with these objects are the Dunkl operators \( \mathcal{D}_k \) (for \( k = 1, \ldots, n \)) that have been introduced in [7] by Dunkl. They are given for \( x \in \mathbb{R}^n \) by

\[
\mathcal{D}_k f(x) = \partial_k f(x) + \sum_{j=1}^n \kappa_j \left( \frac{f(x) - f(\sigma_j(x))}{\langle x, e_j \rangle} \right) \langle e_k, e_j \rangle
\]

(5)

where \( \partial_k \) denotes the usual partial derivative. A fundamental property of these differential-difference operators is their commutativity, that is, to say, \( \mathcal{D}_k \mathcal{D}_l = \mathcal{D}_l \mathcal{D}_k \).

Closely related to them is the so-called intertwining operator \( V_k \) (the subscript means that the operator depends on the parameter \( \kappa_j \), except in the rank-one case, where the subscript is then a single parameter), which is the unique linear isomorphism of \( \mathcal{D}_m \mathcal{D}_n \) such that

\[
V_k (\mathcal{D}_m) = \mathcal{D}_m, \quad V_k (1) = 1, \quad \mathcal{D}_k V_k = V_k \partial_k \text{ for } k = 1, \ldots, n,
\]

\[
V_k f(x) = \int_{|t|=1} \left( \frac{f(x + \kappa_j(t))}{1 - t} \right) \overline{M_k(t)} \mathcal{D}_1 dt,
\]

(6)

with \( M_k(t) = (\Gamma(\kappa_j + (1/2))/\Gamma(\kappa_j)\Gamma(1/2)) \) (where \( \Gamma \) is the well-known Gamma function).

In order to define the Dunkl transform, we also need to introduce the Dunkl kernel \( E_k \) that is given for \( x \in \mathbb{C}^n \) by

\[
E_k(\cdot, x)(y) = V_k e^{\langle \cdot, x \rangle}(y), \quad y \in \mathbb{R}^n,
\]

(7)

and it satisfies the following basic properties: \(|E_k(\cdot, x, y)| \leq 1\) for \( x, y \in \mathbb{R}^n \). Considering the definition of \( E_k \) together with the explicit formula for \( V_k \) gives us

\[
E_k(x, y) = \prod_{j=1}^n E_{\kappa_j}(x_j, y_j).
\]

(8)

In the rank-one case, \( E_k \) is explicitly known. More precisely, it is given for both \( x \) and \( y \) in \( \mathbb{C} \) by

\[
E_k(x, y) = \sum_{j=1}^n \mathcal{D}^2_j f(x) = \sum_{j=1}^n \left\{ \left( \frac{\partial}{\partial x_j} \right)^2 f(x) + \frac{2\kappa_j}{x_j} \frac{\partial}{\partial x_j} f(x) - \frac{k_j^2}{x_j^2} f(x) - f(\sigma_j x) \right\}.
\]

(9)

We are now in a position to introduce the Dunkl transform that is taken with respect to the measure \( \mu_k \) defined by (1). For \( f \in L_k^p(\mathbb{R}^n) \), the Dunkl transform of \( f \), denoted by \( \mathcal{F}_k(f) \), is given by

\[
\mathcal{F}_k(f)(x) = c_k \int_{\mathbb{R}^n} f(y) E_k(x, -iy) d\mu_k(y), \quad x \in \mathbb{R}^n,
\]

(10)

where \( c_k \) is the following constant:

\[
c_k^{-1} = \int_{\mathbb{R}^n} e^{-\langle |f|^2 \rangle/2} d\mu_k(x) = \prod_{j=1}^n c_j^{-1}.
\]

(11)

If \( \kappa_1 = \cdots = \kappa_n = 0 \), then \( V_k = \text{id} \) and the Dunkl transform coincides with the Euclidean Fourier transform.

Let \( x \in \mathbb{R}^n \). The Dunkl translation operator \( \tau_k^x \) is given for \( f \in L_k^p(\mathbb{R}^n) \) by

\[
\mathcal{F}_k(\tau_k^x(f))(y) = E_k(x, y) \mathcal{F}_k(f)(y), \quad y \in \mathbb{R}^n.
\]

(12)

Proposition 1. Let \( x \in \mathbb{R}^n \). The operator \( \tau_k^x \) extends to \( L_k^p(\mathbb{R}^n) \) for \( p \in [1, \infty) \) and for \( f \in L_k^p(\mathbb{R}^n) \), we have

\[
\|\tau_k^x(f)\|_{L_k^p} \leq C \|f\|_{L_k^p},
\]

(13)

where \( C \) is independent of \( x \) and \( f \).

The Dunkl convolution operator is defined for both \( f \) and \( g \) in \( L_k^p(\mathbb{R}^n) \) by

\[
(f \ast_k g)(x) = c_k \int_{\mathbb{R}^n} f(y) \tau_k^y(g)(-y) d\mu_k(y), \quad x \in \mathbb{R}^n.
\]

(14)

Proposition 2. Assume that \( p^{-1} + q^{-1} = 1 + r^{-1} \) with \( p, q, r \in [1, \infty] \). Then, the map \( (f, g) \mapsto f \ast_k g \) defined on \( L_k^p(\mathbb{R}^n) \times L_k^q(\mathbb{R}^n) \) extends to a continuous map from \( L_k^p(\mathbb{R}^n) \times L_k^q(\mathbb{R}^n) \) to \( L_k^r(\mathbb{R}^n) \), and we have

\[
\|f \ast_k g\|_{L_k^r} \leq C \|f\|_{L_k^p} \|g\|_{L_k^q},
\]

(15)

where \( C \) is independent of \( f \) and \( g \).

We finally note that the Dunkl convolution satisfies the properties \( f \ast_k g = g \ast_k f \) and \( \mathcal{F}_k(f \ast_k g) = \mathcal{F}_k(f) \cdot \mathcal{F}_k(g) \).
2.3. Hardy Space. We define the Riesz transforms in the Dunkl setting, putting
\[ \mathcal{R}_j = \mathcal{D}_j (-\mathcal{L})^{-1/2}. \] (17)

The operators \( \mathcal{R}_j \) can be expressed as the Dunkl multiplier operators, namely,
\[ \mathcal{R}_j f (x) = \mathcal{D}_j (-\mathcal{L})^{-1/2} f (x) = \mathcal{D}_j \int_{\mathbb{R}^n} \frac{1}{|ξ|} E(x, iξ) \mathcal{F} f (ξ) dμ(ξ) \]
\[ = \int_{\mathbb{R}^n} \frac{ξ_j}{|ξ|} E(x, iξ) \mathcal{F} f (ξ) dμ(ξ). \] (18)

The Hardy type space \( H^1_κ(\mathbb{R}^n) \) for Dunkl analysis is defined to be
\[ H^1_κ(\mathbb{R}^n) = \left\{ f \in L^1_κ(\mathbb{R}^n) \colon \| f \|_{H^1_κ} = \| f \|_{1,κ} + \sum_{j=1}^n \| \mathcal{R}_j f \|_{1,κ} < ∞ \right\}, \] (19)
endowed with the norm
\[ \| f \|_{H^1_κ} = \| f \|_{1,κ} + \sum_{j=1}^n \| \mathcal{R}_j f \|_{1,κ}. \] (20)

An \( µ_κ \)-atom is a measurable function \( a : \mathbb{R}^n \rightarrow \mathbb{C} \) such that
(i) \( a \) is supported in a ball \( B \)
(ii) \( \| a \|_{L^∞} \leq µ_κ(B)^{-1} \)
(iii) \( \int_{\mathbb{R}^n} a(x) dµ_κ(x) = 0 \)

By definition, the atomic Hardy space \( H^1_{κ,atom} \) consists of all functions \( f \in L^1_κ(\mathbb{R}^n) \) that can be written as \( f = \sum λ_ν a_ν \), where the \( a_ν \)'s are atoms and \( \sum |λ_ν| < +∞ \), and the norm is given by
\[ \| f \|_{H^1_{κ,atom}} = \inf \sum |λ_ν|, \] (21)
where the infimum is taken over all atomic decompositions of \( f \).

The following result was proved in [12].

**Proposition 3.** The spaces \( H^1_κ \) and \( H^1_{κ,atom} \) coincide and the norms \( \| f \|_{H^1_κ} \) and \( \| f \|_{H^1_{κ,atom}} \) are equivalent.

3. Hausdorff Operator

We define the multivariate Dunkl-Hausdorff operator \( \mathcal{H}_κ = \mathcal{H}_κ(Φ, A) \) acting on Borel measurable functions \( f : \mathbb{R}^n \rightarrow \mathbb{C} \) by
\[ \mathcal{H}_κ f (x) = \int_{\mathbb{R}^n} Φ(s) f(s(Φ(s)x)) dµ_κ(s). \] (22)

3.1. Boundedness of \( \mathcal{H}_κ \) on the Spaces \( L^p_κ(\mathbb{R}^n) \). Let \( 1 ≤ p ≤ ∞ \) and denoted by \( p^∗ \), the exponent conjugate to \( p \); that is, let \( 1/(p + 1)/p^∗ = 1 \) with the agreement that \( 1/∞ := 0 \).

For a nonsingular diagonal matrix \( A = diag(a_1, \ldots, a_n) \), we denote by \( ν(A) = \prod_{j} |a_j|^{2s_j} \).

**Theorem 1.** If \( f \in L^p_κ(\mathbb{R}^n) \) for some \( 1 ≤ p ≤ ∞ \), \( \mathcal{A}(s) = diag(a_1(s), \ldots, a_n(s)) \) is a nonsingular diagonal matrix, and if
\[ K_{p,κ} := \int_{\mathbb{R}^n} |Φ(s)| \left( (\mathcal{A}(s)^{-1}) \det(\mathcal{A}(s)^{-1}) \right)^{1/p} dµ_κ(s) < ∞, \] (23)
then
\[ \| \mathcal{H}_κ f \|_{p,κ} ≤ K_{p,κ} \| f \|_{p,κ}. \] (24)

Clearly, the constant \( K_{p,κ} \) depends not only on \( p \) and \( κ \) but also on \( Φ \) and \( \mathcal{A} \), but does not depend on \( f \).

**Proof.** First, assume that \( 1 < p < ∞ \). Making use of Fubini’s theorem and Minkowski’s inequality in the general form and integrating by substitution yields the following:

\[ \| \mathcal{H}_κ f \|_{p,κ} = \left( \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |Φ(s)| f(s(Φ(s)x)) dµ_κ(s) \right)^p dµ_κ(x) \right)^{1/p} \]
\[ = \int_{\mathbb{R}^n} \left( \int_{\mathbb{R}^n} |Φ(s)| f(s(Φ(s)x))^p dµ_κ(x) \right)^{1/p} dµ_κ(s) \]
\[ = \int_{\mathbb{R}^n} |Φ(s)| \left( \int_{\mathbb{R}^n} |f(v)|^p \left( \det(\mathcal{A}(s)^{-1}) \right)^{1/p} dµ_κ(v) \right)^{1/p} dµ_κ(s) \]
\[ = \| f \|_{p,κ} \int_{\mathbb{R}^n} |Φ(s)| \left( \left( \det(\mathcal{A}(s)^{-1}) \right)^{-1/p} dµ_κ(s) \right) \]
\[ = K_{p,κ} \| f \|_{p,κ}, \]
which proves (24) in case \( 1 < p < ∞ \).
In case $p = 1$, the abovementioned argument works without using Minkowski’s inequality. The case $p = \infty$ is trivial.

3.1.1. Examples

(1) If $n = 1$ and $\Phi (s) = (\varphi (1/s)/s)$, then

$$
\mathcal{H}_x f (x) = \int_0^\infty \varphi (s) \frac{x^\alpha}{s^{2\alpha + 1}} \left( \frac{x}{s} \right) ds
$$

(26)

which is the same definition given in [8].

(2) By choosing $\Phi (s_1, s_2) = x^{(1/2)}(s_1, s_2)$ and $\mathcal{A} (s_1, s_2) = \text{diag} (s_1, s_2)$, we get the bivariate ($n = 2$) Dunkl-Cesàro operator $\mathcal{C}_x$:

$$
K_{p, x} (\mathcal{C}_x) = \int_{\mathbb{R}^n} \Phi (s)[\mathcal{A}^{-1}(s)]^{1/p} \left| \det (\mathcal{A}^{-1}(s)) \right|^{1/p} ds
$$

(28)

This implies the case $1 < p \leq \infty$ in the following corollary, while the case $p = 1$ is trivial.

**Corollary 1.** If condition (23) is satisfied for some $1 \leq p \leq \infty$, then the operator $\mathcal{H}_x^*$ is bounded on $L^p_k (\mathbb{R}^n)$:

$$
\left\| \mathcal{H}_x^* f \right\|_{\mathcal{H}_x^*} \leq K_{p, x} \left\| f \right\|_{\mathcal{H}_x^*}.
$$

(29)

We also claim that this $\mathcal{H}_x^*$ is the adjoint operator to $\mathcal{H}_x$ in the sense that we attempt to make clear in the following.

Let $\mathcal{H}_x = \mathcal{H}_x (\Phi; \mathcal{A})$ be a Hausdorff operator, for which condition (23) is satisfied for some $1 \leq p \leq \infty$ and let

$$
\mathcal{C}_x (x_1, x_2) = \int_{[0, 1]^2} f (s_1 x_1, s_2 x_2) d\mu_{x_1} (s_1) d\mu_{x_2} (s_2)
$$

(27)

3.2. The Adjoint Operator to $\mathcal{H}_x$. Consider the Hausdorff operator $\mathcal{H}_x (\Phi; \mathcal{A})$ for which condition (23) is satisfied for some $1 \leq p \leq \infty$. By Theorem 1, the operator $\mathcal{H}_x$ is bounded on $L^p_k (\mathbb{R}^n)$. For $\Psi (s) = \Phi (s) \nu (\mathcal{A}^{-1}(s)) \left| \det (\mathcal{A}^{-1}(s)) \right|$ we claim that the operator $\mathcal{H}_x^* = \mathcal{H}_x (\Psi; \mathcal{A}^{-1})$ is bounded on the conjugate space $L^{p'} (\mathbb{R}^n)$. Indeed, by assumption, we have

$$
\mathcal{H}_x^* (\mathcal{H}_x f) (x) = \int_{\mathbb{R}^n} g (x) \mathcal{H}_x f (x) d\mu_x (x) = \int_{\mathbb{R}^n} f (x) \mathcal{H}_x^* g (x) d\mu_x (x).
$$

(30)

Observe that $\mathcal{H}_x^*$ is also a Hausdorff operator according to our definition.

**Proof.** We have seen above that by virtue of Theorem 1 and Corollary 1 both integrals in (30) exist as Lebesgue integrals. So, we may apply Fubini’s theorem twice as follows:

$$
\int_{\mathbb{R}^n} g (x) \mathcal{H}_x f (x) d\mu_x (x) = \int_{\mathbb{R}^n} \Phi (s) \int_{\mathbb{R}^n} f (\mathcal{A} (s) x) g (x) d\mu_{x} (x) d\mu_x (s)
$$

(31)

$$
= \int_{\mathbb{R}^n} \Phi (s) \left| \det \mathcal{A}^{-1} (s) \right| \left| \det \mathcal{A}^{-1} (s) \right| d\mu_{x} (x) d\mu_x (s)
$$

which is (30).
3.3. Commuting Relations with Dunkl Transform. In the following theorem, we answer the claim indicated in the title of this section.

Theorem 2

(i) If condition (23) is satisfied for $p = 1$, then

$$\mathcal{F}_k(\mathcal{H}_K f)(u) = \mathcal{H}_K^*(\mathcal{F}_k)(u). \quad (32)$$

Proof

The proof of (33) is analogous to that of (32). □

(ii) If condition (23) is satisfied for $p = \infty$, then

$$\mathcal{F}_k(\mathcal{H}_K f)(u) = \mathcal{H}_K^*(\mathcal{F}_k)(u). \quad (33)$$

Proof

(i) Thus, we may apply Fubini’s theorem twice in order to obtain that

$$\tilde{\mu}_N \mathcal{F}_k(\mathcal{H}_K f)(u) = \mathcal{H}_K^*(\mathcal{F}_k)(u).$$

(ii) The proof of (33) is analogous to that of (32).

3.4. Boundedness of $\mathcal{H}_K$ on the Dunkl Hardy Spaces $H^1_{\mathcal{H}}(\mathbb{R}^n)$. Proceeding similarly to the proof of Theorem 1 in [13], we are going to extend the earlier version of $H^1$ estimates, as the strongest and sharpest can be found in [4].

Let $\|B\|_2 = \max_{j=1}^n |Bx_j|$, where $|\cdot|$ denotes the Euclidean norm. We will say that $\Phi \in L^2_{\mathcal{H},B}$ if

$$\|\Phi\|_{L^2_{\mathcal{H},B}} = \int_{\mathbb{R}^n} \|\Phi(x)\|_{B(x)} \mu_k(x) < \infty. \quad (35)$$

Theorem 3. The Hausdorff operator $\mathcal{H}_K f$ is bounded on the real Hardy space $H^1_{\mathcal{H}}(\mathbb{R}^n)$ provided $\Phi \in L^2_{\mathcal{H},B} \cap L^\infty(\mathbb{R}^n)$, and

$$\|\mathcal{H}_K f\|_{H^1_k} \leq \|\Phi\|_{L^2_{\mathcal{H},B}} \leq \|f\|_{H^1_k}. \quad (36)$$

We shall need the following inequalities for volumes of the Euclidean balls (see [12]). Recall that $k_1, \ldots, k_n \geq 0$ and $N = n + \sum_{j=1}^n 2k_j$. On $\mathbb{R}^n$, equipped with the Euclidean distance, the measure $\mu_k$ has the following rescaling properties:
Let
\[ f(\mathcal{A}(s)x) = \sum_k \lambda_k b_k(\mathcal{A}(s)x), \]  
(41)
where \( b_k \) is an \( \mu_k \)-atom supported in the ball \( B(y,r) \).

We will show that multiplying \( b_k(\mathcal{A}(s)x) \) by a constant depending on \( s \) we get an atomic decomposition of \( f \) itself, with no composition in the argument.

We have
\[ \int_{\mathbb{R}^n} b_k(\mathcal{A}(s)x) d\mu_k(x) = \int_{\mathbb{R}^n} b_k(v) \left( \prod_{j=1}^n [a(s)]_{2k_j} \right)^{-1} d\mu_k(v) = 0. \]
(42)

\[ \langle \mathcal{A}(s)x - y, \mathcal{A}(s)x - y \rangle = \langle \mathcal{A}(s)(x - \mathcal{A}^{-1}(s)y), \mathcal{A}(s)(x - \mathcal{A}^{-1}(s)y) \rangle \]
\[ \geq l^2(s) \langle x - \mathcal{A}^{-1}(s)y, \mathcal{A}(s)(x - \mathcal{A}^{-1}(s)y) \rangle, \]
then
\[ \text{supp } b_k(\mathcal{A}(s)\cdot) \subset B\left( \mathcal{A}^{-1}(s)x_0, \frac{r}{l(s)} \right). \]
(45)

Therefore, \( l(s)^N b_k(\mathcal{A}(s)\cdot) \) is an atom, and
\[ \|f(\mathcal{A}(s)\cdot)\|_{H^1} \leq \|\mathcal{A}^{-1}(s)\|_2 \|f\|_{H^1}. \]
(47)

Data Availability

No data were used to support this study.

Disclosure

Part of this article has been presented by the authors in “The 2nd International Conference on Fixed Point Theory & Applications.”

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References
