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Research Article

Some Fixed Point Results for Perov-Ćirić-Prešić Type F-Contractions with Application

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Ćirić and Prešić developed the concept of Prešić contraction to Ćirić-Prešić type contractive mappings in the background of a metric space. On the other hand, Altun and Olgun introduced Perov type F-contractions. In this paper, we extend the concept of Ćirić-Prešić contractions to Perov-Ćirić-Prešić type F-contractions. Our results modify some known ones in the literature. To support our main result, an example and an application to nonlinear operator systems are presented.

1. Introduction

The Banach contraction principle (BCP) [1] is one of the powerful results in nonlinear analysis. It has many applications in the background of ODE and PDE.

Theorem 1 [1]. *Let (Δ, d) be a complete metric space and let $\Upsilon : \Delta \longrightarrow \Delta$ so that*

$$d(\Upsilon\iota, \Upsilon\kappa) \leq \gamma d(\iota, \kappa) \text{ for all } \iota, \kappa \in \Delta, \tag{1}$$

where $\gamma \in [0, 1)$. Then, there is a unique σ in Δ such that $\sigma = \Upsilon\sigma$. Also, for each $\zeta_0 \in \Delta$, the sequence $\zeta_{n+1} = \Upsilon\zeta_n$ converges to σ.

The BCP has been extended and generalized in many directions (see [2–4]).

Prešić [5] gave the following result.

Theorem 2 [5]. *Let (Δ, d) be a complete metric space and let $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Suppose that*

$$d(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \leq \sum_{i=1}^{k} \lambda_i d(\zeta_i, \zeta_{i+1}), \tag{2}$$

for all $\zeta_1, \cdots, \zeta_{k+1}$ in Δ, where $\lambda_i \geq 0$ and $\sum_{i=1}^{k} \lambda_i \in [0, 1)$. Then Υ has a unique fixed point ζ^* (that is $\Upsilon(\zeta^*, \cdots, \zeta^*) = \zeta^*$). Moreover, for all arbitrary points $\zeta_1, \cdots, \zeta_{k+1}$ in Δ, the sequence $\{\zeta_n\}$ defined by $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$, converges to ζ^*.

It is obvious that for $k = 1$, Theorem 2 coincides with the BCP.

Theorem 2 was generalized by Ćirić and Prešić [6] as follows.

Theorem 3 [6]. *Let (Δ, d) be a complete metric space and $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Suppose that*

$$d(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \leq \lambda \max \{d(\zeta_i, \zeta_{i+1}) : 1 \leq i \leq k\}, \tag{3}$$

for all $\zeta_1, \cdots, \zeta_{k+1}$ in Δ, where $\lambda \in [0, 1)$. Then Υ has a fixed point $\zeta^* \in \Delta$. Also, for all points $\zeta_1, \cdots, \zeta_{k+1} \in \Delta$, the sequence $\{\zeta_n\}$ defined by $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$, converges to ζ^*. The fixed point of Υ is unique if

$$d(\Upsilon(\rho, \cdots, \rho), \Upsilon(\rho, \cdots, \rho)) < d(\rho, \rho), \tag{4}$$

for all $\rho, \varrho \in \Delta$ *with* $\rho \neq \varrho$.

For more details on Prešić type contractions, we refer the reader to [2, 5, 7–11].

In this paper, $\mathbb{R}_+ = [0,\infty)$, $\mathbb{R}^m$ denotes the set of $m \times 1$ real matrices, $\mathbb{R}_+^m$ will be the set of $m \times 1$ real matrices with elements in $[0, \infty)$, θ denotes the zero $m \times 1$ matrix, $M_{m,m}(\mathbb{R}_+)$ denotes the set of all $m \times m$ matrices with elements in $[0, \infty)$, and Θ will be the zero $m \times m$ matrix, by I the identity $m \times m$ matrix. If $A \in M_{m,m}(\mathbb{R}_+)$, then A^T states the transpose matrix of A. Let $\pi = (\pi_i)_{i=1}^m$, $\varpi = (\varpi_i)_{i=1}^m \in \mathbb{R}^m$, then by $\pi \preceq \varpi$ (resp. $\pi \prec \varpi$), we suppose $\pi_i \leq \varpi_i$ (resp. $\pi_i < \varpi_i$) for each $i \in \{1, 2, \cdots, m\}$. Also, $\pi \preceq \varpi$ and $\varpi \succeq \pi$ will mean the same.

Let Δ be a nonempty set and let $V : \Delta \times \Delta \longrightarrow \mathbb{R}^m$ be a function. V is called a vector-valued metric, and (Δ, V) is called a vector-valued metric space, if

(1) $V(\iota, \kappa) = \theta$ if and only if $\iota = \kappa$,

(2) $V(\iota, \kappa) = V(\kappa, \iota)$,

(3) $V(\iota, \kappa) \preceq (\iota, \sigma) + V(\sigma, \kappa)$,

for all $\iota, \kappa, \sigma \in \Delta$.

Example 1 (Example 1.3. of [12]). Let $D_1, D_2, \cdots, D_n$ be usual metrics on X.

Then, the mapping $d : X \times X \longrightarrow \mathscr{R}^n$ defined by $d(x, y) = (D_1(x, y), D_2(x, y), \cdots, D_n(x, y))$ is a VVM on X.

From now on, we apply VVMS instead of a vector-valued metric space.

The concepts of convergence, Cauchyness, and completeness in a VVMS will be similar as in a usual metric case. Perov [13] stated the contraction mapping principle in the setting of VVMSs. Before stating this theorem, we must remember the following facts:

Let $A \in M_{m,m}(\mathbb{R}_+)$. Then A is said to converge to zero if and only if $A^n \longrightarrow \Theta$ as $n \longrightarrow \infty$ (see [14]).

Perov [13] proved the following interesting extension of BCP (see more results in [15–20]).

Theorem 4 [13]. *Let* (Δ, V) *be a VVMS and* $\Upsilon : \Delta \longrightarrow \Delta$ *be a mapping such that there exists a matrix* $A \in M_{m,m}(\mathbb{R}_+)$ *such that*

$$V(\Upsilon\iota, \Upsilon\kappa) V A V(\iota, \kappa), \tag{5}$$

for all $\iota, \kappa \in \Delta$. *If A is convergent to zero, then*

(1) Υ *has a unique fixed point* σ *in* Δ

(2) for all $x_0 \in \Delta$, *the sequence* $\{x_n\}$ *defined by* $x_n = \Upsilon^n x_0$ *is convergent to* σ

(3) $V(x_n, \sigma) V A^n (I - A)^{-1} V(x_0, \Upsilon x_0)$.

In this paper, considering the recent approach of Wardowski [21], we present a generalization of Perov fixed point theorem and Ćirić-Prešić fixed point theorem. Some generalization of Wardowski results can be found in [22, 23].

As in [24], let $F : \mathbb{R}_+^m \longrightarrow \mathbb{R}^m$ be a function. Let

(F1) F be strictly increasing in each variable, i.e., $\pi \prec \varpi$; then, $F(\pi) \prec F(\varpi)$, for all $\pi = (\pi_i)_{i=1}^m$ and $\varpi = (\varpi_i)_{i=1}^m \in \mathbb{R}_+^m$,

(F2) For each sequence $\{\pi_n\} = (\pi_n^{(1)}, \pi_n^{(2)}, \cdots, \pi_n^{(m)})$ of $\mathbb{R}_+^m$

$$\lim_{n\to\infty} \pi_n^{(i)} = 0 \text{ if and only if } \lim_{n\to\infty} \varpi_n^{(i)} = -\infty, \tag{6}$$

for each $i \in \{1, 2, \cdots m\}$, where

$$F\left(\left(\pi_n^{(1)}, \pi_n^{(2)}, \cdots, \pi_n^{(m)}\right)\right) = \left(\varpi_n^{(1)}, \varpi_n^{(2)}, \cdots, \varpi_n^{(m)}\right). \tag{7}$$

(F3) There exists $k \in (0, 1)$ such that $\lim_{\pi_i \to 0^+} \pi_i^k \varpi_i = 0$ for each $i \in \{1, 2, \cdots m\}$, where

$$F((\pi_1, \pi_2, \cdots, \pi_m)) = (\varpi_1, \varpi_2, \cdots, \varpi_m). \tag{8}$$

We denote by $\mathscr{F}^m$ the set of all functions F satisfying (*F*1)–(*F*3)

Example 2 [24]. Define $F : \mathbb{R}_+^m \longrightarrow \mathbb{R}^m$ by

$$F((\pi_1, \pi_2, \cdots, \pi_m)) = (\ln \pi_1, \ln \pi_2, \cdots, \ln \pi_m), \tag{9}$$

then $F \in \mathscr{F}^m$.

Note that we can define $G : \mathbb{R}^m \longrightarrow \mathbb{R}_+^m$ by

$$G((\varpi_1, \varpi_2, \cdots, \varpi_m)) = \left(e^{\varpi_1}, e^{\varpi_2}, \cdots, e^{\varpi_m}\right), \tag{10}$$

which we can treat it as the inverse of multivariable function F.

Note that from now on, F is a continuously differentiable function from all open sets of $A \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}^n$, and the Jacobian determinant of F at every $p \in A \subseteq \mathbb{R}^n$ is nonzero; then, according to inverse function theorem, F is invertible near p.

Example 3 [24]. Define $F : \mathbb{R}_+^2 \longrightarrow \mathbb{R}^2$ by

$$F((\pi_1, \pi_2)) = (\ln \pi_1, \pi_2 + \ln \pi_2), \tag{11}$$

then $F \in \mathscr{F}^2$.

Example 4 [24]. Define $F:\mathbb{R}_+^3\longrightarrow\mathbb{R}^3$ by

$$F((\pi_1,\pi_2,\pi_3))=\left(\ln\pi_1,\pi_2+\ln\pi_2,-\frac{1+\pi_1}{\sqrt{\pi_3}}\right),\tag{12}$$

then $F\in\mathscr{F}^3$.

Considering the class $\mathscr{F}^m$, Altun and Olgun [24] introduced the concept of Perov type F-contraction as follows:

Definition 5 [24]. Let (Δ,V) be a VVMS and $\Upsilon:\Delta\longrightarrow\Delta$ be a map. If there exist $F\in\mathscr{F}^m$ and $\varsigma=(\varsigma_i)_{i=1}^m\in\mathbb{R}_+^m$ such that

$$\varsigma+F(V(\Upsilon\iota,\Upsilon y))VF(V(\iota,\kappa)),\tag{13}$$

for all $\iota,\kappa\in\Delta$ with $V(\Upsilon\iota,\Upsilon\kappa)\succ\theta$, then Υ is called a Perov type F-contraction.

If we consider $F:\mathbb{R}_+^m\longrightarrow\mathbb{R}^m$ by

$$F((\pi_1,\pi_2,\cdots,\pi_m))=(\ln\pi_1,\ln\pi_2,\cdots,\ln\pi_m),\tag{14}$$

then (13) turns to Perov contraction [24].

We can present new type contractions in a VVMS, via considering some function $F\in\mathscr{F}^m$ in (13).

Theorem 6 [24]. *Let (Δ,V) be a complete VVMS and let $\Upsilon:\Delta\longrightarrow\Delta$ be a Perov type F-contraction. Then Υ admits a unique fixed point.*

In this paper, we introduce the concept of Perov-Ćirić-Prešić type F-contractions. An illustrative example and an application are given to support our main result.

2. Main Results

In this section, combining the ideas of Perov, Wardowski, and Ćirić-Prešić, we obtain a new extension of BCP.

Our main result is as follows:

Theorem 7. *Let (Δ,V) be a complete VVMS and let $\Upsilon:\Delta^k\longrightarrow\Delta$ (k is a positive integer). Assume that there exist $F\in\mathscr{F}^m$ and $\varsigma=(\varsigma_i)_{i=1}^m\in\mathbb{R}_+^m$ satisfying*

$$\begin{aligned}\varsigma&+F(V(\Upsilon(\zeta_1,\cdots,\zeta_k),\Upsilon(\zeta_2,\cdots,\zeta_{k+1})))\\&\le F(\sup\{V(\zeta_1,\zeta_2),V(\zeta_2,\zeta_3),\cdots,V(\zeta_k,\zeta_{k+1})\}),\end{aligned}\tag{15}$$

for all $\zeta_1,\cdots,\zeta_{k+1}\in\Delta$ with $V(\Upsilon(\zeta_1,\cdots,\zeta_k),\Upsilon(\zeta_2,\cdots,\zeta_{k+1}))\succ\theta$. Moreover, let there exists a sequence $\{\zeta_n\}$ in Δ such that $\zeta_{n+k}=\Upsilon(\zeta_n,\zeta_{n+1},\cdots,\zeta_{n+k-1})$ and $V(\zeta_{n+k},\zeta_{n+k+1})\succ\theta$, for all $n\in\mathbb{N}$. Also, if $\zeta_n\longrightarrow v$, then $V(\zeta_n,v)\succ\theta$, for all $n\in\mathbb{N}$. Then, the sequence $\{\zeta_n\}$ converges to a fixed point of Υ. Moreover, if for all $\rho,\varrho\in\Delta$ with $\rho\neq\varrho$,

$$V(\Upsilon(\rho,\cdots,\rho),\Upsilon(\rho,\cdots,\rho))<V(\rho,\rho),\tag{16}$$

then the fixed point of Υ is unique.

Proof. For any $n\in\mathbb{N}$, we have

$$\begin{aligned}&F(V(\zeta_{n+k},\zeta_{n+k+1}))\\&\quad=F(V(\Upsilon(\zeta_n,\cdots,\zeta_{n+k-1}),\Upsilon(\zeta_{n+1},\cdots,\zeta_{n+k})))\\&\quad\le F(\sup\{V(\zeta_n,\zeta_{n+1}),V(\zeta_{n+1},\zeta_{n+2}),\cdots,V(\zeta_{n+k-1},\zeta_{n+k})\})-\varsigma.\end{aligned}\tag{17}$$

Therefore,

$$\begin{aligned}&F(V(\zeta_{k+1},\zeta_{k+2}))\\&\quad=F(V(\Upsilon(\zeta_1,\cdots,\zeta_k),\Upsilon(\zeta_2,\cdots,\zeta_{k+1})))\\&\quad\le F(\sup\{V(\zeta_1,\zeta_2),V(\zeta_2,\zeta_3),\cdots,V(\zeta_k,\zeta_{k+1})\})-\varsigma\\&\quad=F(\Lambda)-\varsigma,\end{aligned}\tag{18}$$

where $\Lambda=\sup\{V(\zeta_1,\zeta_2),V(\zeta_2,\zeta_3),\cdots,V(\zeta_k,\zeta_{k+1})\}$. Now,

$$\begin{aligned}&F(V(\zeta_{k+2},\zeta_{k+3}))\\&\quad=F(V(\Upsilon(\zeta_2,\cdots,\zeta_{k+1}),\Upsilon(\zeta_3,\cdots,\zeta_{k+2})))\\&\quad\le F(\sup\{V(\zeta_2,\zeta_3),V(\zeta_3,\zeta_4),\cdots,V(\zeta_{k+1},\zeta_{k+2})\})-\varsigma\\&\quad\le F\left(\max\left\{\Lambda,F^{-1}(F(\Lambda)-\varsigma)\right\}\right)-\varsigma=F(\Lambda)-\varsigma.\end{aligned}\tag{19}$$

Continuing this approach, we have

$$\begin{aligned}&F(V(\zeta_{2k},\zeta_{2k+1}))\\&\quad=F(V(\Upsilon(\zeta_k,\cdots,\zeta_{2k-1}),\Upsilon(\zeta_{k+1},\cdots,\zeta_{2k})))\\&\quad\le F(\sup\{V(\zeta_k,\zeta_{k+1}),V(\zeta_{k+1},\zeta_{k+2}),\cdots,V(\zeta_{2k-1},\zeta_{2k})\})-\varsigma\\&\quad\le F\left(\max\left\{\Lambda,F^{-1}(F(\Lambda)-\varsigma)\right\}\right)-\varsigma=F(\Lambda)-\varsigma,F(V(\zeta_{2k+1},\zeta_{2k+2}))\\&\quad=F(V(\Upsilon(\zeta_{k+1},\cdots,\zeta_{2k}),\Upsilon(\zeta_{k+2},\cdots,\zeta_{2k+1})))\\&\quad\le F(\sup\{V(\zeta_{k+1},\zeta_{k+2}),V(\zeta_{k+2},\zeta_{k+3}),\cdots,V(\zeta_{2k},\zeta_{2k+1})\})-\varsigma\\&\quad\le F\left(F^{-1}(F(\Lambda)-\varsigma)\right)-\varsigma=F(\Lambda)-2\varsigma,F(V(\zeta_{3k},\zeta_{3k+1}))\\&\quad=F(V(\Upsilon(\zeta_{2k},\cdots,\zeta_{3k-1}),\Upsilon(\zeta_{2k+1},\cdots,\zeta_{3k})))\\&\quad\le F(\sup\{V(\zeta_{2k},\zeta_{2k+1}),V(\zeta_{2k+1},\zeta_{2k+2}),\cdots,V(\zeta_{3k-1},\zeta_{3k})\})-\varsigma\\&\quad\le F\left(\max\left\{F^{-1}(F(\Lambda)-\varsigma),F^{-1}(F(\Lambda)-2\varsigma)\right\}\right)-\varsigma\\&\quad=F(\Lambda)-2\varsigma,F(V(\zeta_{3k+1},\zeta_{3k+2}))\\&\quad=F(V(\Upsilon(\zeta_{2k+1},\cdots,\zeta_{3k}),\Upsilon(\zeta_{2k+2},\cdots,\zeta_{3k+1})))\\&\quad\le F(\sup\{V(\zeta_{2k+1},\zeta_{2k+2}),V(\zeta_{2k+2},\zeta_{2k+3}),\cdots,V(\zeta_{3k},\zeta_{3k+1})\})-\varsigma\\&\quad\le F\left(F^{-1}(F(\Lambda)-2\varsigma)\right)-\varsigma=F(\Lambda)-3\varsigma.\end{aligned}\tag{20}$$

Continuing this process, we get

$$F\left(V\left(\zeta_{pk+i},\zeta_{pk+i+1}\right)\right)\le F(\Lambda)-p\varsigma,\text{ for all }p\in V\text{ and }i\in\{1,2,\cdots,k\}.\tag{21}$$

Now, taking $V(\zeta_n,\zeta_{n+1})=(\pi_n^1,\pi_n^2,\cdots,\pi_n^m)$ and $F(\pi_n^1,\pi_n^2,\cdots,\pi_n^m)=(\varpi_n^1,\varpi_n^2,\cdots,\varpi_n^m)$, we obtain that

$$\left(\varpi^1_{pk+i}, \varpi^2_{pk+i}, \cdots, \varpi^m_{pk+i}\right) = F\left(\pi^1_{pk+i}, \pi^2_{pk+i}, \cdots, \pi^m_{pk+i}\right) \le F(\Lambda) - p\varsigma = (r_1 - p\varsigma_1, r_2 - p\varsigma_2, \cdots, r_m - p\varsigma_m), \tag{22}$$

where $\Lambda = (\Lambda_i)_{i=1}^m$ and $F((\Lambda_i)_{i=1}^m) = (r_i)_{i=1}^m$. Therefore,

$$\varpi^j_{pk+i} \le r_j - p\varsigma_j \text{ for all } j \in \{1, 2, \cdots, m\}. \tag{23}$$

Passing to the limit, we get $\lim_{p\to\infty}\varpi^j_{pk+i} = -\infty$. Therefore, $\lim_{p\to\infty}\pi^j_{pk+i} = 0$ for all $j \in \{1, 2, \cdots, m\}$. Thus, $\lim_{p\to\infty} V(\zeta_{pk+i}, \zeta_{pk+i+1}) = \theta$. From ($F3$), there exists $\lambda \in (0, 1)$ such that

$$\lim_{p\to\infty}\left[\pi^j_{pk+i}\right]^\lambda \varpi^j_{pk+i} = 0, \text{ for all } j \in \{1, 2, \cdots, m\}. \tag{24}$$

From (23),

$$\left[\pi^j_{pk+i}\right]^\lambda \varpi^j_{pk+i} \le \left[\pi^j_{pk+i}\right]^\lambda r_j - p\left[\pi^j_{pk+i}\right]^\lambda \varsigma_j \text{ for all } j \in \{1, 2, \cdots, m\}. \tag{25}$$

Therefore,

$$\left[\pi^j_{pk+i}\right]^\lambda \varpi^j_{pk+i} - \left[\pi^j_{pk+i}\right]^\lambda r_j \le -p\left[\pi^j_{pk+i}\right]^\lambda \varsigma_j \le 0 \text{ for all } j \in \{1, 2, \cdots, m\}. \tag{26}$$

Thus, $\lim_{p\to\infty} p[\pi^j_{pk+i}]^\lambda = 0$ for all $j \in \{1, 2, \cdots, m\}$. So, for any $j \in \{1, 2, \cdots, m\}$, there exists $p_j \in \mathbb{N}$ such that $p[\pi^j_{pk+i}]^\lambda \le 1$, for all $p \ge p_j$. Thus, $\pi^j_{pk+i} \le 1/p^{1/\lambda}$, for all $p \ge p_j$. Putting $p_0 = \max\{p_j : 1 \le j \le m\}$, we have $\pi^j_{pk+i} \le 1/p^{1/\lambda}$ for all $p \ge p_0$ and all $i \in \{1, 2, \cdots, k\}$. We claim that $\{\zeta_n\}$ is a Cauchy sequence. Consider two elements $m, n \in \mathbb{N}$ so that $p_0 k \le n < m$. Then, there are $p, q \in \mathbb{N}$ and $i, j \in \{1, 2, \cdots, k\}$ such that $p_0 \le p \le q$, $n = pk + i$, and $m = qk + j$. Now, we have

$$V(\zeta_n, \zeta_m) = V\left(\zeta_{pk+i}, \zeta_{qk+j}\right) \le \sum_{r=p}^{q}\sum_{l=1}^{k} V(\zeta_{rk+l}, \zeta_{rk+l+1}) \le \sum_{r=p}^{q} k\left(\frac{1}{r^{1/\lambda}}, \cdots, \frac{1}{r^{1/\lambda}}\right) \le k\left(\sum_{r=p}^{q}\frac{1}{r^{1/\lambda}}, \cdots, \sum_{r=p}^{q}\frac{1}{r^{1/\lambda}}\right). \tag{27}$$

As $n, m \longrightarrow \infty$, we have $p, q \longrightarrow \infty$. Thus, the last term in (27) converges to θ, and so $\{\zeta_n\}$ is a Cauchy sequence in (Δ, V). Since (Δ, V) is a complete VVMS, there is $\nu \in \Delta$ so that $\lim_{n\to\infty}\zeta_n = \nu$. Now, we shall prove that ν is a fixed point of Υ. To see this, we have

$$\begin{aligned} V(\zeta_{n+k}, \Upsilon(\nu, \cdots, \nu)) &= V(\Upsilon(\zeta_n, \zeta_{n+1} \cdots, \zeta_{n+k-1}), \Upsilon(\nu, \cdots, \nu)) \\ &\le V(\Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1}), \Upsilon(\zeta_{n+1}, \zeta_{n+2}, \cdots, \zeta_{n+k-1}, \nu)) \\ &\quad + V(\Upsilon(\zeta_{n+1}, \zeta_{n+2}, \cdots, \zeta_{n+k-1}, \nu), \Upsilon(\zeta_{n+2}, \zeta_{n+3}, \cdots, \zeta_{n+k-1}, \nu, \nu)) \\ &\quad + \cdots + V(\Upsilon(\zeta_{n+k-1}, \nu, \cdots, \nu), \Upsilon(\nu, \nu, \cdots, \nu)) \\ &\le F^{-1}[F(\max\{V(\zeta_n, \zeta_{n+1}), \cdots, V(\zeta_{n+k-2}, \zeta_{n+k-1}), V(\zeta_{n+k-1}, \nu)\}) - \varsigma] \\ &\quad + F^{-1}[F(\max\{V(\zeta_{n+1}, \zeta_{n+2}), \cdots, V(\zeta_{n+k-2}, \zeta_{n+k-1}), \\ &\quad \times V(\zeta_{n+k-1}, \nu)\}) - \varsigma] \cdots + F^{-1}[F(V(\zeta_{n+k-1}, \nu)) - \varsigma] \longrightarrow \theta, \end{aligned} \tag{28}$$

as $n \longrightarrow \infty$. Thus,

$$V(\nu, \Upsilon(\nu, \cdots, \nu)) = \lim_{n\to\infty} V(\zeta_{n+k}, \Upsilon(\nu, \cdots, \nu)) = \theta. \tag{29}$$

Therefore, $\nu = \Upsilon(\nu, \cdots, \nu)$. Suppose that u, ν are two distinct fixed points of Υ. From our hypothesis,

$$V(u, \nu) = V(\Upsilon(u, \cdots, u), \Upsilon(\nu, \cdots, \nu))) < V(u, \nu), \tag{30}$$

which is a contradiction. Thus, the fixed point of Υ is unique.

Note that by taking

$$F((\pi_1, \pi_2, \cdots, \pi_m)) = (\ln \pi_1, \ln \pi_2, \cdots, \ln \pi_m), \tag{31}$$

the above theorem reduces to the following theorem.

Theorem 8. *Let (Δ, V) be a complete VVMS and $\Upsilon : \Delta^k \longrightarrow \Delta$ (k is a positive integer). Suppose that there exist $F \in \mathscr{F}^m$ and $\varsigma = (\varsigma_i)_{i=1}^m \in \mathbb{R}_+^m$ satisfying*

$$V(\Upsilon(\zeta_1, \cdots, \zeta_k), \Upsilon(\zeta_2, \cdots, \zeta_{k+1})) \le A \sup\{V(\zeta_i, \zeta_{i+1}) : i = 1, \cdots, k\}, \tag{32}$$

where

$$A = \begin{pmatrix} e^{-\varsigma_1} & 0 & \cdots & 0 \\ 0 & e^{-\varsigma_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e^{-\varsigma_m} \end{pmatrix}_{m\times m}. \tag{33}$$

Let the sequence $\{\zeta_n\}$ in Δ be such that $\zeta_{n+k} = \Upsilon(\zeta_n, \zeta_{n+1}, \cdots, \zeta_{n+k-1})$ and $V(\zeta_{n+k}, \zeta_{n+k+1}) \succ \theta$, for all $n \in \mathbb{N}$. Also, if $\zeta_n \longrightarrow \nu$, then $V(\zeta_n, \nu) \succ \theta$, for all $n \in \mathbb{N}$. Then, the sequence $\{\zeta_n\}$ converges to a fixed point of Υ. Also, if

$$V(\Upsilon(\rho, \cdots, \rho), \Upsilon(\rho, \cdots, \rho)) < V(\rho, \rho), \quad (34)$$

for all $\rho, \varrho \in \Delta$ with $\rho \neq \varrho$, then the fixed point of Υ is unique.

We present an example to support our main result.

Example 5. Let $\Delta = \{\varsigma_n = 1/n^2 : n = 1, 2, \cdots\} \cup \{\varsigma_0 = 0\}$, $V(\rho, \varrho) = (|\rho - \varrho|, |\rho - \varrho|)$, and define $\Upsilon : \Delta^2 \longrightarrow \Delta$ by

$$\Upsilon(\varsigma_n, \varsigma_m) = \begin{pmatrix} \varsigma_{\max\{m,n\}+1}, & n, m \geq 1, \\ 0, & n = 0 \text{ or } m = 0. \end{pmatrix} \quad (35)$$

Firstly, note that for all $m, n \in \mathbb{N} \cup \{0\}$ with $m \neq n$, from Example 2.3 of [25], we have

$$\begin{aligned} &|\varsigma_{n+1} - \varsigma_{m+1}|^{1/\sqrt{|\varsigma_{n+1}-\varsigma_{m+1}|}} |\varsigma_n - \varsigma_m|^{-1/\sqrt{|\varsigma_n-\varsigma_m|}} \leq \frac{1}{2}, \\ &\frac{1}{\sqrt{|\varsigma_{n+1} - \varsigma_{m+1}|}} - \frac{1}{\sqrt{|\varsigma_n - \varsigma_m|}} \geq 1. \end{aligned} \quad (36)$$

As we know, $\varsigma_{n+2} = \min\{\varsigma_{n+1}, \varsigma_{n+2}\} = \Upsilon(\varsigma_n, \varsigma_{n+1})$, for all $\mathbb{N}$ and

$$\begin{aligned} V(\varsigma_{n+2}, \varsigma_{n+3}) &= (|\varsigma_{n+2} - \varsigma_{n+3}|, |\varsigma_{n+2} - \varsigma_{n+3}|) \\ &= \left(\left| \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2} \right|, \left| \frac{1}{(n+2)^2} - \frac{1}{(n+3)^2} \right| \right) \\ &\succ (0,0) = \theta. \end{aligned} \quad (37)$$

Also, $\varsigma_n = 1/n^2 \longrightarrow 0$ and

$$V(\varsigma_n, 0) = (|\varsigma_n - 0|, |\varsigma_n - 0|) = \left(\frac{1}{n^2}, \frac{1}{n^2}\right) \succ (0,0) = \theta. \quad (38)$$

Define $F : \mathbb{R}_+^2 \longrightarrow \mathbb{R}^2$ by

$$F((\pi_1, \pi_2)) = \begin{cases} \left(\dfrac{\ln \pi_1}{\sqrt{\pi_1}}, \dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 \leq e \\ \left(\dfrac{\pi_1}{e\sqrt{e}}, \dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 > e. \end{cases} \quad (39)$$

Obviously, $F \in \mathcal{F}^2$. Also, take $\varsigma = (\varsigma_1, \varsigma_2) = (\ln 2, 1)$. We have

$$\begin{aligned} &\varsigma + F(V(\Upsilon(\varepsilon, \varepsilon), \Upsilon(\varepsilon, \sigma))) \\ &\quad \leq F(\max\{V(\varepsilon, \varepsilon), V(\varepsilon, \sigma)\}) \\ &\quad \Leftrightarrow (\ln 2, 1) + F(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|, |\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|) \\ &\quad \leq F(\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}) \Leftrightarrow (\ln 2, 1) \\ &\qquad + \left(\frac{\ln(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}}, \frac{-1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} \right) \\ &\quad \leq \left(\frac{\ln(\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\})}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}}, \frac{-1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \right) \\ &\quad \Leftrightarrow \ln 2 + \frac{\ln(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} \leq \frac{\ln(\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\})}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}}, 1 \\ &\qquad + \frac{-1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} \leq \frac{-1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \\ &\quad \Leftrightarrow (|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}} \max \\ &\qquad \times \{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}} \\ &\quad \leq \frac{1}{2}, \frac{1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} - \frac{1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \geq 1, \end{aligned} \quad (40)$$

for any $\varepsilon, \varepsilon, \sigma \in \Delta$. Now, Let $\varepsilon = \varsigma_n$, $\varepsilon = \varsigma_m$, and $\sigma = \varsigma_p$. If $m \geq \max\{n, p\}$, then

$$\begin{aligned} V(\Upsilon(\varepsilon, \varepsilon), \Upsilon(\varepsilon, \sigma)) &= V\left(\Upsilon(\varsigma_n, \varsigma_m), \Upsilon\left(\varsigma_m, \varsigma_p\right)\right) \\ &= V(\varsigma_{m+1}, \varsigma_{m+1}) = (0,0) = \theta. \end{aligned} \quad (41)$$

So, we may assume that either $m < n$ or $m < p$. We consider the following cases:

Case 1. $n \leq m < p$. Let $n = 0$. If $n = m = 0$, then

$$V(\Upsilon(\varepsilon, \varepsilon), \Upsilon(\varepsilon, \sigma)) = (0,0) = \theta. \quad (42)$$

If $0 = n < m$, then

$$\begin{aligned} &(|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}} \\ &\quad \cdot \max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}^{1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}} \\ &= \left(\varsigma_{p+1}\right)^{1/\sqrt{\varsigma_{p+1}}} (\varsigma_m)^{1/\sqrt{\varsigma_m}} = \left(\frac{1}{(p+1)^2}\right)^{p+1} \left(\frac{1}{m^2}\right)^{-m} \\ &\leq \left(\frac{1}{(p+1)^2}\right)^{p+1} \left(\frac{1}{p^2}\right)^{-p} \leq \frac{1}{(p+1)^2} \\ &\leq \frac{1}{2} \frac{1}{\sqrt{|\Upsilon(\varepsilon, \varepsilon) - \Upsilon(\varepsilon, \sigma)|}} - \frac{1}{\sqrt{\max\{|\varepsilon - \varepsilon|, |\varepsilon - \sigma|\}}} \\ &= \frac{1}{\sqrt{\varsigma_{p+1}}} - \frac{1}{\sqrt{\varsigma_m}} = p + 1 - m \geq p + 1 - p = 1, \end{aligned} \quad (43)$$

and if $n > 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{m+1}-\varsigma_{p+1})^{1/\sqrt{\varsigma_{m+1}-\varsigma_{p+1}}}(\varsigma_m-\varsigma_p)^{-1/\sqrt{\varsigma_m-\varsigma_p}}\\&\le\frac{1}{2}\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_m}}=p+1-m\ge p+1-p=1.\end{aligned}\tag{44}$$

Case 2. $m < p \le n$. Here, if $m = 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=(\varsigma_{n+1})^{1/\sqrt{\varsigma_{n+1}}}(\varsigma_n)^{-1/\sqrt{\varsigma_n}}=\left(\frac{1}{(n+1)^2}\right)^{n+1}\left(\frac{1}{n^2}\right)^{-n}\le\frac{1}{(n+1)^2}\le\frac{1}{2},\end{aligned}\tag{45}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}}}-\frac{1}{\sqrt{\varsigma_n}}=n+1-n\ge n+1-n=1,\end{aligned}\tag{46}$$

and if $m > 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\\&\quad\cdot\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1}-\varsigma_{p+1})^{1/\sqrt{\varsigma_n-\varsigma_p}}(\varsigma_n-\varsigma_p)^{-1/\sqrt{\varsigma_n-\varsigma_p}}\le\frac{1}{2},\end{aligned}\tag{47}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}-\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_n-\varsigma_p}}\ge 1.\end{aligned}\tag{48}$$

Case 3. $m \le n < p$. In this case, if $m = 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{p+1})^{1/\sqrt{\varsigma_{p+1}}}(\varsigma_p)^{-1/\sqrt{\varsigma_p}}=\left(\frac{1}{(p+1)^2}\right)^{p+1}\left(\frac{1}{p^2}\right)^{-p}\\&\le\frac{1}{(p+1)^2}\le\frac{1}{2},\end{aligned}\tag{49}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_p}}=p+1-p=1,\end{aligned}\tag{50}$$

and if $m > 0$, then.

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1}-\varsigma_{p+1})^{1/\sqrt{\varsigma_{n+1}-\varsigma_{p+1}}}(\varsigma_n-\varsigma_p)^{-1/\sqrt{\varsigma_n-\varsigma_p}}\le\frac{1}{2},\end{aligned}\tag{51}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}-\varsigma_{p+1}}}-\frac{1}{\sqrt{\varsigma_n-\varsigma_p}}\ge 1.\end{aligned}\tag{52}$$

Case 4. $p \le m < n$. Here, if $p = 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1})^{1/\sqrt{\varsigma_{n+1}}}(\varsigma_n)^{-1/\sqrt{\varsigma_n}}=\left(\frac{1}{(n+1)^2}\right)^{n+1}\left(\frac{1}{n^2}\right)^{-n}\\&\le\frac{1}{(n+1)^2}\le\frac{1}{2},\end{aligned}\tag{53}$$

and

$$\begin{aligned}&\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&=\frac{1}{\sqrt{\varsigma_{n+1}}}-\frac{1}{\sqrt{\varsigma_n}}=n+1-n=1,\end{aligned}\tag{54}$$

and if $p > 0$, then

$$\begin{aligned}&(|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|)^{1/\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}\max\\&\quad\cdot\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}^{-1/\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}\\&\le(\varsigma_{n+1}-\varsigma_{m+1})^{1/\sqrt{\varsigma_{n+1}-\varsigma_{m+1}}}(\varsigma_n-\varsigma_m)^{-1/\sqrt{\varsigma_n-\varsigma_m}}\le\frac{1}{2},\end{aligned}\tag{55}$$

and

$$\frac{1}{\sqrt{|\Upsilon(\varepsilon,\varepsilon)-\Upsilon(\varepsilon,\sigma)|}}-\frac{1}{\sqrt{\max\{|\varepsilon-\varepsilon|,|\varepsilon-\sigma|\}}}=\frac{1}{\sqrt{\varsigma_{n+1}-\varsigma_{m+1}}}-\frac{1}{\sqrt{\varsigma_n-\varsigma_m}}\geq 1. \tag{56}$$

Also, let $\rho, \rho \in \Delta$ with $\rho \neq \rho$. Without loss of any generality, let $\rho = \varsigma_n$, $\rho = \varsigma_m$ with $n < m$. If $n = 0$, then

$$\begin{aligned} V(\Upsilon(\rho,\rho),\Upsilon(\rho,\rho)) &= V(\Upsilon(\varsigma_0,\varsigma_0),\Upsilon(\varsigma_m,\varsigma_m)) = V(0,\varsigma_{m+1}) \\ &= (\varsigma_{m+1},\varsigma_{m+1}) \prec (\varsigma_m,\varsigma_m) = V(0,\varsigma_m) \\ &= V(\rho,\rho), V(\varsigma_1,\varsigma_{m-1}) = \frac{m(m-1)}{2}-1 \\ &< \frac{m(m+1)}{2}-1 = V(\varsigma_1,\varsigma_m) = V(\rho,\rho), \end{aligned} \tag{57}$$

and if $n > 0$, then

$$\begin{aligned} V(\Upsilon(\rho,\rho),\Upsilon(\rho,\rho)) &= V(\Upsilon(\varsigma_n,\varsigma_n),\Upsilon(\varsigma_m,\varsigma_m)) = V(\varsigma_{n+1},\varsigma_{m+1}) \\ &= (|\varsigma_{n+1}-\varsigma_{m+1}|,|\varsigma_{n+1}-\varsigma_{m+1}|) \\ &= \left(\frac{1}{(n+1)^2}-\frac{1}{(m+1)^2},\frac{1}{(n+1)^2}-\frac{1}{(m+1)^2}\right) \\ &\prec \left(\frac{1}{n^2}-\frac{1}{m^2},\frac{1}{n^2}-\frac{1}{m^2}\right) = V(\varsigma_n,\varsigma_m) \\ &= V(\rho,\rho). \end{aligned} \tag{58}$$

We see that all of the conditions of Theorem 2 are satisfied. Thus, Υ has a unique fixed point. Here, $\Upsilon(\varsigma_0,\varsigma_0)=\varsigma_0$, and ς_0 is the unique fixed point.

We present an example in an infinite dimensional sequence space ℓ_1 which is adapted from the above example, and so, we leave the details for the reader.

Let A be the space of all convergent sequences (a_n) for which $a_i = 1/n^2$ (n is an arbitrary natural number) for exactly one i and $a_j = 0$ for other indices.

Let $\Delta = A \cup \{\varsigma_0 = (0,0,0,\cdots)\}$, $V((a_n)_{m=1}^{\infty},(b_n)_{n=1}^{\infty}) = (\Sigma_{m,n=1}^{\infty}|a_n-b_n|,\Sigma_{m,n=1}^{\infty}|a_n-b_n|)$, and define $\Upsilon:\Delta^2 \longrightarrow \Delta$ by

$$\Upsilon((a_n)_{n=1}^{\infty},(b_n)_{n=1}^{\infty}) = \begin{cases} \left(0,\cdots,0,a_{\max\{i,j\}+1}=\dfrac{1}{n^2},0,\cdots\right), & (a_n)\neq(0,0,0,\cdots) \text{ and } (b_n)\neq(0,0,0,\cdots), \\ (0,0,0,\cdots), & (a_n)=(0,0,0,\cdots) \text{ or } (b_n)=(0,0,0,\cdots). \end{cases} \tag{59}$$

Define $F:\mathbb{R}_+^2 \longrightarrow \mathbb{R}^2$ by

$$F((\pi_1,\pi_2)) = \begin{cases} \left(\dfrac{\ln \pi_1}{\sqrt{\pi_1}},\dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 \leq e \\ \left(\dfrac{\pi_1}{e\sqrt{e}},\dfrac{-1}{\sqrt{\pi_2}}\right), & \pi_1 > e. \end{cases} \tag{60}$$

Obviously, $F \in \mathscr{F}^2$. Also, take $\varsigma = (\varsigma_1,\varsigma_2) = (\ln 2, 1)$.

Reviewing the above example, we can show that all of the conditions of Theorem 2 are satisfied. Thus, Υ has a unique fixed point. Here, $\Upsilon((0,0,0,\cdots),(0,0,0,\cdots)) = (0,0,0,\cdots)$ and $(0,0,0,\cdots)$ is the unique fixed point.

3. Application

Let $(E,\|\cdot\|_E)$ be a Banach space and $A_1,\cdots,A_k : E^k \longrightarrow E$ be k nonlinear operators. In this section, motivated by the work in [26], we will present a result on existence of a solution for the following semilinear operator system:

$$\begin{aligned} A_1(\iota_1,\iota_2,\cdots,\iota_k) &= \iota_1, \\ &\vdots \\ A_k(\iota_1,\iota_2,\cdots,\iota_k) &= \iota_k. \end{aligned} \tag{61}$$

Similar systems which appear in various branches of mathematics could be seen in [27].

Let $\Delta = E^k$ and define $V:\Delta\times\Delta \longrightarrow \mathbb{R}^k$, for $u=(\iota_1,\cdots,\iota_k)$, $v=(\varepsilon_1,\cdots,\varepsilon_k)\in\Delta$ by $V(u,v)=(\|\iota_1-\varepsilon_1\|_E,\cdots,\|\iota_k-\varepsilon_k\|_E)$. Evidently, (Δ, V) is a complete VVMS.

If we define a mapping $\Upsilon:\Delta^k \longrightarrow \Delta$ by

$$\Upsilon(u,u,\cdots,u) = (A_1(\iota_1,\iota_2,\cdots,\iota_k),\cdots,A_k(\iota_1,\iota_2,\cdots,\iota_k)), \tag{62}$$

then the system (61) can be written as a fixed point problem such as

$$\Upsilon(u,u,\cdots,u) = (\iota_1,\iota_2,\cdots,\iota_k) = u, \tag{63}$$

in the space Δ. Therefore, applying Theorem 2, we investigate the sufficient hypothesis which leads to the existence of a solution of problem (63).

Theorem 9. *Assume that there exist positive real numbers ς_i ($i = 1,\cdots,k$) such that*

$$\|A_i(\iota_1,\iota_2,\cdots,\iota_k)-A_i(\varepsilon_1,\varepsilon_2,\cdots,\varepsilon_k)\|_E \leq e^{-\varsigma_i}\|\iota_i-\varepsilon_i\|_E, \tag{64}$$

for all $u=(\iota_1,\cdots,\iota_k)$, $v=(\varepsilon_1,\cdots,\varepsilon_k)\in E^k$ with $\iota_i\neq\varepsilon_i$. Then, the system (61) has a unique solution in E^k.

Proof. By the inequality (64), we have

$$\left(\varsigma_i + \ln \|A_i(\iota_1, \iota_2, \cdots, \iota_k) - A_i(\kappa_1, \kappa_2, \cdots, \kappa_k)\|_E\right) \le \ln \|\iota_i - \kappa_i\|_E, \tag{65}$$

for all $i = 1, \cdots, k$. Hence, we get

$$\begin{aligned}&\left(\varsigma_1 + \ln \|A_1(u_1) - A_1(u_2)\|_E, \cdots, \varsigma_k + \ln \|A_k(u_1) - A_k(u_2)\|_E\right)\\&\quad \preceq (\ln \|\iota_1 - \kappa_1\|_E, \cdots, \ln \|\iota_k - \kappa_k\|_E).\end{aligned} \tag{66}$$

Taking the function $F \in \mathscr{F}^k$ as $F(\pi_1, \cdots, \pi_k) = (\ln \pi_1, \cdots, \ln \pi_k)$, the above inequality can be written as

$$\begin{aligned}&(\varsigma_1, \cdots, \varsigma_k) + F((\|A_1u_1 - A_1u_2\|_E, \cdots, \|A_ku_1 - A_ku_2\|_E))\\&\quad \le F((\|\iota_1 - \kappa_1\|_E, \cdots, \|\iota_k - \kappa_k\|_E))\\&\quad = F(V(u_1, u_2)), VF(\sup \{V(u_1, u_2), V(u_1, u_2), \cdots, V(u_1, u_2)\}),\end{aligned} \tag{67}$$

or, equivalently,

$$\begin{aligned}&(\varsigma_1, \cdots, \varsigma_k) + F(V(\Upsilon(u_1, \cdots, u_k), \Upsilon(u_2, \cdots, u_{k+1})))\\&\quad \preceq F(\sup \{V(u_1, u_2), V(u_2, u_3), \cdots, V(u_k, u_{k+1})\}),\end{aligned} \tag{68}$$

where $\varsigma = (\varsigma_1, \cdots, \varsigma_k)$. Thus, applying Theorem 2, Υ possesses a unique fixed point in $\Delta = E^k$, or, equivalently, the semilinear operator system (61) has a unique solution in E^k.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no competing interests.

Authors' Contributions

All authors read and approved the manuscript.

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