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Research Article

Regularity of Commutators of the One-Sided Hardy-Littlewood Maximal Functions

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In this paper, the regularity properties of two classes of commutators of the one-sided Hardy-Littlewood maximal functions and their fractional variants are investigated. Some new bounds for the derivatives of the above commutators and the boundedness and continuity for the above commutators on the Sobolev spaces will be presented. The corresponding results for the discrete analogues are also considered.

1. Introduction

The regularity theory of maximal operators has been the subject of many recent articles in harmonic analysis. One of the driving questions in this theory is whether a given maximal operator improves, preserves, or destroys a priori regularity of an initial datum f. The question was first studied by Kinnunen [1], who showed that the usual centered Hardy-Littlewood maximal function $\mathscr{M}$ is bounded on the first order Sobolev spaces $W^{1,p}(\mathbb{R}^d)$ for all $1 < p \leq \infty$. Recall that the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$, $1 \leq p \leq \infty$, are defined by

$$W^{1,p}\left(\mathbb{R}^d\right) := \left\{f : \mathbb{R}^d \longrightarrow \mathbb{R} : \|f\|_{1,p} = \|f\|_{L^p(\mathbb{R}^d)} + \|\nabla f\|_{L^p(\mathbb{R}^d)} < \infty\right\}, \tag{1}$$

where ∇f is the weak gradient of f. It was noted that the $W^{1,p}$-bound for the uncentered maximal operator $\tilde{\mathscr{M}}$ also holds by a simple modification of Kinnunen's arguments or Theorem 1 of [12]. Later on, Kinnunen's result was extended to a local version in [2], to a fractional version in [3], to a multisublinear version in [4, 5], and to a one-sided version in [6]. Due to the lack of sublinearity for $\mathscr{M}$ at the derivative level, the continuity of $\mathscr{M} : W^{1,p}(\mathbb{R}^d) \longrightarrow W^{1,p}(\mathbb{R}^d)$ for $1 < p < \infty$ is certainly a nontrivial issue. This problem was addressed by Luiro [7] in the affirmative and was later extended to the local version in [8] and the multisublinear version in [4, 9]. Other works on the regularity of maximal operators can be consulted in [10, 11]. Since the map $\mathscr{M} : L^1(\mathbb{R}^d) \longrightarrow L^1(\mathbb{R}^d)$ is not bounded, the $W^{1,1}$-regularity for the maximal operator seems to be a deeper issue. A crucial question was posed by Hajłasz and Onninen in [12]: *Is the map $f \mapsto |\nabla \mathscr{M} f|$ bounded from $W^{1,1}(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$?* A complete solution was obtained only in dimension $d = 1$ (see [13–16] for an example), and partial progress on the general dimension $d \geq 2$ was given by Hajłasz and Malý [17] and Luiro [18]. For other interesting works related to this theory, we suggest the readers to consult [19–22], among others.

Very recently, Liu et al. [23] investigated the regularity of commutators of the Hardy-Littlewood maximal function. Precisely, let b be a locally integrable function defined on $\mathbb{R}^n$, we define the commutator of the Hardy-Littlewood maximal function $[b, \mathscr{M}]$ by

$$[b, \mathscr{M}](f)(x) = b(x)\mathscr{M}f(x) - \mathscr{M}(bf)(x), \quad x \in \mathbb{R}^n. \tag{2}$$

The maximal commutator of $\mathscr{M}$ with b is defined by

$$\mathscr{M}_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x) - b(y)||f(y)|dy, \tag{3}$$

where $B(x, r)$ is the open ball in $\mathbb{R}^d$ centered at x with radius r and volume $|B(x, r)|$.

We now list the main result of [23] as follows:

Theorem 1 (see [23]). *Let $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. If $b \in W^{1,p_2}(\mathbb{R}^d)$, then*

(i) The map $[b, \mathcal{M}]: W^{1,p_1}(\mathbb{R}^d) \longrightarrow W^{1,p}(\mathbb{R}^d)$ is bounded and continuous. In particular, if $f \in W^{1,p_1}(\mathbb{R}^d)$, then

$$|\nabla[b, \mathcal{M}](f)(x)| \le |\nabla b(x)|\mathcal{M}f(x) + |b(x)|\mathcal{M}|\nabla f|(x) + \mathcal{M}|\nabla(bf)|(x), \tag{4}$$

for almost every $x \in \mathbb{R}^d$. Moreover,

$$\|[b, \mathcal{M}](f)\|_{1,p} \le C_{p_1,p_2,d}\|b\|_{1,p_2}\|f\|_{1,p_1}. \tag{5}$$

(ii) The map $\mathcal{M}_b : W^{1,p_1}(\mathbb{R}^d) \longrightarrow W^{1,p}(\mathbb{R}^d)$ is bounded. Moreover, if $f \in W^{1,p_1}(\mathbb{R}^d)$, then

$$\|\mathcal{M}_b f\|_{1,p} \le C_{p_1,p_2,d}\|b\|_{1,p_2}\|f\|_{1,p_1}. \tag{6}$$

The main motivations of this work not only extend Theorem 1 to a one-sided setting but also investigate the regularity properties of the discrete analogue for commutators of the one-sided Hardy-Littlewood maximal functions and their fractional variants. Let us recall some definitions and backgrounds. For $0 \le \beta < 1$, the one-sided fractional maximal operators $\mathcal{M}_\beta^+$ and $\mathcal{M}_\beta^-$ are defined by

$$\begin{aligned} \mathcal{M}_\beta^+ f(x) &= \sup_{s>0} \frac{1}{s^{1-\beta}} \int_x^{x+s} |f(y)|dy, \\ \mathcal{M}_\beta^- f(x) &= \sup_{r>0} \frac{1}{r^{1-\beta}} \int_{x-r}^{x} |f(y)|dy. \end{aligned} \tag{7}$$

When $\beta = 0$, the operators $\mathcal{M}_\beta^+$ (resp., $\mathcal{M}_\beta^-$) reduce to the one-sided Hardy-Littlewood maximal functions $\mathcal{M}^+$ (resp., $\mathcal{M}^-$). The study of the one-sided maximal operators originated ergodic maximal operator (see [24]). The one-sided fractional maximal operators have a close connection with the well-known Riemann-Liouville fractional integral operator and the Weyl fractional integral operator (see [25]). It was known that $\mathcal{M}_\beta^+$ is of type (p, q) for $1 < p < \infty$, $0 \le \beta < 1/p$ and $q = p/(1 - p\beta)$. For $p = 1$ we have $\mathcal{M}_\beta^+ : L^1(\mathbb{R}) \longrightarrow L^{1/(1-\beta),\infty}(\mathbb{R})$ bounded. The same conclusions hold for $\mathcal{M}_\beta^-$.

In order to establish the $W^{1,1}$-regularity for the one-dimensional uncentered Hardy-Littlewood maximal function, Tanaka [16] first studied the regularity of $\mathcal{M}^+$ and $\mathcal{M}^-$. Precisely, Tanaka proved that if $f \in W^{1,1}(\mathbb{R})$, then the distributional derivatives of $\mathcal{M}^+ f$ and $\mathcal{M}^- f$ are integrable functions, and

$$\begin{aligned} \|(\mathcal{M}^+ f)'\|_{L^1(\mathbb{R})} &\le \|f'\|_{L^1(\mathbb{R})}, \\ \|(\mathcal{M}^- f)'\|_{L^1(\mathbb{R})} &\le \|f'\|_{L^1(\mathbb{R})}. \end{aligned} \tag{8}$$

A combination of arguments in [15, 16] yields that both $\mathcal{M}^+ f$ and $\mathcal{M}^- f$ are absolutely continuous on $\mathbb{R}$. Later on, Liu and Mao [6] proved that both $\mathcal{M}^+$ and $\mathcal{M}^-$ map $W^{1,p}(\mathbb{R}) \longrightarrow W^{1,p}(\mathbb{R})$ boundedly and continuously for $1 < p < \infty$. Similar arguments to those in Remark (iii) in [1] can be used to conclude that both $\mathcal{M}^+$ and $\mathcal{M}^-$ map $W^{1,\infty}(\mathbb{R}) \longrightarrow W^{1,\infty}(\mathbb{R})$ boundedly. Recently, the main result of [6] was extended to the fractional version in [26] and to the multilinear case in [27]. We now introduce the partial result of [26] as follows:

Theorem 2 (see [26]). *Let $1 < p \le \infty$, $0 \le \beta < 1/p$, and $1/q = 1/p - \beta$. Then, the map $\mathcal{M}_\beta^+ : W^{1,p}(\mathbb{R}) \longrightarrow W^{1,q}(\mathbb{R})$ is bounded and continuous. Moreover, if $f \in W^{1,p}(\mathbb{R})$, then*

$$\left|\left(\mathcal{M}_\beta^+ f\right)'(x)\right| \le \mathcal{M}_\beta^+ f'(x), \tag{9}$$

for almost every $x \in \mathbb{R}$. The same conclusions hold for the operator $\mathcal{M}_\beta^-$.

Now we introduce two classes of commutators of the one-sided fractional maximal functions.

Definition 3. Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally integrable function and $\beta \in [0, 1)$. The commutators of the one-sided fractional maximal function $[b, \mathcal{M}_\beta^+]$ and $[b, \mathcal{M}_\beta^-]$ are defined by

$$\begin{aligned} \left[b, \mathcal{M}_\beta^+\right](f)(x) &= b(x)\mathcal{M}_\beta^+ f(x) - \mathcal{M}_\beta^+(bf)(x), \quad x \in \mathbb{R}, \\ \left[b, \mathcal{M}_\beta^-\right](f)(x) &= b(x)\mathcal{M}_\beta^- f(x) - \mathcal{M}_\beta^-(bf)(x), \quad x \in \mathbb{R}. \end{aligned} \tag{10}$$

Definition 4. Let $b : \mathbb{R} \longrightarrow \mathbb{R}$ be a locally integrable function and $\beta \in [0, 1)$. The maximal commutators of $\mathcal{M}_\beta^+$ and $\mathcal{M}_\beta^-$ with b are defined, respectively, by

$$\begin{aligned} \mathcal{M}_{b,\beta}^+ f(x) &= \sup_{r>0} \frac{1}{r^{1-\beta}} \int_x^{x+r} |b(x) - b(y)||f(y)|dy, \quad x \in \mathbb{R}, \\ \mathcal{M}_{b,\beta}^- f(x) &= \sup_{r>0} \frac{1}{r^{1-\beta}} \int_{x-r}^{x} |b(x) - b(y)||f(y)|dy, \quad x \in \mathbb{R}. \end{aligned} \tag{11}$$

It should be pointed out that the following facts are useful in proving our main results.

Remark 5. (i) The operator $[b, \mathcal{M}_\beta^+]$ is neither positive nor sublinear. By Hölder's inequality and the L^p-bounds and continuity for $\mathcal{M}_\beta^+$, we have that the map $[b, \mathcal{M}_\beta^+]: L^{p_1}(\mathbb{R}) \longrightarrow L^q(\mathbb{R})$ is bounded and continuous, provided that $1 < p_1, p_2$,

$q<\infty$, $0\le\beta<1/p_1$, $1/q=1/p_1+1/p_2-\beta$, $1/p_1+1/p_2<1$, and $b\in L^{p_2}(\mathbb{R})$. Moreover,

$$\left\|\left[b,\mathcal{M}_\beta^+\right](f)\right\|_{L^q(\mathbb{R})}\le C_{p_1,p_2,\beta}\|f\|_{L^{p_1}(\mathbb{R})}\|b\|_{L^{p_2}(\mathbb{R})}. \tag{12}$$

The same conclusions also hold for $[b,\mathcal{M}_\beta^-]$.

(ii) The operator $\mathcal{M}_{b,\beta}^+$ is positive and sublinear. Clearly

$$\mathcal{M}_{b,\beta}^+f(x)\le|b(x)|\mathcal{M}_\beta^+f(x)+\mathcal{M}_\beta^+(bf)(x),\quad \text{for all } x\in\mathbb{R}. \tag{13}$$

Inequality (13) together with Hölder's inequality, the bounds, and sublinearity of $\mathcal{M}_\beta^+$ yields that the map $\mathcal{M}_{b,\beta}^+:L^{p_1}(\mathbb{R})\longrightarrow L^q(\mathbb{R})$ is bounded if $1<p_1,p_2,q<\infty$, $0\le\beta<1/p_1$, $1/p_1+1/p_2<1$, $1/q=1/p_1+1/p_2-\beta$, and $b\in L^{p_2}(\mathbb{R})$. Moreover,

$$\left\|\mathcal{M}_{b,\beta}^+f\right\|_{L^q(\mathbb{R})}\le C_{p_1,p_2,\beta}\|f\|_{L^{p_1}(\mathbb{R})}\|b\|_{L^{p_2}(\mathbb{R})}. \tag{14}$$

The same conclusions also hold for $\mathcal{M}_{b,\beta}^-$.

Based on the above, it is a natural question to ask whether the commutators $[b,\mathcal{M}_\beta^+]$, $[b,\mathcal{M}_\beta^-]$, $\mathcal{M}_{b,\beta}^+$, and $\mathcal{M}_{b,\beta}^-$ have somewhat regularity properties. This is one main motivation of this paper, which can be addressed by the following results.

Theorem 6. *Let $1<p_1,p_2,q<\infty$, $0\le\beta<1/p_1$, $1/p_1+1/p_2<1$, and $1/q=1/p_1+1/p_2-\beta$. If $b\in W^{1,p_2}(\mathbb{R})$, then the map $[b,\mathcal{M}_\beta^+]$: $W^{1,p_1}(\mathbb{R})\longrightarrow W^{1,q}(\mathbb{R})$ is bounded and continuous. In particular, if $f\in W^{1,p_1}(\mathbb{R})$, it holds that*

$$\begin{aligned}\left|\left(\left[b,\mathcal{M}_\beta^+\right](f)\right)'(x)\right|&\le|b(x)|\mathcal{M}_\beta^+f'(x)+\left|b'(x)\right|\mathcal{M}_\beta^+f(x)\\&\quad+\mathcal{M}_\beta^+\left(bf'\right)(x)+\mathcal{M}_\beta^+\left(b'f\right)(x),\end{aligned} \tag{15}$$

for almost every $x\in\mathbb{R}$. Moreover,

$$\left\|\left[b,\mathcal{M}_\beta^+\right](f)\right\|_{1,q}\le C_{p_1,p_2,\beta}\|b\|_{1,p_2}\|f\|_{1,p_1}. \tag{16}$$

The same conclusions also hold for the operator $[b,\mathcal{M}_\beta^-]$.

Theorem 7. *Let $1<p_1,p_2,q<\infty$, $0\le\beta<1/p_1$, $1/p_1+1/p_2<1$, and $1/q=1/p_1+1/p_2-\beta$. If $b\in W^{1,p_2}(\mathbb{R})$ and $f\in W^{1,p_1}(\mathbb{R})$, then*

$$\begin{aligned}\left|\left(\mathcal{M}_{b,\beta}^+f\right)'(x)\right|&\le|b(x)|\mathcal{M}_\beta^+f'(x)+\left|b'(x)\right|\mathcal{M}_\beta^+f(x)\\&\quad+\mathcal{M}_\beta^+\left(bf'\right)(x)+\mathcal{M}_\beta^+\left(b'f\right)(x),\end{aligned} \tag{17}$$

for almost every $x\in\mathbb{R}$. Moreover,

$$\left\|\mathcal{M}_{b,\beta}^+f\right\|_{1,q}\le C_{p_1,p_2,\beta}\|b\|_{1,p_2}\|f\|_{1,p_1}. \tag{18}$$

The same conclusions also hold for the operator $\mathcal{M}_{b,\beta}^-$.

On the other hand, the investigation on the regularity of discrete maximal operators has also attracted the attention of many authors (see [6, 19, 28–33]). Let $1\le p<\infty$ and $f:\mathbb{Z}\longrightarrow\mathbb{R}$ be a discrete function, we define the ℓ^p-norm and the ℓ^∞-norm of f by

$$\begin{aligned}\|f\|_{\ell^p(\mathbb{Z})}&=\left(\sum_{n\in\mathbb{Z}}|f(n)|^p\right)^{1/p},\\\|f\|_{\ell^\infty(\mathbb{Z})}&=\sup_{n\in\mathbb{Z}}|f(n)|.\end{aligned} \tag{19}$$

Formally, we define the discrete analogue of the Sobolev spaces by

$$W^{1,p}(\mathbb{Z}):=\left\{f:\mathbb{Z}\longrightarrow\mathbb{R}\mid\|f\|_{1,p}=\|f\|_{\ell^p(\mathbb{Z})}+\left\|f'\right\|_{\ell^p(\mathbb{Z})}<\infty\right\}, \tag{20}$$

where $f'(n)=f(n+1)-f(n)$ is the first derivative of f. It is clear that

$$\|f\|_{\ell^p(\mathbb{Z})}\le\|f\|_{1,p}\le3\|f\|_{\ell^p(\mathbb{Z})},\quad \text{for all } 1\le p\le\infty. \tag{21}$$

Estimate (21) implies that the discrete Sobolev space $W^{1,p}(\mathbb{Z})$ is just the classical $\ell^p(\mathbb{Z})$ with an equivalent norm. Hence, the $W^{1,p}(\mathbb{Z})$ $(1<p\le\infty)$ regularity for discrete maximal operators is trivial. However, the situation $p=1$ is highly nontrivial. We define the total variation of f by

$$\mathrm{Var}(f)=\left\|f'\right\|_{\ell^1(\mathbb{Z})}. \tag{22}$$

We also write

$$\mathrm{Var}(f;[a,b])=\left\|f'\right\|_{\ell^1([a,b-1])}=\sum_{n=a}^{b-1}|f(n+1)-f(n)| \tag{23}$$

for the variation of f on the interval $[a,b]$, where a and b are integers (or possibly $a=-\infty$, or $b=\infty$). It is clear that $\mathrm{Var}(f;(-\infty,\infty))=\mathrm{Var}(f)$. Denote by $BV(\mathbb{Z})$ the set of functions of bounded variation defined on $\mathbb{Z}$, which is a Banach space with the norm

$$\|f\|_{BV(\mathbb{Z})}:=|f(-\infty)|+\mathrm{V}ar(f), \tag{24}$$

where $f(-\infty):=\lim_{n\to-\infty}f(n)$. Clearly,

$$\|f\|_{\ell^\infty(\mathbb{Z})}\le\|f\|_{BV(\mathbb{Z})}\le3\|f\|_{\ell^1(\mathbb{Z})}. \tag{25}$$

The study of regularity properties of discrete maximal operators began with Bober et al. [28] who studied the endpoint regularity of one dimensional discrete centered and

uncentered Hardy-Littlewood maximal operators M and $\tilde{M}$, which are defined by

$$\begin{aligned} Mf(n) &= \sup_{r\in\mathbb{N}} \frac{1}{2r+1}\sum_{k=-r}^{r} |f|(n+k)| \quad \text{and} \quad \tilde{M}f(n) \\ &= \sup_{r,s\in\mathbb{N}} \frac{1}{r+s+1}\sum_{k=-r}^{s} |f(n+k)|, \end{aligned} \tag{26}$$

where $\mathbb{N} = \{0, 1, 2, 3, \cdots\}$. It was shown in [28] that

$$\mathrm{Var}\left(\tilde{M}f\right) \le \mathrm{Var}(f), \quad \text{if } \mathrm{Var}(f) < \infty, \tag{27}$$

$$\mathrm{Var}(Mf) \le \left(2 + \frac{146}{315}\right)\|f\|_{\ell^1(\mathbb{Z})}, \quad \text{if } f \in \ell^1(\mathbb{Z}). \tag{28}$$

It was noted that inequality (27) is sharp and inequality (27) for M was proven by Temur in [33] (with constant $C = 294{,}912{,}004$). Inequality (28) was improved by Madrid [32] who obtained the sharp constant $C = 2$. Recently, Carneiro and Madrid [19] extended (28) to the fractional setting and showed that if $0 \le \beta < 1$, $q = 1/(1-\beta)$, and $f : \mathbb{Z} \longrightarrow \mathbb{R}$ is a discrete function such that $\mathrm{Var}(f) < \infty$ and $\tilde{M}_\beta f \equiv \infty$, then

$$\left\|\left(\tilde{M}_\beta f\right)'\right\|_{\ell^q(\mathbb{Z})} \le 4^{1/q}\mathrm{Var}(f), \tag{29}$$

where $\tilde{M}_\beta$ is the discrete uncentered fractional maximal operator defined by

$$\tilde{M}_\beta f(n) = \sup_{r,s\in\mathbb{N}} \frac{1}{(r+s+1)^{1-\beta}}\sum_{k=-r}^{s} |f(n+k)|. \tag{30}$$

It is currently unknown whether inequality (29) holds for the discrete centered fractional maximal operator

$$M_\beta f(n) = \sup_{r\in\mathbb{N}} \frac{1}{(2r+1)^{1-\beta}}\sum_{k=-r}^{r} |f(n+k)|. \tag{31}$$

It was pointed out in [30] that both the maps $f \mapsto (M_\beta f)'$ and $f \mapsto (\tilde{M}_\beta f)'$ (for $0 \le \beta < 1$) are bounded and continuous from $\ell^1(\mathbb{Z})$ to $\ell^1(\mathbb{Z})$. Moreover, if $f \in \ell^1(\mathbb{Z})$, then

$$\begin{aligned} \mathrm{Var}\left(M_\beta f\right) &\le 2\|f\|_{\ell^1(\mathbb{Z})}, \\ \mathrm{Var}\left(\tilde{M}_\beta f\right) &\le 2\|f\|_{\ell^1(\mathbb{Z})}, \end{aligned} \tag{32}$$

and the constants $C = 2$ are the best possible. Liu [30] pointed out that the operator $f' \mapsto (M_\beta f)'$ is not bounded from $\ell^1(\mathbb{Z})$ to $\ell^r(\mathbb{Z})$ for all $1 \le r < 1/(1-\beta)$. The continuity of $M : BV(\mathbb{Z}) \longrightarrow BV(\mathbb{Z})$ and $\tilde{M} : BV(\mathbb{Z}) \longrightarrow BV(\mathbb{Z})$ was proven by Madrid [34] and Carneiro et al. [20], respectively. Recently, Liu and Mao [6] studied the regularity of the discrete one-sided Hardy-Littlewood maximal operators and proved them.

Theorem 8 (see [6]). *Let $f : \mathbb{Z} \longrightarrow \mathbb{R}$ be a discrete function such that $Var(f) < \infty$, then*

$$Var(M^+f) \le Var(f). \tag{33}$$

Moreover, the map $f \mapsto (M^+f)'$ is continuous from $\ell^1(\mathbb{Z})$ to $\ell^1(\mathbb{Z})$. The same results also hold for M^-. Here

$$\begin{aligned} M^+f(n) &= \sup_{s\in\mathbb{N}} \frac{1}{s+1}\sum_{k=0}^{s} |f(n+k)|, \\ M^-f(n) &= \sup_{r\in\mathbb{N}} \frac{1}{r+1}\sum_{k=-r}^{0} |f(n+k)|. \end{aligned} \tag{34}$$

Very recently, Liu [26] extended Theorem 8 to the fractional setting.

Theorem 9 (see [26]). *Let $0 \le \beta < 1$. Then, the map $f \mapsto (M_\beta^+ f)'$ is bounded and continuous from $\ell^1(\mathbb{Z})$ to $\ell^1(\mathbb{Z})$. Moreover, if $f \in \ell^1(\mathbb{Z})$, then*

$$Var\left(M_\beta^+ f\right) \le 2\|f\|_{\ell^1(\mathbb{Z})}, \tag{35}$$

and the constant $C = 2$ is the best possible. The same results hold for M_β^-. Here

$$\begin{aligned} M_\beta^+ f(n) &= \sup_{s\in\mathbb{N}} \frac{1}{(s+1)^{1-\beta}}\sum_{k=0}^{s} |f(n+k)|, \\ M_\beta^- f(n) &= \sup_{r\in\mathbb{N}} \frac{1}{(r+1)^{1-\beta}}\sum_{k=-r}^{0} |f(n+k)|. \end{aligned} \tag{36}$$

The second aim of this paper is to study the regularity of the discrete analogues of $[b, \mathcal{M}_\beta^+]$ and $\mathcal{M}_{b,\beta}^+$. Let us introduce some definitions.

Definition 10. Let $b : \mathbb{Z} \longrightarrow \mathbb{R}$ be a discrete function and $\beta \in [0, 1)$. The commutators of the discrete one-sided fractional maximal function $[b, M_\beta^+]$ and $[b, M_\beta^-]$ are defined by

$$\begin{aligned} \left[b, M_\beta^+\right](f)(n) &= b(n)M_\beta^+ f(n) - M_\beta^+(bf)(n), \quad n \in \mathbb{Z}, \\ \left[b, M_\beta^-\right](f)(n) &= b(n)M_\beta^- f(n) - M_\beta^-(bf)(n), \quad n \in \mathbb{Z}. \end{aligned} \tag{37}$$

Definition 11. Let $b : \mathbb{Z} \longrightarrow \mathbb{R}$ be a discrete function. For $\beta \in [0, 1)$, the maximal commutators of $M_{b,\beta}^+$ and $M_{b,\beta}^-$ with b are defined, respectively, by

$$M_{b,\beta}^{+}f(n)=\sup_{r\in\mathbb{N}}\frac{1}{(r+1)^{1-\beta}}\sum_{k=0}^{r}|b(n)-b(n+k)|\cdot|f(n+k)|,\quad n\in\mathbb{Z},$$
$$M_{b,\beta}^{-}f(n)=\sup_{r\in\mathbb{N}}\frac{1}{(r+1)^{1-\beta}}\sum_{k=0}^{r}|b(n)-b(n-k)|\cdot|f(n-k)|,\quad n\in\mathbb{Z}. \tag{38}$$

We now formulate the rest of main results as follows:

Theorem 12. *Let $b\in BV(\mathbb{Z})$. Then, the map $[b,M^{+}]$: $BV(\mathbb{Z})\longrightarrow BV(\mathbb{Z})$ is bounded. Moreover,*

$$Var([b,M^{+}](f))\le 6\|b\|_{BV(\mathbb{Z})}\|f\|_{BV(\mathbb{Z})},\quad \text{for all } f\in BV(\mathbb{Z}). \tag{39}$$

The same conclusions also hold for the operator $[b,M^{-}]$.

Theorem 13. *Let $b\in BV(\mathbb{Z})$ and $\beta\in[0,1)$. Then, the map $[b,M_{\beta}^{+}]$: $\ell^{1}(\mathbb{Z})\longrightarrow BV(\mathbb{Z})$ is bounded and continuous. Moreover, for any $\beta\in(0,1)$, it holds that*

$$Var\left(\left[b,M_{\beta}^{+}\right](f)\right)\le 5\|b\|_{BV(\mathbb{Z})}\|f\|_{\ell^{1}(\mathbb{Z})},\quad \text{for all } f\in\ell^{1}(\mathbb{Z}). \tag{40}$$

The same conclusions also hold for the operator $[b,M_{\beta}^{-}]$.

Theorem 14. *Let $b:\mathbb{Z}\longrightarrow\mathbb{R}$ be a discrete function such that $Var(b)<\infty$ and $\beta\in[0,1)$. Then, the map $f\mapsto(M_{b,\beta}^{+}f)'$ is bounded and continuous from $\ell^{1}(\mathbb{Z})$ to $\ell^{1}(\mathbb{Z})$. Moreover,*

$$\left\|\left(M_{b,\beta}^{+}f\right)'\right\|_{\ell^{1}(\mathbb{Z})}\le 2Var(b)\|f\|_{\ell^{1}(\mathbb{Z})},\quad \text{for all } f\in\ell^{1}(\mathbb{Z}). \tag{41}$$

The same conclusions also hold for the operator $M_{b,\beta}^{-}$.

This paper will be organized as follows. Section 2 is devoted to proving Theorems 6 and 7. The proofs of Theorems 12–14 will be given in Section 3. We remark that the proofs of Theorems 6 and 7 are motivated by [21, 28]. The main ideas in the proofs of Theorems 12–14 are motivated by [6, 26], but some techniques are needed. In particular, in the proof of the continuity part of Theorem 14, we give a useful application of the Brezis-Lieb lemma in [35].

Throughout this paper, the letter C will stand for positive constants, not necessarily the same one at each occurrence but independent of the essential variables. In particular, the letter $C_{\alpha,\beta}$ denotes the positive constants that depend on the parameters α,β.

2. Proofs of Theorems 6 and 7

In this section, we shall prove Theorems 6 and 7. Before giving our proofs, let us give some notations and lemmas. Let $f\in L^{p}(\mathbb{R})$ with $p\ge 1$. For all $h\in\mathbb{R}$ with $h\ne 0$, we define

$$f_{h}(x)=\frac{f_{\tau(h)}-f(x)}{|h|}\ \text{and}\ f_{\tau(h)}(x)=f(x-h). \tag{42}$$

For convenience, we set

$$G(f;p)=\limsup_{h\to 0}\frac{\left\|f_{\tau(h)}-f\right\|_{L^{p}(\mathbb{R})}}{|b|}. \tag{43}$$

According to Section 7.11 in [36], one has that for $1<p<\infty$,

$$f\in W^{1,p}(\mathbb{R})\Leftrightarrow f\in L^{p}(\mathbb{R})\ \text{and}\ G(f;p)<\infty. \tag{44}$$

Moreover, for functions $f\in W^{1,p}(\mathbb{R})$ for $1<p\le\infty$, we have (see [36], Section 7.11) that $f_{h}\longrightarrow f'$ in $L^{p}(\mathbb{R})$ when $|h|\longrightarrow 0$.

In order to prove Theorems 6 and 7, we need the following lemma, which follows from [23].

Lemma 15. (see [23]). *Let $1<p_{1},p_{2},p<\infty$ and $1/p=1/p_{1}+1/p_{2}$. If $f\in W^{1,p_{1}}(\mathbb{R})$ and $g\in W^{1,p_{2}}(\mathbb{R})$, then $fg\in W^{1,p}(\mathbb{R})$. Moreover,*

$$(fg)'(x)=f'(x)g(x)+f(x)g'(x), \tag{45}$$

for almost every $x\in\mathbb{R}$. In particular, it holds that

$$\|fg\|_{1,p}\le\|f\|_{1,p_{1}}\|g\|_{1,p_{2}}. \tag{46}$$

Now we are in a position to prove Theorems 6 and 7.

Proof of Theorem 6. We only prove Theorem 6 for $[b,\mathscr{M}_{\beta}^{+}]$ since another case can be obtained similarly. Let $1<p_{1},p_{2},q<\infty$, $0\le\beta<1/p_{1}$, $1/q=1/p_{1}+1/p_{2}-\beta$, $1/p_{1}+1/p_{2}<1$, $b\in W^{1,p_{2}}(\mathbb{R})$, and $f\in W^{1,p_{1}}(\mathbb{R})$. Let r,p be such that $1/p=1/p_{1}+1/p_{2}$ and $1/r=1/p_{1}-\beta$. It is clear that $1/q=1/p_{2}+1/r=1/p-\beta$, $p\in(1,\infty)$, and $0\le\beta<1/p$.

(i) We first prove the bounds for $[b,\mathscr{M}_{\beta}^{+}]$. By Theorem 2, we have $\mathscr{M}_{\beta}^{+}f\in W^{1,r}(\mathbb{R})$. By Lemma 15, we have that $b\mathscr{M}_{\beta}^{+}f\in W^{1,q}(\mathbb{R})$ and $bf\in W^{1,p}(\mathbb{R})$. By Theorem 2 again, we see that $\mathscr{M}_{\beta}^{+}(bf)\in W^{1,q}(\mathbb{R})$. Hence, $[b,\mathscr{M}_{\beta}^{+}](f)\in W^{1,q}(\mathbb{R})$

(ii) We now prove (16). Applying Theorem 2, one has

$$\left\|\mathscr{M}_{\beta}^{+}f\right\|_{1,r}\le C_{p_{1},\beta}\|f\|_{1,p_{1}}, \tag{47}$$

which together with Lemma 15 yields that

$$\left\|b\mathcal{M}_\beta^+ f\right\|_{1,q} \le 2\|b\|_{1,p_2}\left\|\mathcal{M}_\beta^+ f\right\|_{1,r} \le C_{p_1,\beta}\|b\|_{1,p_2}\|f\|_{1,p_1}. \tag{48}$$

By Lemma 15 again, it holds that

$$\|bf\|_{1,p} \le 2\|b\|_{1,p_2}\|f\|_{1,p_1}, \tag{49}$$

which together with Theorem 2 implies that

$$\left\|\mathcal{M}_\beta^+(bf)\right\|_{1,q} \le C_{p,\beta}\|bf\|_{1,p} \le C_{p_1,p_2,\beta}\|b\|_{1,p_2}\|f\|_{1,p_1}. \tag{50}$$

Combining (50) with (48) leads to (16).

(iii) We now prove the continuity part. Let $f_j \longrightarrow f$ in $W^{1,p_1}(\mathbb{R})$. It suffices to show that

$$\left\|\left[b,\mathcal{M}_\beta^+\right]\left(f_j\right) - \left[b,\mathcal{M}_\beta^+\right](f)\right\|_{1,q} \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{51}$$

By Lemma 15 and applying the continuity result in Theorem 2, one can get

$$\begin{aligned}\left\|b\mathcal{M}_\beta^+ f_j - b\mathcal{M}_\beta^+ f\right\|_{1,q} &= \left\|b\left(\mathcal{M}_\beta^+ f_j - \mathcal{M}_\beta^+ f\right)\right\|_{1,q}\\ &\le 2\|b\|_{1,p_2}\left\|\mathcal{M}_\beta^+ f_j - \mathcal{M}_\beta^+ f\right\|_{1,r}\\ &\longrightarrow 0 \text{ as } j \longrightarrow \infty,\end{aligned} \tag{52}$$

$$\left\|bf_j - bf\right\|_{1,p} = \left\|b\left(f_j - f\right)\right\|_{1,p} \le 2\|b\|_{1,p_2}\left\|f_j - f\right\|_{1,p_1}. \tag{53}$$

Combining (53) with the continuity result in Theorem 2 implies that

$$\left\|\mathcal{M}_\beta^+\left(bf_j\right) - \mathcal{M}_\beta^+(bf)\right\|_{1,q} \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{54}$$

Then, (51) follows from (52) and (54).

(iv) It remains to prove (15). By Lemma 15, we have

$$(bf)'(x) = b'(x)f(x) + b(x)f'(x), \tag{55}$$

$$\begin{aligned}\left(\left[b,\mathcal{M}_\beta^+\right](f)\right)'(x) &= \left(b\mathcal{M}_\beta^+ f\right)'(x) - \left(\mathcal{M}_\beta^+(bf)\right)'(x)\\ &= b'(x)\mathcal{M}_\beta^+ f(x) + b(x)\left(\mathcal{M}_\beta^+ f\right)'(x)\\ &\quad - \left(\mathcal{M}_\beta^+(bf)\right)'(x),\end{aligned} \tag{56}$$

for almost every $x \in \mathbb{R}$. By Theorem 2, one can get

$$\left|\left(\mathcal{M}_\beta^+ f\right)'(x)\right| \le \mathcal{M}_\beta^+ f'(x), \tag{57}$$

$$\left|\left(\mathcal{M}_\beta^+(bf)\right)'(x)\right| \le \mathcal{M}_\beta^+\left((bf)'\right)(x), \tag{58}$$

for almost every $x \in \mathbb{R}$. It follows from (55)–(58) that

$$\begin{aligned}\left|\left(\left[b,\mathcal{M}_\beta^+\right](f)\right)'(x)\right| &\le |b'(x)|\mathcal{M}_\beta^+ f(x) + |b(x)|\left|\left(\mathcal{M}_\beta^+ f\right)'(x)\right|\\ &\quad + \left|\left(\mathcal{M}_\beta^+(bf)\right)'(x)\right| \le |b'(x)|\mathcal{M}_\beta^+ f(x)\\ &\quad + |b(x)|\mathcal{M}_\beta^+ f'(x) + \mathcal{M}_\beta^+\left(b'f\right)(x)\\ &\quad + \mathcal{M}_\beta^+\left(bf'\right)(x),\end{aligned} \tag{59}$$

for almost every $x \in \mathbb{R}$. This proves (15) and completes the Proof of Theorem 6.

Proof of Theorem 7. We only prove Theorem 7 for $\mathcal{M}_{b,\beta}^+$ since another case can be obtained similarly. Let $1 < p_1, p_2, q < \infty$, $0 \le \beta < 1/p_1$, $1/q = 1/p_1 + 1/p_2 - \beta$, $1/p_1 + 1/p_2 < 1$, $b \in W^{1,p_2}(\mathbb{R})$, and $f \in W^{1,p_1}(\mathbb{R})$. Let r, p be such that $1/p = 1/p_1 + 1/p_2$ and $1/r = 1/p_1 - \beta$. It is clear that $1/q = 1/p_2 + 1/r = 1/p - \beta$, $p \in (1,\infty)$ and $0 \le \beta < 1/p$.

(i) We first prove that $\mathcal{M}_{b,\beta}^+ f \in W^{1,q}(\mathbb{R})$. Fix $x, h \in \mathbb{R}$, it is easy to see that

$$\left(\mathcal{M}_{b,\beta}^+ f\right)_{\tau(h)}(x) = \sup_{r>0}\frac{1}{r^{1-\beta}}\int_x^{x+r}\left|b_{\tau(h)}(x) - b_{\tau(h)}(y)\right|\left|f_{\tau(h)}(y)\right|dy, \tag{60}$$

which gives that

$$\begin{aligned}&\left|\left(\mathcal{M}_{b,\beta}^+ f\right)_{\tau(h)}(x) - \mathcal{M}_{b,\beta}^+ f(x)\right|\\ &\quad\le \sup_{r>0}\frac{1}{r^{1-\beta}}\int_x^{x+r}\left|\left(b_{\tau(h)}(x) - b_{\tau(h)}(y)\right)f_{\tau(h)}(y)\right.\\ &\qquad\left. - (b(x) - b(y))f(y)\right|dy\\ &\quad\le \mathcal{M}_{b_{\tau(h)},\beta}^+\left(f_{\tau(h)} - f\right)(x) + \left|b_{\tau(h)}(x) - b(x)\right|\mathcal{M}_\beta^+ f(x)\\ &\qquad + \mathcal{M}_\beta^+\left(\left(b_{\tau(h)} - b\right)f\right)(x).\end{aligned} \tag{61}$$

By (61), Hölder's inequality, and the bounds for $\mathcal{M}_{b,\beta}^+$ and $\mathcal{M}_\beta^+$, one can get

$$\begin{aligned}
&\left\|\left(\mathcal{M}_{b,\beta}^{+}f\right)_{\tau(h)}-\mathcal{M}_{b,\beta}^{+}f\right\|_{L^q(\mathbb{R})}\\
&\quad\le\left\|\mathcal{M}_{b_{\tau(h)},\beta}^{+}\left(f_{\tau(h)}-f\right)\right\|_{L^q(\mathbb{R})}+\left\|\left(b_{\tau(h)}-b\right)\mathcal{M}_{\beta}^{+}f\right\|_{L^q(\mathbb{R})}\\
&\qquad+\left\|\mathcal{M}_{\beta}^{+}\left(\left(b_{\tau(h)}-b\right)f\right)\right\|_{L^q(\mathbb{R})}\\
&\quad\le C_{p_1,p_2,\beta}\left(\left\|b_{\tau(h)}\right\|_{L^{p_2}(\mathbb{R})}\left\|f_{\tau(h)}-f\right\|_{L^{p_1}(\mathbb{R})}\right.\\
&\qquad\left.+\left\|b_{\tau(h)}-b\right\|_{L^{p_2}(\mathbb{R})}\left\|\mathcal{M}_{\beta}^{+}f\right\|_{L^r(\mathbb{R})}+\left\|\left(b_{\tau(h)}-b\right)f\right\|_{L^p(\mathbb{R})}\right)\\
&\quad\le C_{p_1,p_2,\beta}\left(\|b\|_{L^{p_2}(\mathbb{R})}\left\|f_{\tau(h)}-f\right\|_{L^{p_1}(\mathbb{R})}\right.\\
&\qquad\left.+\left\|b_{\tau(h)}-b\right\|_{L^{p_2}(\mathbb{R})}\|f\|_{L^{p_1}(\mathbb{R})}\right).
\end{aligned}\tag{62}$$

Because of $f\in W^{1,p_1}(\mathbb{R})$ and $b\in W^{1,p_2}(\mathbb{R})$, then by (44), we can get

$$G(f;p_1)<\infty,\quad G(b;p_2)<\infty.\tag{63}$$

It follows from (62) and (63) that

$$G\left(\mathcal{M}_{b,\beta}^{+}f;q\right)<\infty.\tag{64}$$

Combining (64) with (44) and the bounds for $\mathcal{M}_{b,\beta}^{+}$ yields $\mathcal{M}_{b,\beta}^{+}f\in W^{1,q}(\mathbb{R})$.

(ii) We now prove (17). Since $\mathcal{M}_{b,\beta}^{+}f\in W^{1,q}(\mathbb{R})$, $f\in W^{1,p_1}(\mathbb{R})$ and $b\in W^{1,p_2}(\mathbb{R})$, then $f_h\longrightarrow f'$ in $L^{p_1}(\mathbb{R})$ as $h\longrightarrow 0$, $b_{\tau(h)}\longrightarrow b$ and $b_h\longrightarrow b'$ in $L^{p_2}(\mathbb{R})$ as $h\longrightarrow 0$, $b_{\tau(h)}f_h\longrightarrow bf'$ in $L^p(\mathbb{R})$ as $h\longrightarrow 0$, $\mathcal{M}_{\beta}^{+}(f_h-f')\longrightarrow 0$ in $L^r(\mathbb{R})$ as $h\longrightarrow 0$, $\mathcal{M}_{\beta}^{+}(b_hf-b'f)\longrightarrow 0$ in $L^q(\mathbb{R})$ as $h\longrightarrow 0$, $\mathcal{M}_{\beta}^{+}(b_{\tau(h)}f_h-bf')\longrightarrow 0$ in $L^q(\mathbb{R})$ as $h\longrightarrow 0$, $(\mathcal{M}_{b,\beta}^{+}f)_h\longrightarrow(\mathcal{M}_{b,\beta}^{+}f)'$ in $L^q(\mathbb{R})$ when $h\longrightarrow 0$. Therefore, there exists a sequence of real numbers $\{h_k\}$ satisfying $\lim_{k\to\infty}h_k=0$ and a measurable set E satisfying $|\mathbb{R}\setminus E|=0$ such that $b_{\tau(h_k)}(x)\longrightarrow b(x)$, $b_{h_k}(x)\longrightarrow b'(x)$, $\mathcal{M}_{\beta}^{+}(f_{h_k}-f')(x)\longrightarrow 0$, $\mathcal{M}_{\beta}^{+}(b_{\tau(h_k)}f_{h_k}-bf')(x)\longrightarrow 0$, and $(\mathcal{M}_{b,\beta}^{+}f)_{h_k}(x)\longrightarrow(\mathcal{M}_{b,\beta}^{+}f)'(x)$ as $k\longrightarrow\infty$ for all $x\in E$. From (61) and (13) we have that for all $h\in\mathbb{R}$

$$\begin{aligned}
&\left|\left(\mathcal{M}_{b,\beta}^{+}f\right)_{\tau(h)}(x)-\mathcal{M}_{b,\beta}^{+}f(x)\right|\\
&\quad\le\left|b_{\tau(h)}(x)\right|\mathcal{M}_{\beta}^{+}\left(f_{\tau(h)}-f\right)(x)\\
&\qquad+\mathcal{M}_{\beta}^{+}\left(b_{\tau(h)}\left(f_{\tau(h)}-f\right)\right)(x)\\
&\qquad+\left|b_{\tau(h)}(x)-b(x)\right|\mathcal{M}_{\beta}^{+}f(x)\\
&\qquad+\mathcal{M}_{\beta}^{+}\left(\left(b_{\tau(h)}-b\right)f\right)(x).
\end{aligned}\tag{65}$$

It follows that

$$\begin{aligned}
\left|\left(\mathcal{M}_{b,\beta}^{+}f\right)'(x)\right|&=\left|\lim_{k\to\infty}\left(\mathcal{M}_{b,\beta}^{+}f\right)_{h_k}(x)\right|\\
&\le\lim_{k\to\infty}\left(\left|b_{\tau(h_k)}(x)\right|\mathcal{M}_{\beta}^{+}f_{h_k}(x)\right.\\
&\qquad+\left|b_{h_k}(x)\right|\mathcal{M}_{\beta}^{+}f(x)+\mathcal{M}_{\beta}^{+}\left(b_{\tau(h_k)}f_{h_k}\right)(x)\\
&\qquad\left.+\mathcal{M}_{\beta}^{+}\left(b_{h_k}f\right)(x)\right)=|b(x)|\mathcal{M}_{\beta}^{+}f'(x)\\
&\qquad+\left|b'(x)\right|\mathcal{M}_{\beta}^{+}f(x)+\mathcal{M}_{\beta}^{+}\left(bf'\right)(x)\\
&\qquad+\mathcal{M}_{\beta}^{+}\left(b'f\right)(x),
\end{aligned}\tag{66}$$

for any $x\in E$, which gives (17).

(iii) By (17), the bounds for $\mathcal{M}_{\beta}^{+}$, and Hölder's inequality, one can get

$$\begin{aligned}
\left\|\left(\mathcal{M}_{b,\beta}^{+}f\right)'\right\|_{L^q(\mathbb{R})}&\le\left\|b\mathcal{M}_{\beta}^{+}f'\right\|_{L^q(\mathbb{R})}+\left\|b'\mathcal{M}_{\beta}^{+}f\right\|_{L^q(\mathbb{R})}\\
&\qquad+\left\|\mathcal{M}_{\beta}^{+}\left(bf'\right)\right\|_{L^q(\mathbb{R})}+\left\|\mathcal{M}_{\beta}^{+}\left(b'f\right)\right\|_{L^q(\mathbb{R})}\\
&\le\|b\|_{L^{p_2}(\mathbb{R})}\left\|\mathcal{M}_{\beta}^{+}f'\right\|_{L^r(\mathbb{R})}\\
&\qquad+\|b'\|_{L^{p_2}(\mathbb{R})}\left\|\mathcal{M}_{\beta}^{+}f\right\|_{L^r(\mathbb{R})}\\
&\qquad+C_{p,\beta}\left(\|bf'\|_{L^p(\mathbb{R})}+\|b'f\|_{L^p(\mathbb{R})}\right)\\
&\le C_{p_1,p_2,\beta}\|b\|_{1,p_2}\|f\|_{1,p_1},
\end{aligned}\tag{67}$$

which together with (14) yields (18).

3. Proofs of Theorems 12–14

This section is devoted to presenting the proofs of Theorems 12–14.

Proof of Theorem 12. It is clear that

$$\begin{aligned}
(bf)'(n)&=b(n+1)f(n+1)-b(n)f(n)\\
&=b'(n)f(n+1)+b(n)f'(n),
\end{aligned}\tag{68}$$

for all $n\in\mathbb{Z}$. By (68) one has

$$\left\|(bf)'\right\|_{\ell^1(\mathbb{Z})}\le\mathrm{Var}(b)\|f\|_{\ell^\infty(\mathbb{Z})}+\|b\|_{\ell^\infty(\mathbb{Z})}\mathrm{Var}(f).\tag{69}$$

By (69) and Theorem 8, it holds that

$$\begin{aligned}\operatorname{Var}([b,M^+](f)) &= \left\|([b,M^+](f))'\right\|_{\ell^1(\mathbb{Z})}\\ &\le \left\|(bM^+f)'\right\|_{\ell^1(\mathbb{Z})} + \left\|(M^+(bf))'\right\|_{\ell^1(\mathbb{Z})}\\ &\le \operatorname{Var}(b)\|M^+f\|_{\ell^\infty(\mathbb{Z})} + \|b\|_{\ell^\infty(\mathbb{Z})}\operatorname{Var}(M^+f)0\\ &\quad + \operatorname{Var}(b)\|f\|_{\ell^\infty(\mathbb{Z})} + \|b\|_{\ell^\infty(\mathbb{Z})}\operatorname{Var}(f)\\ &\le 2\left(\operatorname{Var}(b)\|f\|_{\ell^\infty(\mathbb{Z})} + \|b\|_{\ell^\infty(\mathbb{Z})}\operatorname{Var}(f)\right).\end{aligned} \tag{70}$$

On the other hand, one can easily check that

$$\begin{aligned}&\left|\operatorname{Var}(b)\|f\|_{\ell^\infty(\mathbb{Z})} + \|b\|_{\ell^\infty(\mathbb{Z})}\operatorname{Var}(f)\right|\\ &\quad\le |b(n)|M^+f(n) + M^+(bf)(n)\\ &\quad\le 2\|b\|_{\ell^\infty(\mathbb{Z})}\|f\|_{\ell^\infty(\mathbb{Z})},\end{aligned} \tag{71}$$

which together with (70) and (25) yields that

$$\|[b,M^+](f)\|_{BV(\mathbb{Z})} \le 6\|b\|_{BV(\mathbb{Z})}\|f\|_{BV(\mathbb{Z})}. \tag{72}$$

Proof of Theorem 13. (i) It is clear that

$$\|bf\|_{\ell^1(\mathbb{Z})} \le \|b\|_{\ell^\infty(\mathbb{Z})}\|f\|_{\ell^1(\mathbb{Z})}. \tag{73}$$

One can easily check that

$$M_\beta^+f(n) \le \|f\|_{\ell^1(\mathbb{Z})}, \tag{74}$$

for all $n \in \mathbb{Z}$. In light of (73) and (74) we would have

$$\begin{aligned}\left\|\left[b,M_\beta^+\right](f)\right\|_{\ell^\infty(\mathbb{Z})} &\le \|b\|_{\ell^\infty(\mathbb{Z})}\left\|M_\beta^+f\right\|_{\ell^\infty(\mathbb{Z})} + \left\|M_\beta^+(bf)\right\|_{\ell^\infty(\mathbb{Z})}\\ &\le \|b\|_{\ell^\infty(\mathbb{Z})}\|f\|_{\ell^1(\mathbb{Z})} + \|bf\|_{\ell^1(\mathbb{Z})}\\ &\le 2\|b\|_{\ell^\infty(\mathbb{Z})}\|f\|_{\ell^1(\mathbb{Z})}.\end{aligned} \tag{75}$$

On the other hand, by Theorem 9 and (73), (69), and (25), we have

$$\begin{aligned}&\operatorname{Var}\left(\left[b,M_\beta^+\right](f)\right) = \left\|\left(\left[b,M_\beta^+\right](f)\right)'\right\|_{\ell^1(\mathbb{Z})}\\ &\quad\le \left\|\left(bM_\beta^+f\right)'\right\|_{\ell^1(\mathbb{Z})} + \left\|\left(M_\beta^+(\mathrm{b}f)\right)'\right\|_{\ell^1(\mathbb{Z})}\\ &\quad\le \operatorname{Var}(b)\left\|M_\beta^+f\right\|_{\ell^\infty(\mathbb{Z})} + \|b\|_{\ell^\infty(\mathbb{Z})}\operatorname{Var}\left(M_\beta^+f\right) + 2\|bf\|_{\ell^1(\mathbb{Z})}\\ &\quad\le 5\|b\|_{BV(\mathbb{Z})}\|f\|_{\ell^1(\mathbb{Z})}.\end{aligned} \tag{76}$$

Combining (76) with (75) yields that the map $[b,M_\beta^+]$: $\ell^1(\mathbb{Z}) \longrightarrow BV(\mathbb{Z})$ is bounded.

(iii) We now prove the continuity result. Let $f \in \ell^1(\mathbb{Z})$, $\{f_j\}_{j=1}^\infty \subset \ell^1(\mathbb{Z})$, and $f_j \longrightarrow f$ in $\ell^1(\mathbb{Z})$ as $j \longrightarrow \infty$. We want to show that

$$\left\|\left[b,M_\beta^+\right]\left(f_j\right) - \left[b,M_\beta^+\right](f)\right\|_{BV(\mathbb{Z})} \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{77}$$

By the sublinearity of M_β^+ and (25), (73), and (74), it holds that

$$\begin{aligned}&\left|\left[b,M_\beta^+\right]\left(f_j\right)(n) - \left[b,M_\beta^+\right](f)(n)\right|\\ &\quad\le |b(n)|\left|M_\beta^+f_j(n) - M_\beta^+f(n)\right| + \left|M_\beta^+\left(bf_j\right)(n) - M_\beta^+(bf)(n)\right|\\ &\quad\le |b(n)|M_\beta^+\left(f_j - f\right)(n) + M_\beta^+\left(b\left(f_j - f\right)\right)(n)\\ &\quad\le \|b\|_{\ell^\infty(\mathbb{Z})}\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})} + \left\|b\left(f_j - f\right)\right\|_{\ell^1(\mathbb{Z})}\\ &\quad\le 2\|b\|_{BV(\mathbb{Z})}\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})},\end{aligned} \tag{78}$$

for all $n \in \mathbb{Z}$. From (78), we see that $[b,M_\beta^+](f_j)(n) \longrightarrow [b,M_\beta^+](f)(n)$ uniformly for $n \in \mathbb{Z}$. Therefore, to prove (77), it suffices to show that

$$\operatorname{Var}\left(\left[b,M_\beta^+\right]\left(f_j\right) - \left[b,M_\beta^+\right](f)\right) \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{79}$$

By (73) and (25), we get

$$\begin{aligned}\left\|bf_j - bf\right\|_{\ell^1(\mathbb{Z})} &= \left\|b\left(f_j - f\right)\right\|_{\ell^1(\mathbb{Z})} \le 2\|b\|_{\ell^\infty(\mathbb{Z})}\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})}\\ &\le 2\|b\|_{BV(\mathbb{Z})}\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})},\end{aligned} \tag{80}$$

which yields that $bf_j \longrightarrow bf$ in $\ell^1(\mathbb{Z})$ as $j \longrightarrow \infty$. This together with Theorem 9 implies that

$$\operatorname{Var}\left(M_\beta^+\left(bf_j\right) - M_\beta^+(bf)\right) \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{81}$$

On the other hand, by Theorem 9 again, we get

$$\operatorname{Var}\left(M_\beta^+f_j - M_\beta^+f\right) \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{82}$$

By (69), (74), and (25) and the sublinearity for M_β^+, we have

$$\begin{aligned}
&\operatorname{Var}\left(bM_\beta^+ f_j - bM_\beta^+ f\right)\\
&\quad = \left\|\left(bM_\beta^+ f_j - bM_\beta^+ f\right)'\right\|_{\ell^1(\mathbb{Z})}\\
&\quad = \left\|\left(b\left(M_\beta^+ f_j - M_\beta^+ f\right)\right)'\right\|_{\ell^1(\mathbb{Z})}\\
&\quad \le \operatorname{Var}(b)\left\|M_\beta^+ f_j - M_\beta^+ f\right\|_{\ell^\infty(\mathbb{Z})}\\
&\qquad + \|b\|_{\ell^\infty(\mathbb{Z})}\operatorname{Var}\left(M_\beta^+ f_j - M_\beta^+ f\right)\\
&\quad \le \|b\|_{BV(\mathbb{Z})}\left(\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})} + \operatorname{Var}\left(M_\beta^+ f_j - M_\beta^+ f\right)\right).
\end{aligned} \tag{83}$$

This together with (82) yields that

$$\operatorname{Var}\left(bM_\beta^+ f_j - bM_\beta^+ f\right) \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{84}$$

Combining (84) with (81) implies that

$$\begin{aligned}
&\operatorname{Var}\left(\left[b, M_\beta^+\right]\left(f_j\right) - \left[b, M_\beta^+\right](f)\right)\\
&\quad = \left\|\left(\left[b, M_\beta^+\right]\left(f_j\right) - \left[b, M_\beta^+\right](f)\right)'\right\|_{\ell^1(\mathbb{Z})}\\
&\quad \le \left\|\left(bM_\beta^+ f_j - bM_\beta^+ f\right)'\right\|_{\ell^1(\mathbb{Z})}\\
&\qquad + \left\|\left(M_\beta^+\left(bf_j\right) - M_\beta^+(bf)\right)'\right\|_{\ell^1(\mathbb{Z})}\\
&\quad = \operatorname{Var}\left(bM_\beta^+ f_j - bM_\beta^+ f\right) + \operatorname{Var}\left(M_\beta^+\left(bf_j\right) - M_\beta^+(bf)\right)\\
&\quad \longrightarrow 0 \text{ as } j \longrightarrow \infty,
\end{aligned} \tag{85}$$

which proves (79) and finishes the Proof of Theorem 13.

Proof of Theorem 14. We only prove Theorem 14 for $M_{b,\beta}^+$ since another one is analogous. The proof will be divided into two steps:

Step 1. Proof of the boundedness part. Let $f \in \ell^1(\mathbb{Z})$. Without loss of generality we may assume $f \ge 0$ since $M_{b,\beta}^+ f = M_{b,\beta}^+ |f|$. For convenience, we define the function $\Gamma : [0, \infty) \longrightarrow \mathbb{R}$ by $\Gamma(x) = (x+1)^{\beta-1} - (x+2)^{\beta-1}$ for any $x \ge 0$. It is clear that $\Gamma(x)$ is decreasing on $[0, \infty)$ and $\sum_{n\in\mathbb{N}} \Gamma(n) = 1$. Fix $n \in \mathbb{Z}$ and $r \in \mathbb{N}$, it holds that

$$\begin{aligned}
&\frac{1}{(r+1)^{1-\beta}}\sum_{k=0}^{r} |b(n) - b(n+k)||f(n+k)|\\
&\qquad \le \operatorname{Var}(b)\|f\|_{\ell^1(\mathbb{Z})}(r+1)^{\beta-1}.
\end{aligned} \tag{86}$$

This yields that for any fixed $n \in \mathbb{Z}$, there exists $r_n \in \mathbb{N}$ such that

$$M_{b,\beta}^+ f(n) = A_{r_n} f(n) := (r_n+1)^{\beta-1}\sum_{k=0}^{r_n} |b(n) - b(n+k)| f(n+k). \tag{87}$$

Let

$$\begin{aligned}
X^+ &= \left\{n \in \mathbb{Z} : M_{b,\beta}^+ f(n+1) > M_{b,\beta}^+ f(n)\right\},\\
X^- &= \left\{n \in \mathbb{Z} : M_{b,\beta}^+ f(n) > M_{b,\beta}^+ f(n+1)\right\}.
\end{aligned} \tag{88}$$

Then, we have

$$\begin{aligned}
\left\|\left(M_{b,\beta}^+ f\right)'\right\|_{\ell^1(\mathbb{Z})} &= \sum_{n\in X^+}\left(M_{b,\beta}^+ f(n+1) - M_{b,\beta}^+ f(n)\right)\\
&\quad + \sum_{n\in X^-}\left(M_{b,\beta}^+ f(n) - M_{b,\beta}^+ f(n+1)\right).
\end{aligned} \tag{89}$$

We can write

$$\begin{aligned}
&\sum_{n\in X^+}\left(M_{b,\beta}^+ f(n+1) - M_{b,\beta}^+ f(n)\right)\\
&\quad \le \sum_{n\in X^+}\left(A_{r_{n+1}} f(n+1) - A_{r_{n+1}+1} f(n)\right)\\
&\quad = \sum_{n\in X^+}\Bigg((r_{n+1}+1)^{\beta-1}\sum_{k\in\mathbb{Z}} |b(n+1)\\
&\qquad - b(k)|f(k)\chi_{[n+1,n+r_{n+1}+1]}(k)\\
&\qquad - (r_{n+1}+2)^{\beta-1}\sum_{k\in\mathbb{Z}} |b(n) - b(k)|f(k)\chi_{[n,n+r_{n+1}+1]}(k)\Bigg)\\
&\quad \le \sum_{n\in X^+}|b(n+1) - b(n)|(r_{n+1}+1)^{\beta-1}\sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_{n+1}+1]}(k)\\
&\qquad + \operatorname{Var}(b)\sum_{n\in X^+}\Gamma(r_{n+1})\sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_{n+1}+1]}(k),
\end{aligned} \tag{90}$$

$$\begin{aligned}
&\sum_{n\in X^-}\left(M_{b,\beta}^+ f(n) - M_{b,\beta}^+ f(n+1)\right)\\
&\quad \le \sum_{n\in X^-}\left(A_{r_n}(f)(n) - A_{r_n+1}(f)(n+1)\right)\\
&\quad = \sum_{n\in X^-}\Bigg((r_n+1)^{\beta-1}\sum_{k\in\mathbb{Z}} |b(n) - b(k)|f(k)\chi_{[n,n+r_n]}(k)\\
&\qquad - (r_n+2)^{\beta-1}\sum_{k\in\mathbb{Z}} |b(n+1) - b(k)|f(k)\chi_{[n+1,n+r_n+2]}(k)\Bigg)\\
&\quad \le \sum_{n\in X^-}\Bigg((r_n+1)^{\beta-1}\sum_{k\in\mathbb{Z}} |b(n) - b(k)|f(k)\chi_{[n+1,n+r_n+1]}(k)\\
&\qquad - (r_n+2)^{\beta-1}\sum_{k\in\mathbb{Z}} |b(n+1) - b(k)|f(k)\chi_{[n+1,n+r_n+2]}(k)\Bigg)\\
&\quad \le \sum_{n\in X^-}|b(n+1) - b(n)|(r_n+1)^{\beta-1}\sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_n+1]}(k)\\
&\qquad + \sum_{n\in X^-}\Gamma(r_n)\sum_{k\in\mathbb{Z}} |b(n+1) - b(k)|f(k)\chi_{[n+1,n+r_n+1]}(k).
\end{aligned} \tag{91}$$

By Hölder's inequality with exponents $p = 1/\beta$ and $p' = 1/(1-\beta)$ and the fact that $\ell^q(\mathbb{Z}) \subseteq \ell^p(\mathbb{Z})$ for all $1 \le q \le p \le \infty$, it holds that

$$(r_{n+1}+1)^{\beta-1} \sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_{n+1}+1]}(k) \le \|f\|_{\ell^{1/\beta}(\mathbb{Z})} \le \|f\|_{\ell^1(\mathbb{Z})}, \tag{92}$$

$$(r_n+1)^{\beta-1} \sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_n+1]}(k) \le \|f\|_{\ell^{1/\beta}(\mathbb{Z})} \le \|f\|_{\ell^1(\mathbb{Z})}. \tag{93}$$

In light of (90) and (92), we would have

$$\begin{aligned}\sum_{n\in X^+} &\left(M^+_{b,\beta}f(n+1) - M^+_{b,\beta}f(n)\right)\\ &\le \|f\|_{\ell^1(\mathbb{Z})} \sum_{n\in X^+} |b(n+1)-b(n)|\\ &\quad + \mathrm{Var}(b) \sum_{n\in X^+} \Gamma(r_{n+1}) \sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_{n+1}+1]}(k).\end{aligned} \tag{94}$$

By (91) and (93), we have

$$\begin{aligned}\sum_{n\in X^-} &\left(M^+_{b,\beta}f(n) - M^+_{b,\beta}f(n+1)\right)\\ &\le \|f\|_{\ell^1(\mathbb{Z})} \sum_{n\in X^-} |b(n+1)-b(n)|\\ &\quad + \mathrm{Var}(b) \sum_{n\in X^-} \Gamma(r_n) \sum_{k\in\mathbb{Z}} f(k)\chi_{[n+1,n+r_n+1]}(k).\end{aligned} \tag{95}$$

It follows from (89), (94), and (95) that

$$\begin{aligned}&\left\|\left(M^+_{b,\beta}f\right)'\right\|_{\ell^1(\mathbb{Z})}\\ &\le \sum_{n\in\mathbb{Z}} |b(n+1)-b(n)| \|f\|_{\ell^1(\mathbb{Z})}\\ &\quad + \mathrm{Var}(b) \sum_{n\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} f(k)\Gamma(k-n-1)\chi_{[n+1,\infty)}(k)\\ &\le \mathrm{Var}(b)\|f\|_{\ell^1(\mathbb{Z})} + \mathrm{Var}(b) \sum_{k\in\mathbb{Z}} f(k) \sum_{n\in\mathbb{Z}} \Gamma(k-n-1)\chi_{[n+1,\infty)}(k).\end{aligned} \tag{96}$$

Fix $k \in \mathbb{Z}$, one can easily check that

$$\sum_{n\in\mathbb{Z}} \Gamma(k-n-1)\chi_{[n+1,\infty)}(k) = \sum_{n=-\infty}^{k-1} \Gamma(k-n-1) = 1. \tag{97}$$

This together with (96) yields that

$$\left\|\left(M^+_{b,\beta}f\right)'\right\|_{\ell^1(\mathbb{Z})} \le 2\mathrm{Var}(b)\|f\|_{\ell^1(\mathbb{Z})}. \tag{98}$$

This proves (41).

Step 2. Proof of the continuity part. Let $f \in \ell^1(\mathbb{Z})$, $\{f_j\}_{j=1}^{\infty} \subset \ell^1(\mathbb{Z})$, and $f_j \longrightarrow f$ in $\ell^1(\mathbb{Z})$ when $j \longrightarrow \infty$. Without loss of generality, we may assume that all $f_j \ge 0$ and $f \ge 0$ since $||f_j| - |f|| \le |f_j - f|$. We want to show that

$$\left\|\left(M^+_{b,\beta}f_j\right)' - \left(M^+_{b,\beta}f\right)'\right\|_{\ell^1(\mathbb{Z})} \longrightarrow 0 \text{ as } j \longrightarrow \infty. \tag{99}$$

By the sublinearity of $M^+_{b,\beta}$, we can get

$$\begin{aligned}\left|M^+_{b,\beta}f_j(n) - M^+_{b,\beta}f(n)\right| &\le M^+_{b,\beta}\left(f_j - f\right)(n)\\ &\le \mathrm{Var}(b)\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})},\end{aligned} \tag{100}$$

which yields that $M^+_{b,\beta}f_j \longrightarrow M^+_{b,\beta}f$ uniformly in $\mathbb{Z}$. Fix $\varepsilon \in (0,1)$, there exists $N_1 > 0$ depending only on ε, f such that

$$\left\|f_j - f\right\|_{\ell^1(\mathbb{Z})} < \varepsilon, \quad \text{for all } j \ge N_1; \tag{101}$$

$$\begin{aligned}\left\|f_j\right\|_{\ell^1(\mathbb{Z})} &\le \left\|f_j - f\right\|_{\ell^1(\mathbb{Z})} + \|f\|_{\ell^1(\mathbb{Z})}\\ &\le \|f\|_{\ell^1(\mathbb{Z})} + 1, \quad \text{for all } j \ge \mathrm{N}_1.\end{aligned} \tag{102}$$

It follows from (100) that

$$\left(M^+_{b,\beta}f_j\right)'(n) \longrightarrow \left(M^+_{b,\beta}f\right)'(n) \text{ as } j \longrightarrow \infty, \tag{103}$$

for all $n \in \mathbb{Z}$. By the boundedness part, we have that $(M^+_{b,\beta}f)' \in \ell^1(\mathbb{Z})$. This together with the classical Brezis-Lieb lemma in [3] implies that (99) reduces to the following:

$$\lim_{j\to\infty} \left\|\left(M^+_{b,\beta}f_j\right)'\right\|_{\ell^1(\mathbb{Z})} = \left\|\left(M^+_{b,\beta}f\right)'\right\|_{\ell^1(\mathbb{Z})}. \tag{104}$$

By (103) and Fatou's lemma, one can get

$$\left\|\left(M^+_{b,\beta}f\right)'\right\|_{\ell^1(\mathbb{Z})} \le \liminf_{j\to\infty} \left\|\left(M^+_{b,\beta}f_j\right)'\right\|_{\ell^1(\mathbb{Z})}. \tag{105}$$

Therefore, to prove (104), it suffices to show that

$$\limsup_{j\to\infty} \left\|\left(M^+_{b,\beta}f_j\right)'\right\|_{\ell^1(\mathbb{Z})} \le \left\|\left(M^+_{b,\beta}f\right)'\right\|_{\ell^1(\mathbb{Z})}. \tag{106}$$

We now prove (106). Since $f \in \ell^1(\mathbb{Z})$, then there exists a sufficiently large positive integer $\Lambda_1 > 0$ depending only on ε, f such that

$$\sum_{|n|\ge\Lambda_1} f(n) < \varepsilon. \tag{107}$$

There exists an integer $\Lambda_2 > 0$ such that $s^{\beta-1} < \varepsilon$ if $s \ge \Lambda_2$. Since $\mathrm{Var}(b) < \infty$, then there exists a large positive integer Λ_3 depending only on b, ε such that

$$\sum_{|n|\ge\Lambda_3} |b(n+1) - b(n)| < \varepsilon. \tag{108}$$

Let $\Lambda = \max\{\Lambda_1, \Lambda_2, \Lambda_3\}$. By (103), there exists $N_2 > 0$ depending only on ε, Λ such that

$$\begin{aligned}&\left|\left(M_{b,\beta}^{+}f_j\right)'(n) - \left(M_{b,\beta}^{+}f\right)'(n)\right| \\ &\quad \le \frac{\varepsilon}{4\Lambda+2}, \quad \text{for all } |n| \le 2\Lambda,\ j \ge N_2.\end{aligned} \tag{109}$$

It follows from (109) that

$$\begin{aligned}\left\|\left(M_{b,\beta}^{+}f_j\right)'\right\|_{\ell^1(\mathbb{Z})} &\le \sum_{|n|\le 2\Lambda}\left|\left(M_{b,\beta}^{+}f_j\right)'(n) - \left(M_{b,\beta}^{+}f\right)'(n)\right| \\ &\quad + \left\|\left(M_{b,\beta}^{+}f\right)'\right\|_{\ell^1(\mathbb{Z})} \\ &\quad + \sum_{|n|\ge 2\Lambda}\left|\left(M_{b,\beta}^{+}f_j\right)'(n)\right| \\ &\le \varepsilon + \left\|\left(M_{b,\beta}^{+}f\right)'\right\|_{\ell^1(\mathbb{Z})} \\ &\quad + \sum_{|n|\ge 2\Lambda}\left|\left(M_{b,\beta}^{+}f_j\right)'(n)\right|,\end{aligned} \tag{110}$$

for all $j \ge N_2$. Fix $j \ge N_2$, we set

$$\begin{aligned}X_j^{+} &= \left\{|n| \ge 2\Lambda : M_{b,\beta}^{+}f_j(n+1) > M_{b,\beta}^{+}f_j(n)\right\}, \\ X_j^{-} &= \left\{|n| \ge 2\Lambda : M_{b,\beta}^{+}f_j(n) \ge M_{b,\beta}^{+}f_j(n+1)\right\}.\end{aligned} \tag{111}$$

Since $f_j \in \ell^1(\mathbb{Z})$, then for $n \in \mathbb{Z}$, there exists $r_n \in \mathbb{N}$ such that $M_{b,\beta}^{+}f_j(n) = A_{r_n}f_j(n)$. Then, we can write

$$\begin{aligned}\sum_{|n|\ge 2\Lambda}\left|\left(M_{b,\beta}^{+}f_j\right)'(n)\right| &= \sum_{n\in X_j^{+}}\left(M_{b,\beta}^{+}f_j(n+1) - M_{b,\beta}^{+}f_j(n)\right) \\ &\quad + \sum_{n\in X_j^{-}}\left(M_{b,\beta}^{+}f_j(n) - M_{b,\beta}^{+}f_j(n+1)\right).\end{aligned} \tag{112}$$

Similar arguments to those used in deriving (94) and (95) may yield that

$$\begin{aligned}&\sum_{n\in X_j^{+}}\left(M_{b,\beta}^{+}f_j(n+1) - M_{b,\beta}^{+}f_j(n)\right) \\ &\quad \le \left\|f_j\right\|_{\ell^1(\mathbb{Z})}\sum_{n\in X_j^{+}}|b(n+1) - b(n)| \\ &\qquad + \operatorname{Var}(b)\sum_{n\in X_j^{+}}\Gamma(r_{n+1})\sum_{k\in\mathbb{Z}}f_j(k)\chi_{[n+1,n+r_{n+1}+1]}(k),\end{aligned} \tag{113}$$

$$\begin{aligned}&\sum_{n\in X_j^{-}}\left(M_{b,\beta}^{+}f_j(n) - M_{b,\beta}^{+}f_j(n+1)\right) \\ &\quad \le \left\|f_j\right\|_{\ell^1(\mathbb{Z})}\sum_{n\in X_j^{-}}|b(n+1) - b(n)| \\ &\qquad + \operatorname{Var}(b)\sum_{n\in X_j^{-}}\Gamma(r_n)\sum_{k\in\mathbb{Z}}f_j(k)\chi_{[n+1,n+r_n+1]}(k).\end{aligned} \tag{114}$$

It follows from (102), (108), and (112)–(114) that

$$\begin{aligned}&\sum_{|n|\ge 2\Lambda}\left|\left(M_{b,\beta}^{+}f_j\right)'(n)\right| \\ &\quad \le \sum_{|n|\ge 2\Lambda}|b(n+1) - b(n)|\left\|f_j\right\|_{\ell^1(\mathbb{Z})} \\ &\qquad + \sum_{|n|\ge 2\Lambda}\operatorname{Var}(b)\sum_{k\in\mathbb{Z}}f_j(k)\Gamma(k-n-1)\chi_{[n+1,\infty)}(k) \\ &\quad \le \left(\|f\|_{\ell^1(\mathbb{Z})} + 1\right)\varepsilon \\ &\qquad + \operatorname{Var}(b)\sum_{k\in\mathbb{Z}}f_j(k)\sum_{|n|\ge 2\Lambda}\Gamma(k-n-1)\chi_{[n+1,\infty)}(k).\end{aligned} \tag{115}$$

By (101) and (107), we can get

$$\begin{aligned}&\sum_{k\in\mathbb{Z}}f_j(k)\sum_{|n|\ge 2\Lambda}\Gamma(k-n-1)\chi_{[n+1,\infty)}(k) \\ &\quad \le \sum_{|k|\ge\Lambda}f_j(k)\sum_{n=-\infty}^{k-1}\Gamma(k-n-1) \\ &\qquad + \sum_{|k|<\Lambda}f_j(k)\sum_{|n|\ge 2\Lambda,\ n<k}\Gamma(k-n-1) \\ &\quad \le \sum_{|k|\ge\Lambda}f(k) + \left\|f_j - f\right\|_{\ell^1(\mathbb{Z})} + \sum_{|k|<\Lambda}f_j(k)\sum_{n\le -2\Lambda}\Gamma(k-n-1) \\ &\quad \le 2\varepsilon + \sum_{|k|<\Lambda}f_j(k)\Lambda^{\beta-1} \le \left(3 + \|f\|_{\ell^1(\mathbb{Z})}\right)\varepsilon.\end{aligned} \tag{116}$$

It follows from (115) and (116) that

$$\begin{aligned}\sum_{|n|\ge 2\Lambda}\left|\left(M_{b,\beta}^{+}f_j\right)'(n)\right| &\le \left(\left(3 + \|f\|_{\ell^1(\mathbb{Z})}\right)\operatorname{Var}(b)\right. \\ &\quad \left. + \left(\|f\|_{\ell^1(\mathbb{Z})} + 1\right)\right)\varepsilon.\end{aligned} \tag{117}$$

Inequality (117) together with (110) implies that

$$\left\|\left(M_{b,\beta}^{+}f_j\right)'\right\|_{\ell^1(\mathbb{Z})} \le C\varepsilon + \left\|\left(M_{b,\beta}^{+}f\right)'\right\|_{\ell^1(\mathbb{Z})}, \tag{118}$$

for all $j \ge \max\{N_1, N_2\}$. Here, $C > 0$ depends only on b, f. This leads to (106) and completes the Proof of Theorem 14.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that she have no conflicts of interest.

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