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Research Article

Some New Results for the Sobolev-Type Fractional Order Delay Systems with Noncompact Semigroup

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The topological structure of solution sets for the Sobolev-type fractional order delay systems with noncompact semigroup is studied. Based on a fixed point principle for multivalued maps, the existence result is obtained under certain mild conditions. With the help of multivalued analysis tools, the compactness of the solution set is also obtained. Finally, we apply the obtained abstract results to the partial differential inclusions.

1. Introduction

Let $(X, \|\cdot\|)$ and $(Y, \|\cdot\|)$ be Banach spaces. Given two operators $L: D(L) \subset X \longrightarrow Y$ and $A: D(A) \subset X \longrightarrow Y$, consider the Sobolev-type fractional order delay system:

$$\begin{cases} {}^{0}D_{c}^{q}(Lx(t)) + Ax(t) \in H(t, x_t), & t \in J := [0, T], \\ x(t) = \varphi(t), & t \in [-\tau, 0], \end{cases} \tag{1}$$

where ${}^{0}D_{c}^{q}$ $(0 < q < 1)$ is the Caputo fractional order derivative, $x_t \in C([-\tau, 0]; D(L))$, $x_t(\zeta) = x(t+\zeta)$ $(\zeta \in [-\tau, 0])$, $\varphi \in C([-\tau, 0]; D(L))$, and $H: [0, T] \times C([-\tau, 0]; X) \longrightarrow 2^{Y}$ is a multivalued map.

Fractional calculus (an extension of ordinary calculus) is an important branch of mathematical analysis, and its theoretical and practical applications have been greatly developed. Many mathematicians have devoted themselves to the study of fractional differential equations for a long time and have made important contributions to theory and application of fractional differential equations. For more details about fractional calculus and fractional differential equations, we refer to the monographs [1–3] and the papers [4–9]. At present, much interest has been developing in the fractional differential inclusions, and we refer readers to [10, 11] and the references therein.

Recently, the solvability and controllability of fractional order systems have been investigated by lots of authors. For a class of Sobolev-type fractional order evolution equations with nonlocal conditions, Debbouche and Nieto [12] obtained that the equation had at least one mild solution, which was proved to be unique. For the Sobolev-type fractional functional evolution equations, Fečkan et al. [13] gave the controllability results by applying the Schauder fixed point theorem [14, 15]. For the fractional evolution inclusions, by the multivalued analysis techniques, Wang et al. [10] proved the existence results. Wang and Zhou [16] investigated the existence and controllability results for the fractional semilinear differential inclusion with the Caputo fractional derivative by means of the Bohnenblust–Karlin's fixed point theorem. The readers can see [17–19] for more results of the fractional order system. Summarizing the above settings, one cause is that the nonlinearity with the compact value is upper semicontinuous, which has been proved to be too harsh and difficult to meet in practical application. The other cause is that the semigroup is compact.

Our interest in system (1) is to analyze the topological properties (such as compactness, acyclicity, and R_{δ}-structure) of its solution set. Unfortunately, the results on this direction are less known. Considering these, we aim to establish the topological properties of the set of all mild solutions for system (1). In this article, when the semigroup is noncompact and H is weakly upper semicontinuous to the solution variable, we firstly concern with the existence of mild solution of system (1), which can be obtained by using a fixed point theorem of multivalued maps (see Lemma 5). And then, by means of the theory of measure of noncompactness and a singular version of Gronwall inequality, we prove that the set of all mild solutions of system (1) is compact and the corresponding solution map is the upper semicontinuity. We should like to emphasize that system (1) involve the delay term, which makes it difficult for us to estimate the noncompactness measure of the solution set.

This paper is structured as follows: Section 2 presents some concepts and facts. Section 3 deals with the nonemptiness and compactness of the set of all mild solutions for system (1) and the upper semicontinuity of the corresponding solution map. We illustrate our abstract results by an example of partial differential inclusion in Section 4.

2. Preliminaries

In this section, we will introduce some notations and describe some results, which are used later in the paper.

Let $C([\alpha, \beta]; X)$ be the Banach space of all continuous functions from $[\alpha, \beta]$ to X, equipped with the sup-norm and $L^p(J; X)$ $(1 \le p < \infty)$ be the Banach space of all Bochner integrable functions f from $[0, T]$ to X satisfying $\int_0^T \|f(t)\| dt < \infty$, endowed with its standard norm.

Lemma 1 (see [20]). *Assume that $D \subset C([-\tau, T]; X)$ is a bounded subset. If D is equicontinuous, then the convex closure of D, denoted by $\overline{conv}(D)$, is bounded and equicontinuous.*

We introduce some facts about measures of noncompactness (MNC). Let $\chi(\cdot)$ be the Hausdorff MNC in X; that is,

$$\chi(D) = \inf\{\varepsilon > 0: D \text{ has a finite } \varepsilon - \text{net}\}, \quad D \subset X. \tag{2}$$

Then, the Hausdorff MNC satisfies the following properties (see [21] for details).

(i) Let $E: X \longrightarrow X$ be a bounded linear operator and $D \subset X$, then the following inequality holds:

$$\chi(ED) \le \|E\| \chi(D). \tag{3}$$

(ii) Assume that $B \subset X$ is bounded; then, for every $\epsilon > 0$, we can find $\{B_n\} \subset B$ such that

$$\chi(B) \le 2\chi(\{B_n\}) + \varepsilon. \tag{4}$$

We recall some additional properties of the Hausdorff MNC.

Lemma 2 (see [22]). *Let $\{h_n\} \subset L^1(J; X)$ be a sequence satisfying that, for a.e. $t \in J$,*

$$\|h_n(t)\| \le \theta(t) \text{ uniformly for } h_n, \tag{5}$$

where $\theta \in L^1(J; \mathbb{R}^+)$. Then, $\chi(\{h_n(t)\}) \in L^1(J; \mathbb{R}^+)$ and

$$\chi\left(\left\{\int_0^t h_n(s) ds\right\}\right) \le 2 \int_0^t \chi(\{h_n(s)\}) ds \tag{6}$$

for each $t \in J$.

Definition 1 (see [10]). If $f \in L^1(J; Y)$, $q \ge 0$, then the Riemann–Liouville fractional integral of order q of f is defined by

$$L_t^q f(t) := w_q * f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds, \quad t \in J, q > 0, \tag{7}$$

with $L_t^0 f(t) = f(t)$, where

$$w_q(s) = \begin{cases} \dfrac{1}{\Gamma(q)} s^{q-1}, & t > 0, \\ 0, & s \le 0, \end{cases} \tag{8}$$

$w_0(s) = 0$, and $\Gamma(\cdot)$ is the Gamma function.

Definition 2 (see [10]). If $f \in C^{k-1}(J; Y)$, $w_{k-q} * f \in W^{k,1}(J; Y)$ with $k \in \mathbb{N}$ and $q \in (k-1, k)$. Then, the q-order Caputo derivative of the function f is defined by

$$^0D_c^q f(t) := \frac{d^k}{dt^k} L_t^{k-q} \left(f(t) - \sum_{i=0}^{k-1} f^{(i)}(0) w_{i+1}(t) \right). \tag{9}$$

See [2, 11] for more results of the fractional calculus.

Let $p > 1$ and $pq > 1$ and $h \in L^p(J; Y)$. Consider the following system:

$$\begin{cases} ^0D_c^q (Lx(t)) + Ax(t) = h(t), & t \in J, \\ x(0) = x_0 \in D(E), \end{cases} \tag{10}$$

where A and L satisfy that

(F_1) A is linear and closed

(F_2) L is linear, closed, and bijective, and $D(L) \subset D(A)$

(F_3) $L^{-1}: Y \longrightarrow D(L)$ is continuous

Combining (F_1) and (F_2) with the closed graph theorem, we know that the linear operator $-AL^{-1}: Y \longrightarrow Y$ is bounded. From this, we deduce that $-AL^{-1}$ generates a semigroup $\{T(t), t \ge 0\}$. It is easy to know that $T(t)$ is equicontinuous for $t > 0$ (see [23], Theorem 1.2]). In this paper, we suppose sup $\{\|T(t)\|, t \ge 0\} \le M$ with some $M > 0$.

Based on the semigroup $\{T(t), t \ge 0\}$ and operator L, we introduce two characteristic solution operators $\mathscr{B}(\cdot)$ and $\mathscr{R}(\cdot)$, which are given by

$$\begin{aligned} \mathscr{B}(t) &= \int_0^\infty L^{-1} \rho_q(\theta) T(t^q \theta) d\theta, \\ \mathscr{R}(t) &= q \int_0^\infty \theta L^{-1} \rho_q(\theta) T(t^q \theta) d\theta, \end{aligned} \tag{11}$$

where

$$\rho_q(t) = \frac{1}{\pi q}\sum_{m=1}^{\infty}(-t)^{m-1}\frac{\Gamma(1+qm)}{m!}\sin(m\pi q), \quad t \in (0,\infty), \tag{12}$$

with $\rho_q(t) \ge 0$ and $\int_0^\infty \rho_q(t)dt = 1$.

Obviously, $\mathscr{B}(t)$ and $\mathscr{R}(t)$ are equicontinuous for $t > 0$. According to (F_1)–(F_3), we also have

$$\begin{aligned} \|\mathscr{B}(t)y\| &\le M\|L^{-1}\|\|y\|, \\ \|\mathscr{R}(t)y\| &\le \frac{qM\|L^{-1}\|}{\Gamma(1+q)}\|y\|, \\ t \ge 0,& \quad y \in Y. \end{aligned} \tag{13}$$

Definition 3 (see [13]). A mild solution of (10) is that a function $x \in C(J; X)$, which satisfies

$$x(t) = \mathscr{B}(t)Lx_0 + \int_0^t (t-s)^{q-1}\mathscr{R}(t-s)h(s)ds, \quad t \in J. \tag{14}$$

For each $x_0 \in D(L)$, $\varphi \in C([-\tau, 0]; D(L))$, and $h \in L^p(J; Y)$, denote $u(\cdot, x_0, h)$ by the mild solution of (10), which is also unique, and define the multivalued map $G_\varphi : L^p(J; Y) \longrightarrow C([-\tau, T]; X)$ by

$$G_\varphi h(t) = \begin{cases} \varphi(t), & t \in [-\tau, 0], \\ u(t, \varphi(0), h), & t \in J. \end{cases} \tag{15}$$

Obviously, $G_\varphi h$ is the unique mild solution of the equation:

$$\begin{cases} {}^0D_c^q(Lx(t)) + Ax(t) = h(t), & t \in J, \\ x(t) = \varphi(t), & t \in [-\tau, 0]. \end{cases} \tag{16}$$

The following assertion can be derived from the proof of Lemmas 2.4 and 3.1 of [10] (see also Lemma 2.6 of [24]).

Lemma 3. *Assume that $p > 1, pq > 1$, and the conditions (F_1) – (F_3) hold.*

(i) If $U \subset C([-\tau, 0]; D(L))$ is relatively compact and $K \subset L^p(J; Y)$ satisfy that $\|h(t)\| \le k(t)$ for a.e. $t \in J$ and all $h \in K$, where $k \in L^p(J; \mathbb{R}^+)$. Then $G_U(K)$ is equicontinuous in $C([-\tau, T]; X)$.

(ii) Let $\{h_n\} \subset L^p(J; Y)$ and $\{x_n\} \subset C([-\tau, T); X)$ with $x_n = G_\varphi f_n$ be two sequences, and assume in addition that $h_n \longrightarrow h$ weakly in $L^p(J; Y)$ and $x_n \longrightarrow x$ in $C([-\tau, T); X)$, we have $x = G_\varphi h$.

In what follows, we introduce some notations and results about multivalued analysis. The definitions of a multivalued map to be upper semicontinuous (u.s.c.), lower semicontinuous (l.s.c.), and weakly upper semicontinuous (weakly u.s.c.), closed and quasi-compact, one can see [10] for details.

We recall the characterization about u.s.c. maps.

Lemma 4 (see [21], Theorem 1.1.12). *If ϕ with convex, closed, and compact values is closed and quasi-compact, we have that ϕ is u.s.c.*

The following assertion provides us with a fixed point theorem.

Lemma 5 (see [25], Lemma 1). *Let U be a Banach space and $D \subset U$ be a compact and convex subset. Assume that the multivalued map $\phi : D \longrightarrow 2^D$ with closed contractible values is u.s.c., we have that ϕ has at least one fixed point.*

3. Main Results

Suppose that $H : J \times C([-\tau, 0]; X) \longrightarrow Y$ has convex closed values and

(H_1) $H(t, \cdot)$ is weakly u.s.c. for a.e. $t \in J$ and $H(\cdot, w)$ has a L^p-integral selection for each $w \in C([-\tau, 0]; X)$.

(H_2) H is uniformly L^p-integrable bounded, that is,

$$\|H(t, w)\| := \sup\{\|h\|; h \in H(t, w)\} \le \eta(t), \tag{17}$$

for a.e $t \in J$ and each $w \in C([-\tau, 0]; X)$, where $\eta \in L^p(J; \mathbb{R}^+)$.

(H_3) There exists $\mu \in C(J; \mathbb{R}^+)$ such that $\chi(H(t, B)) \le \mu(t)\sup_{s\in[-\tau,0]}\chi(B(s))$ for a.e. $t \in J$ and all bounded subsets $B \subset C([-\tau, 0]; X)$.

Let us define two multivalued maps Sel_H: $C([-\tau, T]; X) \longrightarrow 2^{L^p(J;Y)}$ by

$$\mathrm{Sel}_H(x) := \{h \in L^p(J; Y) \text{ and } h(t) \in H(t, x_t) \text{ for a.e.} t \in J\}, \tag{18}$$

$S : C([-\tau, T]; X) \longrightarrow 2^{C([-\tau,T];X)}$ by

$$S(x) = G_\varphi \circ \mathrm{Sel}_H(x). \tag{19}$$

In this paper, $x \in C([-\tau, T]; X)$ is a mild solution of (1) which means that x is a mild solution of (16) with $h \in \mathrm{Sel}_H(x)$.

The following result gives the properties of Sel_H.

Lemma 6. *Let Y be reflexive. If the hypotheses (H_1) and (H_2) are satisfied, then Sel_H is well defined, and it is weakly u.s.c. with weakly compact and convex values.*

Proof. The proof can be obtained by the same argument as in Lemma 3.3 in [10].

For the sake of simplicity, write

$$\begin{aligned} a &= M\|L^{-1}\|\|L\|, \\ b &= \frac{qM\|L^{-1}\|}{\Gamma(1+q)}, \\ c &= T^{q-(1/p)}\left(\frac{p-1}{pq-1}\right)^{1-(1/p)}. \end{aligned} \tag{20}$$

Theorem 1. *Let Y be reflexive and $p>1$ and $pq>1$. If (F_1)–(F_3) and (H_1)–(H_3) are satisfied, then system (1) has at least a mild solution.*

Proof. Let $\varphi \in C([-\tau, 0]; D(L))$ be given and set

$$\begin{aligned} D_0 &= \left\{x \in C([-\tau,T];X);\, x(t)=\varphi(t) \text{ for } t \in [-\tau,0] \text{ and } \|x(t)\| \le C_{\varphi,\eta} \text{ for all } t \in J\right\}, \\ D_{n+1} &= \overline{\operatorname{conv}} S(D_n), \quad n=0,1,\ldots, \\ D &= \bigcap_{n=0}^{\infty} D_n, \end{aligned} \tag{21}$$

where $C_{\varphi,\eta} = a\|\varphi\|_{C([-\tau,0];D(L))} + bc\|\eta\|_{L^p(0,T)}$.

One sees that D_0 is bounded, convex, and closed. For each $x \in D_0$ and $y \in S(D_0)$, there exists $h \in \mathrm{Sel}_H(x)$ such that $y = G_\varphi h$. By (H_2), it follows that

$$\|y(t)\| \le a\|\varphi\|_{C(([-\tau,0];D(L))} + b\int_0^t (t-s)^{q-1}\eta(s)\mathrm{d}s = C_{\varphi,\eta}, \quad t \in J. \tag{22}$$

This implies that $S(D_0) \subset D_0$, which enable us to obtain that $D_{n+1} \subset D_n \subset \cdots \subset D_0$. Hence, D is convex and closed.

It follows from (H_2) that

$$\|h(t)\| \le \eta(t), \quad \text{for a.e. } t \in J \text{ and } h \in \mathrm{Sel}_H(D_n). \tag{23}$$

We also see that this inequality remains true uniformly for $h \in \mathrm{Sel}_H(D_n)$. Applying Lemma 3 (i), we obtain that $S(D_n) = G_\varphi \mathrm{Sel}_H(D_n)$ is equicontinuous. And thus, D_{n+1} is also equicontinuous due to Lemma 1. Therefore, D is equicontinuous.

For given $\epsilon > 0$, thanks to (4); it follows that there exists a sequence $\{h_n\} \subset \mathrm{Sel}_H(D_n)$ such that

$$\chi(D_{n+1}(t)) = \chi(S(D_n)(t)) \le 2\chi\left(\left(G_\varphi\{h_n\}\right)(t)\right) + \varepsilon, \quad t \in [-\tau, T]. \tag{24}$$

From the definition of G_φ, we know that

$$G_\varphi h_n(t) = \begin{cases} \varphi(t), & t \in [-\tau,0], \\ \mathscr{B}(t)L\varphi(0) + \displaystyle\int_0^t (t-s)^{q-1}\mathscr{R}(t-s)h_n(s)\mathrm{d}s, & t \in J. \end{cases} \tag{25}$$

For $t \in [-\tau, 0]$, it is clear that $\chi(D_{n+1}(t)) = 0$. Also, for $t \in J$ and $s < t$, it follows from (H_2) that

$$\left\|(t-s)^{q-1}\mathscr{R}(t-s)h_n(s)\right\| \le b(t-s)^{q-1}\eta(s). \tag{26}$$

Taking $x_n \in D_n$ such that $h_n \in \mathrm{Sel}_H(x_n)$ and $x_n = G_\varphi h_n$, thanks to (H_3), one has

$$\chi\left(\left\{(t-s)^{q-1}\mathscr{R}(t-s)h_n(s)\right\} \le b(t-s)^{q-1}\mu(s)\sup_{\sigma\in[0,s]}\chi(\{x_n(\sigma)\}), \quad \text{for } t \in J, s<t. \tag{27}$$

Taking inequality (27) and Lemma 2 into account, we have that for $t \in J$,

$$\begin{aligned} \chi(D_{n+1}(t)) &\le 2\chi\left(\left(G_\varphi\{h_n\}\right)(t)\right) + \epsilon \\ &\le 4b\int_0^t (t-s)^{q-1}\mu(s)\sup_{\sigma\in[0,s]}\chi(\{x_n(\sigma)\})\mathrm{d}s + \epsilon \\ &\le 4b\mu_0\int_0^t (t-s)^{q-1}\sup_{\sigma\in[0,s]}\chi(D_n(\sigma))\mathrm{d}s + \epsilon, \end{aligned} \tag{28}$$

where $\mu_0 = \sup_{t\in J}\mu(t)$. Since ϵ is arbitrary, then for $t \in J$,

$$\chi(D_{n+1}(t)) \le 4b\mu_0\int_0^t (t-s)^{q-1}\sup_{\sigma\in[0,s]}\chi(D_n(\sigma))\mathrm{d}s. \tag{29}$$

Set

$$g(t) = 4b\mu_0\int_0^t (t-s)^{q-1}\sup_{\sigma\in[0,s]}\chi(D_n(\sigma))\mathrm{d}s. \tag{30}$$

For each $0 \le t_1 \le t$,

$$g(t)-g(t_1)=4b\mu_0\left(\int_0^{t_1}\xi^{q-1}\left(\sup_{\sigma\in[0,t-\xi]}\chi(D_n(\sigma))-\sup_{\sigma\in[0,t_1-\xi]}\chi(D_n(\sigma))\right)\mathrm{d}\xi+\int_{t_1}^{t}(t-s)^{q-1}\sup_{\sigma\in[0,s]}\chi(D_n(\sigma))\mathrm{d}s\right)\geq 0. \quad (31)$$

This means that $g(t)$ is nondecreasing in $[0, t]$. Hence,

$$\sup_{\sigma\in[0,t]}\chi(D_{n+1}(\sigma))\leq 4b\mu_0\int_0^t(t-s)^{q-1}\sup_{\sigma\in[0,s]}\chi(D_n(\sigma))\mathrm{d}s,\quad t\in J. \quad (32)$$

Taking $n\longrightarrow\infty$, we get

$$\sup_{\sigma\in[0,t]}\left(\lim_{n\longrightarrow\infty}\chi(D_{n+1}(\sigma))\right)\leq 4b\mu_0\int_0^t(t-s)^{q-1}\sup_{\sigma\in[0,s]}\left(\lim_{n\longrightarrow\infty}\chi(D_n(\sigma))\right)\mathrm{d}s,\quad t\in J. \quad (33)$$

Now, applying Gronwall's inequality with singularity, one has

$$\lim_{n\longrightarrow\infty}\chi(D_n(t))=0,\quad t\in J. \quad (34)$$

Therefore, $\chi(D(t))=0$ for each $t\in[-\tau, T]$, which yields that $D(t)$ is relatively compact in X for each $t\in[-\tau, T]$.

Thus, D is compact due to the Arzela–Ascoli theorem.

Now, we will verify that S is u.s.c. and has closed contractible values. It is easy to see that $S(D)\subset D$. Moreover, by the compactness of D, we see that G is quasi-compact. Taking $\{(x_n, y_n)\}\subset \mathrm{Gra}(S)$ with $(x_n, y_n)\longrightarrow(x, y)$, we can find $\{h_n\}\subset L^p(J; Y)$ such that $h_n\in \mathrm{Sel}_H(x_n)$ and $y_n=G_\varphi h_n$. Combining Lemma 2(ii) of [24] with Lemma 6, one can find $h\in\mathrm{Sel}_H(x)$ and a subsequence $\{h_{n_k}\}$ of $\{h_n\}$, such that $h_{n_k}\longrightarrow h$ weakly in $L^p(J; Y)$. By Lemma 3(ii), we get that $y=G_\varphi h$ which implies $y\in S(x)$. From this, we verify that S is closed. Thus, thanks to Lemma 4, one concludes that S is u.s.c.

As a consequence of the closedness of S, we can deduce that S has a closed value. Given $x\in D$ and $h_0\in\mathrm{Sel}_H(x)$. For each $y\in S(x)$, there exists $h\in\mathrm{Sel}_H(x)$ such that $y=G_\varphi h$ and put

$$x_y(t)=\mathscr{B}(t)L\varphi(0)+\int_0^{\lambda T}(t-s)^{q-1}\mathscr{R}(t-s)h(s)\mathrm{d}s+\int_{\lambda T}^{t}(t-s)^{q-1}\mathscr{R}(t-s)h_0(s)\mathrm{d}s,\quad \lambda\in[0,1], t\in(\lambda T, T]. \quad (35)$$

Let us define a function $F: [0, 1]\times S(x)\longrightarrow S(x)$ by

$$F(\lambda, y)(t)=\begin{cases} y(t), & t\in[-\tau,\lambda T],\\ x_y(t), & t\in(\lambda T, T].\end{cases} \quad (36)$$

Denoting by $h*(t):=h(t)\chi_{[0,\lambda T]}(t)+h_0(t)\chi_{[\lambda T,T]}(t)$, it follows that $h*\in\mathrm{Sel}_H(x)$ and thus $x_y\in S(x)$. From this, we conclude that F is well defined, i.e., $F(\lambda, y)\in S(x)$ for each $(\lambda, y)\in[0, 1]\times S(x)$. One can see that F is continuous, and $F(0, y)=G_\varphi h_0$ and $F(1, y)=y$ for every $y\in S(x)$. Accordingly, S has a contractible value.

Therefore, by applying the fixed point theorem of Lemma 5 on S, the conclusion of the theorem holds, which completes the proof.

Denoted by

$$\Phi(\varphi)=\{x\in C([-\tau,T]; X); x \text{ is a mild solution of (1) with } x(t) = \varphi(t), t\in[-\tau,0]\}. \quad (37)$$

We state the results of the compactness of $\Phi(\varphi)$ and the property of the map Φ.

Theorem 2. *Let all conditions in Theorem 1 hold, we have that the set $\Phi(\varphi)$ is compact and Φ is u.s.c.*

Proof. Let φ be given and $\{x_n\}\subset\Phi(\varphi)$, then there exists $h_n\in\mathrm{Sel}_H(x_n)$ such that $x_n=G_\varphi h_n$. We verify that $\{x_n\}$ is relatively compact and closed. As argued in the proof of Theorem 1, the sequence $\{x_n\}$ is equicontinuous. Moreover, we know that

$$\sup_{\sigma\in[0,t]}\chi(\{x_n(\sigma)\})\leq 2b\mu_0\int_0^t(t-s)^{q-1}\sup_{\sigma\in[0,s]}\chi(\{x_n(\sigma)\})\mathrm{d}s,\quad t\in J. \quad (38)$$

From this, we know $\chi(\{x_n(t)\})=0$ for each $t\in J$. Since $\{x_n|_{[-\tau,0]}\}=\{\varphi\}$, one can deduce that $\chi(\{x_n(t)\})=0$ for each $t\in[-\tau, 0]$. Hence, $\{x_n(t)\}$ is relatively compact for each $t\in[-\tau, T]$. Therefore, $\{x_n\}$ is relatively compact in $C([-\tau, T]; X)$. Then, there exists $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k}\longrightarrow x$. This together with Lemma 2(ii) of [24] and Lemma 6 implies that we can find a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ such that $h_{n_k}\longrightarrow h$ weakly in $L^p(J; Y)$ with $h\in\mathrm{Sel}_H(x)$. Again by Lemma 3(ii), we obtain that $x=G_\varphi h$ and thus $x\in\Phi(\varphi)$. This enable us to get that $\{x_n\}$ is closed in $C([-\tau, T]; X)$. Therefore, the compactness of $\Phi(\varphi)$ remains true.

Let $K\subset C([-\tau, 0]; D(L))$ be an compact set and $\{x_n\}\subset\Phi(K)$, take $\{\varphi_n\}\subset K$ and $h_n\in\mathrm{Sel}_H(x_n)$ such that $x_n=G_\varphi h_n$ and $x_n\in\Phi(\varphi_n)$. To apply Lemma 4, it is suffice to show that Φ is closed and quasi-compact. It is easy to know that Φ is closed. So, we only show that Φ is quasi-compact. Following the same method as above, we get $\{x_n\}$ is equicontinuous. On the one hand, one has $\chi(\{x_n(t)\}) = \chi(\{\varphi_n(t)\}) = 0$ for each $t \in [-\tau, 0]$ since the compactness of K. On the other hand, it follows that for each $t \in J$,

$$\begin{aligned}\chi\left(\{x_n(t)\}\right) &\le a\chi\left(\{\varphi_n(0)\}\right) + 2b\mu_0 \int_0^t (t-s)^{q-1} \sup_{\sigma\in[0,s]} \chi\left(\{x_n(\sigma)\}\right) ds \\ &= 2b\mu_0 \int_0^t (t-s)^{q-1} \sup_{\sigma\in[0,s]} \chi\left(\{x_n(\sigma)\}\right) ds.\end{aligned} \tag{39}$$

The last inequality implies that $\chi(\{x_n(t)\}) = 0$ for each $t \in J$ due to Gronwall's inequality with singularity. Therefore, $\chi(\{x_n(t)\}) = 0$ for each $[-\tau, T]$, which yields that $\{x_n(t)\}$ is a relatively compact subset of X. Hence, $\{x_n\}$ is compact due to Arzela–Ascoli theorem. This completes the proof. □

4. An Example

We apply the abstract results to the system of partial differential inclusion as follows:

$$\begin{cases} {}^0D_c^q\left(u(t,x) - u_{xx}(t,x)\right) - u_{xx}(t,x) \in \left[h_1\left(t,x,u_t(x)\right), h_2\left(t,x,u_t(x)\right)\right], & (t,x) \in [0,d]\times[0,\pi], \\ u(t,0) = u(t,\pi) = 0, & t \in [0,d], \\ u(t,x) = \varphi(t,x), & (t,x) \in [-\tau,0]\times[0,\pi], \end{cases} \tag{40}$$

where $(1/2) < q < 1$, $d > 0$, $\varphi \in C([-\tau, 0]; L^2(0, \pi))$, $u_t(\xi, x) = u(t+\xi, x)$, and h_i: $[0,d] \times [0,\pi] \times C([-\tau,0]; L^2(0,\pi)) \longrightarrow \mathbb{R}$ $(i = 1,2)$ are given functions such that h_1 is l.s.c., h_2 is u.s.c., and $h_1(t,x,v) \le h_2(t,x,v)$ for each $(t,x,v) \in [0,d]\times[0,\pi]\times C([-\tau,0]; L^2(0,\pi))$, and there exists $l \in L^\infty(0,T;\mathbb{R}^+)$ such that

$$\max\left\{\left|h_1(t,x,v)\right|, \left|h_2(t,x,v)\right|\right\} \le l(t), \tag{41}$$

for each $(t,x,v) \in [0,d]\times[0,\pi]\times C([-\tau,0]; L^2(0,\pi))$.

Taking $X = Y = L^2(0, \pi)$, we can rewrite system (40) abstractly in the form of system (1), where the operators A and L are defined by $Au = -u_{xx}$ and $Lu = u - u_{xx}$ with $D(A) = D(L) = \{u \in X: u,\ u_x$ are absolutely continuous, $u_{xx} \in X$ and $u(t, 0) = u(t, \pi) = 0\}$, $H: [0, d] \times C([-\tau, 0]; X) \longrightarrow 2^Y$ is given by

$$H(t,v) = \left\{h \in Y: h(x) \in \left[h_1(t,x,v), h_2(t,x,v)\right] \text{a.e. in} [0,\pi]\right\}, \tag{42}$$

for each $(t,v) \in [0,d]\times C([-\tau,0]; X)$. If we assume that H satisfies (H_3), then assumptions (H_1), (H_2) $(\eta(t) = \sqrt{\pi} l(t))$, and (H_3) hold.

From the definition of A, it follows that the spectrum $\sigma(A)$ consists of the simple eigenvalues $\lambda_k = k^2, k \in \mathbb{N}^+$ with the corresponding eigenfunctions $e_k(x) = \sqrt{2/\pi}\sin(kx)$, $x \in [0, \pi]$. Thus, we can see that

$$Ae_k = \lambda_k e_k = k^2 e_k, \left(e_k, e_j\right) = \delta_{kj}, \quad \text{for } k, j = 1,2,3,\ldots, \tag{43}$$

where $\delta_{kj} = 0$ if $k \ne j$ and $\delta_{kj} = 1$ if $k = j$. Define the projection operator P_k by

$$P_k u = (u, e_k)e_k, \quad u \in X, \tag{44}$$

then

$$I = \sum_{k=1}^{\infty} P_k. \tag{45}$$

Therefore, operator A can be explicitly expressed by

$$\begin{aligned} Au = AIu = A\left(\sum_{k=1}^{\infty} P_k\right)u &= \sum_{k=1}^{\infty} A(u,e_k)e_k \\ &= \sum_{k=1}^{\infty} k^2(u,e_k)e_k, \quad u \in D(A), \end{aligned} \tag{46}$$

and operator L is given by

$$Lu = (I + A)u = \sum_{k=1}^{\infty}\left(1+k^2\right)(u,e_k)e_k, \quad u \in D(L). \tag{47}$$

One can easily verify the following assertions:

(i) A and L are linear and closed

(ii) L^{-1} is continuous

(iii) For any $u \in X$,

$$\begin{aligned} L^{-1}u &= \sum_{k=1}^{\infty} \frac{1}{1+k^2}(u,e_k)e_k, \\ -AL^{-1}u &= \sum_{k=1}^{\infty} \frac{-k^2}{1+k^2}(u,e_k)e_k. \end{aligned} \tag{48}$$

From the definition of $-AL^{-1}$, we have

$$T(t)u = \sum_{k=1}^{\infty} e^{\left(-k^2/\left(1+k^2\right)\right)t} (u, e_k) e_k, \quad u \in X. \tag{49}$$

Obviously, $T(t)$ is equicontinuous for $t > 0$ and

$$\begin{aligned}\|T(t)u\|^2 &\leq \left\| \sum_{k=1}^{\infty} e^{-(1/2)t} (u, e_k) e_k \right\|^2 = e^{-t} \sum_{k=1}^{\infty} \left\| (u, e_k) e_k \right\|^2 \\ &= e^{-t} \|u\|^2,\end{aligned} \tag{50}$$

which yields that $\|T(t)\| \leq e^{-(1/2)t} \leq 1$.

Accordingly, all hypotheses in Theorems 1 and 2 are satisfied. Therefore, system (40) has at least a mild solution and the set of all mild solution of system (40) is compact.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors read and approved the final manuscript.

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References

[1] R. Hilfer, *Applications of Fractional Calcus in Physics*, World Scientific, Singapore, 2000.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, "Theory and applications of fractional differential equations," in *North-Holland Mathematics Studies*, Vol. 204, Elsevier Science BV., Amsterdam, The Netherlands, 2006.

[3] J. Sabatier, O. P. Agrawal, and J. A. TenreiroMachado, *Advance in Fractional Calculus*, Springer, Berlin, Germany, 2007.

[4] F. Abdolrazaghi and A. Razani, "On the weak solutions of an overdetermined system of nonlinear fractional partial integro-differential equations," *Miskolc Mathematical Notes*, vol. 20, no. 1, pp. 3–16, 2019.

[5] R. P. Agarwal, M. Benchohra, and S. Hamani, "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions," *Acta Applicandae Mathematicae*, vol. 109, no. 3, pp. 973–1033, 2010.

[6] S. D. Eidelman and A. N. Kochubei, "Cauchy problem for fractional diffusion equations," *Journal of Differential Equations*, vol. 199, no. 2, pp. 211–255, 2004.

[7] V. Lakshmikantham and A. S. Vatsala, "Basic theory of fractional differential equations," *Nonlinear Anal*, vol. 69, pp. 1677–1682, 2008.

[8] A. Razani, "Subsonic detonation waves in porous media," *Physica Scripta*, vol. 94, no. 8, p. 6, 2019.

[9] Y. Zhou, "Attractivity for fractional differential equations in Banach space," *Applied Mathematics Letters*, vol. 75, pp. 1–6, 2018.

[10] R.-N. Wang, P.-X. Zhu, and Q.-H. Ma, "Multi-valued nonlinear perturbations of time fractional evolution equations in Banach spaces," *Nonlinear Dynamics*, vol. 80, no. 4, pp. 1745–1759, 2015.

[11] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[12] A. Debbouche and J. J. Nieto, "Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls," *Applied Mathematics and Computation*, vol. 245, pp. 74–85, 2014.

[13] M. Fečkan, J. R. Wang, and Y. Zhou, "Controllability of fractional functional evolution equations of Sobolev type via characteristic solution operators," *Journal of Optimization Theory and Applications*, vol. 156, no. 1, pp. 79–95, 2013.

[14] M. R. Mokhtarzadeh, M. R. Pournaki, and A. Razani, "A note on periodic solutions of Riccati equations," *Nonlinear Dynamics*, vol. 62, no. 1-2, pp. 119–125, 2010.

[15] A. Razani, *Results in Fixed Point Theory*, Andisheh Zarrin, Ghazvin, Iran, 2010.

[16] J. Wang and Y. Zhou, "Existence and controllability results for fractional semilinear differential inclusions," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 6, pp. 3642–3653, 2011.

[17] Y. K. Chang, A. Pereira, and R. Ponce, "Approximate controllability for fractional differential equations of Sobolev type via properties on resolvent operators," *Fractional Calculus and Applied Analysis*, vol. 20, pp. 963–987, 2017.

[18] A. Cernea, "On the existence of mild solutions for nonconvex fractional semilinear differential inclusions," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 64, no. 64, pp. 1–15, 2012.

[19] R.-N. Wang, D.-H. Chen, and T.-J. Xiao, "Abstract fractional Cauchy problems with almost sectorial operators," *Journal of Differential Equations*, vol. 252, no. 1, pp. 202–235, 2012.

[20] J. Banaś and K. Goebel, "Measure of noncompactness in Banach spaces," in *Lecture Notes in Pure and Applied Mathematics*, Vol. 60, Dekker, New York. NY, USA, 1980.

[21] M. Kamenskii, V. Obukhovskii, and P. Zecca, *Condensing Multi-Valued Maps and Semilinear Differential Inclusions in Banach Spaces*, Vol. 7, Walter de Gruyter, New York, NY, USA, 2001.

[22] H. Mönch, "Boundary value problems for nonlinear ordinary differential equations of second order in Banach spaces," *Nonlinear Analysis: Theory, Methods & Applications*, vol. 4, no. 5, pp. 985–999, 1980.

[23] A. Pazy, *Semigroup of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, NY, USA, 1983.

[24] D.-H. Chen, R.-N. Wang, and Y. Zhou, "Nonlinear evolution inclusions: topological characterizations of solution sets and applications," *Journal of Functional Analysis*, vol. 265, no. 9, pp. 2039–2073, 2013.

[25] D. Bothe, "Multivalued perturbations ofm-accretive differential inclusions," *Israel Journal of Mathematics*, vol. 108, no. 1, pp. 109–138, 1998.