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Research Article

Multiple Positive Solutions for a System of Nonlinear Caputo-Type Fractional Differential Equations

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By using fixed-point index theory, we consider the existence of multiple positive solutions for a system of nonlinear Caputo-type fractional differential equations with the Riemann-Stieltjes boundary conditions.

1. Introduction

Fractional order calculus is more widely used than integer order calculus (see [1]). Based on it, many researchers have been focused on the study of the existence of positive solutions for various fractional differential equations with some boundary conditions. We can refer to [2–35] for some recent works about fractional boundary value problems.

In [14], Ma and Cui discussed the following fractional boundary value problem:

$$\begin{cases} {}^cD^{\theta}_{0+}p(y) + \mu f(y, p(y)) = 0, & y \in [0,1], \\ p(0) = p''(0) = 0, \\ p(1) = \displaystyle\int_0^1 p(y)\,dA(y), \end{cases} \tag{1}$$

where $\theta \in (2,3)$, ${}^cD^{\theta}_{0+}$ is the Caputo fractional derivative, $f \in C([0,1] \times [0,+\infty), [0,+\infty)), \mu > 0$ is a parameter. By using the Guo-Krasnosel'skii fixed-point theorem, they proved that the fractional differential equation has at least a positive solution when μ satisfies some conditions.

A few researchers have studied the existence of solutions for the systems of nonlinear fractional differential equations (see [23–35]). For example, in [24], by monotone iterative technique and cone theory, Zhang et al. have studied the uniqueness of a solution for fractional systems with the Riemann-Stieltjes integral boundary condition. Hao et al. [32] have studied a system of fractional boundary value problems with two parameters; by using the Guo-Krasnosel'skii fixed-point theorem, they obtained the existence of positive solutions for the system in terms of different values of parameters. Meanwhile, a few researchers have been considering the systems for nonlinear integer order boundary value problems, such as in [33], where the researchers considered the existence of multiple positive solutions for a system of nonlinear third-order differential equations. In [34], by Banach's contraction principle, the researchers considered the existence of a unique solution for a system of second order differential equations with coupled integral boundary conditions.

Inspired by [14, 23–35], in this paper, we want to study the following system of fractional differential equations:

$$\begin{cases} -{}^cD^{\theta_1}_{0+}x(t) = f_1(t, x(t), y(t)), & t \in [0,1], \\ -{}^cD^{\theta_2}_{0+}y(t) = f_2(t, x(t), y(t)), & t \in [0,1], \\ x(0) = x''(0) = 0, \quad x(1) = \displaystyle\int_0^1 x(t)\,dA_1(t), \\ y(0) = y''(0) = 0, \quad y(1) = \displaystyle\int_0^1 y(t)\,dA_2(t), \end{cases} \tag{2}$$

where $f_i : [0,1] \times [0,+\infty) \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous; $\theta_i \in (2,3)$; ${}^cD_{0+}^{\theta_i}$ is the Caputo fractional derivative; A_i is a bounded variation function with positive measure with $B_i = \int_0^1 t dA_i(t) < 1, i = 1, 2$.

The main purpose of this paper is that we prove that system (2) has one and multiple positive solutions by the fixed-point index theory.

2. Preliminaries

In this section, we only give the definition of Caputo's fractional derivative. For some of the other definitions and properties of Caputo's fractional derivative, we can refer to [2]. We mainly give the relevant Green functions and the used lemmas.

Definition 1 (see [2, 14]). For a function $x \in C^n[0,+\infty)$, we define Caputo's fractional derivative of order $\theta > 0$ as follows:

$$ {}^cD_{0+}^{\theta} x(t) = \frac{1}{\Gamma(n-\theta)} \int_0^t (t-s)^{n-\theta-1} x^n(s) ds, \quad n-1 < \theta < n, \tag{3}$$

where n is the smallest integer greater than or equal to θ.

From [14], we have the following lemmas.

Lemma 2 (see [14]). *Let $u \in C[0,1]$ and $\theta_1, \theta_2 \in (2,3)$. Then p is the solution of the linear Caputo fractional differential equation*

$$\begin{cases} {}^cD_{0+}^{\theta i} p(t) + u(t) = 0, & t \in [0,1], \\ p(0) = p''(0) = 0, \\ p(1) = \int_0^1 p(t) dA_i(t), \end{cases} \tag{4}$$

if and only if p is the solution of the integral equation

$$ p(t) = \int_0^1 K_i(t,s) u(s) ds, \tag{5}$$

where

$$ K_i(t,s) = \frac{1}{\Gamma(\theta_i)} \begin{cases} \frac{t}{1-B_i}\left[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1} dA_i(t)\right] - (t-s)^{\theta_i-1}, & 0 \le s, t \le 1, \\ \frac{t}{1-B_i}\left[(1-s)^{\theta_i-1} - \int_s^1 (t-s)^{\theta_i-1} dA_i(t)\right], & 0 \le t \le s \le 1, \end{cases} \tag{6}$$

and $B_i = \int_0^1 t dA_i(t) < 1, i = 1, 2$.

Lemma 3 (see [14]). *The above Green's function $K_i(t,s) (i = 1,2)$ has the following properties:*

(i) $\Gamma(\theta_i) K_i(t,s) \le (1/(1-B_i))(1-s)^{\theta_i-1}$, for $t, s \in [0,1]$

(ii) $\Gamma(\theta_i) K_i(t,s) \ge N_i (1-s)^{\theta_i-1}$, for $t \in [(1/4),(3/4)]$, $s \in [0,1]$

where

$$ N_i = \min\left\{ \frac{1 - \int_0^1 t^{\theta_i-1} dA_i(t)}{4(1-B_i)}, \min_{t\in[(1/4),(3/4)]} t\left(1 - t^{\theta_i-2}\right) \right\}, \quad i = 1, 2. \tag{7}$$

Lemma 4 (see [36]). *Let E be a Banach space and $P \subset E$ be a cone. Define $D_r = \{u \in P : \|u\| < r\}$, where $r > 0$. Suppose that $T : \bar{D}_r \longrightarrow P$ is completely continuous, and $Tu \neq u$, for $u \in \partial D_r$.*

(i) If $\|Tu\| \le \|u\|, \forall u \in \partial D_r$, then $i(T, D_r, P) = 1$

(ii) If $\|Tu\| \ge \|u\|, \forall u \in \partial D_r$, then $i(T, D_r, P) = 0$

3. Main Results

Let $E = C[0,1] \times C[0,1]$ be a Banach space with the norm $\|(u,v)\|_E = \|u\| + \|v\|$ on E, where $\|u\| = \max_{0\le t\le 1} |u(t)|$.

Define the cone

$$ P = \left\{ (u,v) \in E : u \ge 0, v \ge 0, \min_{(1/4)\le t\le (3/4)} (u(t) + v(t)) \ge K \|(u,v)\|_E \right\}, \tag{8}$$

where

$$ K = \min\{N_1(1-B_1), N_2(1-B_2)\} < 1. \tag{9}$$

N_1 and N_2 are defined by (7), and the nonlinear operators T_1, T_2, and T are defined by

$$T_1(x,y)(t)=\int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds, \quad t\in[0,1],$$
$$T_2(x,y)(t)=\int_0^1 K_2(t,s)f_2(s,x(s),y(s))ds, \quad t\in[0,1],$$
$$T(x,y)=(T_1(x,y),T_2(x,y)), \quad (x,y)\in E, \tag{10}$$

where $K_i(t,s)(i=1,2)$ is defined by (6).

It is known that fixed points of the operator T in P are positive solutions of the system (2).

Lemma 5. *The operator $T: P\longrightarrow P$ is completely continuous.*

Proof. For $(x,y)\in P$, when $t\in[0,1]$, we have $T_1(x,y)(t)\geq 0$, $T_2(x,y)(t)\geq 0$.

When

$$t\in\left[\frac{1}{4},\frac{3}{4}\right], \tag{11}$$

by Lemma 3, we obtain

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds \\ &\geq \frac{N_1}{\Gamma(\theta_1)}\int_0^1 (1-s)^{\theta_1-1}f_1(s,x(s),y(s))ds \\ &= \frac{N_1(1-B_1)}{\Gamma(\theta_1)}\int_0^1 \frac{(1-s)^{\theta_1-1}}{(1-B_1)}f_1(s,x(s),y(s))ds \\ &\geq N_1(1-B_1)\max_{t\in[0,1]}\int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds \\ &= N_1(1-B_1)\|T_1(x,y)\|. \end{aligned} \tag{12}$$

Similar to the proof of (12), for

$$(x,y)\in P, \quad t\in\left[\frac{1}{4},\frac{3}{4}\right], \tag{13}$$

we have

$$T_2(x,y)(t)\geq N_2(1-B_2)\|T_2(x,y)\|. \tag{14}$$

By (12) and (14), we get

$$\begin{aligned} &\min_{t\in[(1/4),(3/4)]}(T_1(x,y)(t)+T_2(x,y)(t)) \\ &\geq N_1(1-B_1)\|T_1(x,y)\|+N_2(1-B_2)\|T_2(x,y)\| \\ &\geq K\|T(x,y)\|_E. \end{aligned} \tag{15}$$

By (15), $T(P)\subset P$. Similar to the proof of Lemma 6 in [14], we know that $T_1, T_2: C[0,1]\longrightarrow C[0,1]$ are completely continuous. So $T: P\longrightarrow P$ is completely continuous.

For convenience, some marks and assumptions are given. Some of the marks are as follows:

$$\begin{aligned} g_0 &= \lim_{(x,y)\to(0^+,0^+)}\frac{f_1(t,x,y)}{x+y}, \\ \bar{g}_0 &= \lim_{(x,y)\to(0^+,0^+)}\frac{f_2(t,x,y)}{x+y}, \\ g_\infty &= \lim_{(x,y)\to(+\infty,+\infty)}\frac{f_1(t,x,y)}{x+y}, \\ \bar{g}_\infty &= \lim_{(x,y)\to(+\infty,+\infty)}\frac{f_2(t,x,y)}{x+y}, \\ k_1 &= \frac{N_1K}{\Gamma(\theta_1)}\int_{(1/4)}^{(3/4)}(1-s)^{\theta_1-1}ds, \\ k_3 &= \frac{N_2K}{\Gamma(\theta_2)}\int_{(1/4)}^{(3/4)}(1-s)^{\theta_2-1}ds, \end{aligned} \tag{16}$$

where K is defined by (9).

$$\begin{aligned} k_2 &= \frac{1}{\Gamma(\theta_1)(1-B_1)}\int_0^1(1-s)^{\theta_1-1}ds, \\ k_4 &= \frac{1}{\Gamma(\theta_2)(1-B_2)}\int_0^1(1-s)^{\theta_2-1}ds, \\ k &= \min\{k_1,k_3\}, h=\max\{k_2,k_4\}. \end{aligned} \tag{17}$$

Some of the assumptions are as follows:

S_1. There exist $a>0$ and $b_2>b_1>0$ such that

(i) $f_1(t,x,y)\geq a(x+y), f_2(t,x,y)\geq a(x+y)$, $\forall t\in[0,1]$, $0\leq x,y\leq b_1$

(ii) $f_1(t,x,y)<(b_2/2h), f_2(t,x,y)<(b_2/2h)$, $\forall t\in[0,1]$, $0\leq x,y\leq b_2$

where a also satisfies $ak>(1/2)$, k and h are defined by (17).

S_2. There exists $q>0$ such that

$$\begin{aligned} f_1(t,x,y) &> \frac{Kq}{2k}, \\ f_2(t,x,y) &> \frac{Kq}{2k}, \\ &\forall t\in[0,1], x+y\in[Kq,q], \end{aligned} \tag{18}$$

where K is defined by (9) and k is defined by (17).

Theorem 6. *Suppose that $g_0=\bar{g}_0=\infty$ and $g_\infty=\bar{g}_\infty=0$ uniformly on $t\in[0,1]$. Then, system (2) has a positive solution.*

Proof. By $g_0 = \bar{g}_0 = \infty$, we know that there exist $r_1 > 0$ and $m_1 > 0$ such that

$$\begin{aligned} f_1(t,x,y) &\geq m_1(x+y), \\ f_2(t,x,y) &\geq m_1(x+y), \\ &\forall 0 \leq x, y \leq r_1, t \in [0,1], \end{aligned} \tag{19}$$

and $m_1 k_1 > (1/2)$, $m_1 k_3 > (1/2)$.

Set $P_{r_1} = \{(x,y) \in P : \|(x,y)\|_E < r_1\}$. By Lemma 3 and (19), for $(x,y) \in \partial P_{r_1}$, $t \in [(1/4),(3/4)]$, we have

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))\,ds \\ &\geq \frac{N_1 m_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1-s)^{\theta_1-1}(x(s)+y(s))\,ds \\ &\geq m_1 \frac{N_1 K}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1-s)^{\theta_1-1} ds (\|x\| + \|y\|) \\ &= m_1 k_1 \|(x,y)\|_E > \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{20}$$

Similar to the proof of (20), we have

$$T_2(x,y)(t) > \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_1}. \tag{21}$$

So by (20) and (21), we have

$$\begin{aligned} \|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &> \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_1}. \end{aligned} \tag{22}$$

By Lemma 4 and (22), we have

$$i(T, P_{r_1}, P) = 0. \tag{23}$$

From $g_\infty = \bar{g}_\infty = 0$, we know that there exist $\tilde{r} > r_1 > 0$ and $m_2 > 0$ such that

$$\begin{aligned} f_1(t,x,y) &\leq m_2(x+y), \\ f_2(t,x,y) &\leq m_2(x+y), \\ &\forall x + y \geq \tilde{r}, t \in [0,1], \end{aligned} \tag{24}$$

and $m_2 h < (1/4)$. In this part, we divide two cases. One case is that f_1 and f_2 are bounded. Namely, there exists $W > 0$ such that

$$\begin{aligned} f_1(t,x,y) &\leq W, \\ f_2(t,x,y) &\leq W, \\ &\forall x, y \in [0,+\infty), t \in [0,1]. \end{aligned} \tag{25}$$

Take $r_2 > \max\{2Wh, r_1\}$. Set $P_{r_2} = \{(x,y) \in P : \|(x,y)\|_E < r_2\}$. For $(x,y) \in \partial P_{r_2}$ and $t \in [0,1]$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))\,ds \\ &\leq W \int_0^1 K_1(t,s)\,ds \leq \frac{W}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} ds \\ &\leq Wh < \frac{r_2}{2} = \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{26}$$

The other case is that f_1 and f_2 are unbounded. By the continuity of f_1, there exists $r_2 > \max\{r_1, \tilde{r}\}$ such that

$$f_1(t,x,y) \leq f_1(t,r_2,r_2), \quad \text{for } 0 < x, y \leq r_2. \tag{27}$$

Hence, set $P_{r_2} = \{(x,y) \in P : \|(x,y)\|_E < r_2\}$. For $(x,y) \in \partial P_{r_2}$ and $t \in [0,1]$ we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))\,ds \\ &\leq \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} f_1(s,r_2,r_2)\,ds \\ &\leq \frac{1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1} m_2(r_2+r_2)\,ds \\ &= 2m_2 k_2 r_2 = 2m_2 k_2 \|(x,y)\|_E < 2m_2 h \|(x,y)\|_E \\ &< \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{28}$$

So in either case, there always exists $r_2 > r_1$ such that

$$\|T_1(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_2}. \tag{29}$$

Similarly, we have

$$\|T_2(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_2}. \tag{30}$$

So by (29) and (30), we have

$$\begin{aligned} \|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &< \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{r_2}. \end{aligned} \tag{31}$$

By Lemma 4 and (31), we have

$$i(T, P_{r_2}, P) = 1. \tag{32}$$

From (23) and (32), we obtain

$$i(T, P_{r_2} \setminus \bar{P}_{r_1}, P) = i(T, P_{r_2}, P) - i(T, P_{r_1}, P) = 1. \tag{33}$$

So T has a fixed point (x, y) in $P_{r_2} \setminus \bar{P}_{r_1}$. It is obvious that (x, y) is a positive solution of system (2).

Theorem 7. *Suppose that S_1 holds and $g_\infty = \bar{g}_\infty = \infty$. Then, the system (2) has two positive solutions.*

Proof. Set $P_{b_1} = \{(x, y) \in P : \|(x, y)\|_E < b_1\}$. By Lemma 3 and S_1 (i), for $(x, y) \in \partial P_{b_1}$ and $t \in [(1/4), (3/4)]$, we get

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s)) ds \\ &\geq \frac{N_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} f_1(s, x(s), y(s)) ds \\ &\geq \frac{N_1 a}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} (x(s) + y(s)) ds \\ &\geq \frac{N_1 a K}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} ds (\|x\| + \|y\|) \\ &= a k_1 \|(x, y)\|_E \geq a k \|(x, y)\|_E \geq \frac{1}{2} \|(x, y)\|_E. \end{aligned} \tag{34}$$

So

$$\|T_1(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_1}. \tag{35}$$

Similar to the proof of (35), we have

$$\|T_2(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_1}. \tag{36}$$

So by (35) and (36), we have

$$\begin{aligned} \|T(x, y)\|_E &= \|T_1(x, y)\| + \|T_2(x, y)\| \\ &> \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_1}. \end{aligned} \tag{37}$$

By Lemma 4 and (37), we have

$$i(T, P_{b_1}, P) = 0. \tag{38}$$

Set $P_{b_2} = \{(x, y) \in P : \|(x, y)\|_E < b_2\}$. By Lemma 3 and S_1 (ii), for $(x, y) \in \partial P_{b_2}$ and $t \in [0, 1]$, we get

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s)) ds \\ &\leq \frac{1}{\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1} f_1(s, x(s), y(s)) ds \\ &< \frac{b_2}{2h\Gamma(\theta_1)(1 - B_1)} \int_0^1 (1 - s)^{\theta_1 - 1} ds = \frac{b_2 k_2}{2h} \\ &\leq \frac{b_2}{2} = \frac{1}{2} \|(x, y)\|_E. \end{aligned} \tag{39}$$

So

$$\|T_1(x, y)\| < \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_2}. \tag{40}$$

Similarly,

$$\|T_2(x, y)\| < \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_2}. \tag{41}$$

So by (40) and (41), we have

$$\begin{aligned} \|T(x, y)\|_E &= \|T_1(x, y)\| + \|T_2(x, y)\| \\ &< \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_2}. \end{aligned} \tag{42}$$

By Lemma 4 and (42), we have

$$i(T, P_{b_2}, P) = 1. \tag{43}$$

By $g_\infty = \bar{g}_\infty = \infty$, there exist $m_3 > 0$ and $b > b_2$ such that

$$\begin{gathered} f_1(t, x, y) \geq m_3(x + y), \\ f_2(t, x, y) \geq m_3(x + y), \\ \forall x + y \geq b, t \in [0, 1], \end{gathered} \tag{44}$$

and $m_3 k > (1/2)$.

Choose $b_3 > \max\{b_2, (b/K)\}$. Set $P_{b_3} = \{(x, y) \in P : \|(x, y)\|_E < b_3\}$. For $(x, y) \in \partial P_{b_3}$

$$\min_{t \in [(1/4), (3/4)]} (x(t) + y(t)) \geq K(\|x\| + \|y\|) = K b_3 > b. \tag{45}$$

Then, by (44) and Lemma 3, we get

$$\begin{aligned} T_1(x, y)(t) &= \int_0^1 K_1(t, s) f_1(s, x(s), y(s)) ds \\ &\geq \frac{N_1}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} f_1(s, x(s), y(s)) ds \\ &\geq \frac{N_1 m_3}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} (x(s) + y(s)) ds \\ &\geq \frac{N_1 m_3 K}{\Gamma(\theta_1)} \int_{1/4}^{3/4} (1 - s)^{\theta_1 - 1} ds (\|x\| + \|y\|) \\ &= m_3 k_1 \|(x, y)\|_E \geq m_3 k \|(x, y)\|_E > \frac{1}{2} \|(x, y)\|_E. \end{aligned} \tag{46}$$

So

$$\|T_1(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_3}. \tag{47}$$

Similarly,

$$\|T_2(x, y)\| > \frac{1}{2} \|(x, y)\|_E, \quad \forall (x, y) \in \partial P_{b_3}. \tag{48}$$

So by (47) and (48), we have

$$\begin{aligned}\|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &> \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{b_3}.\end{aligned} \tag{49}$$

By Lemma 4 and (49), we have

$$i(T, P_{b_3}, P) = 0. \tag{50}$$

Since $0 < b_1 < b_2 < b_3$, by (38), (43), and (50), we get

$$i(T, P_{b_3} \setminus \bar{P}_{b_2}, P) = i(T, P_{b_3}, P) - i(T, P_{b_2}, P) = -1, \tag{51}$$

$$i(T, P_{b_2} \setminus \bar{P}_{b_1}, P) = i(T, P_{b_2}, P) - i(T, P_{b_1}, P) = 1. \tag{52}$$

From (51), we know that T has a fixed point $(x_1, y_1) \in (P_{b_3} \setminus \bar{P}_{b_2})$. From (52), we know that T has another fixed point$(x_2, y_2) \in (P_{b_2} \setminus P_{b_1})$. So the system (2) has two positive solutions(x_1, y_1) and (x_2, y_2), with $0 < \|(x_2, y_2)\|_E < b_2 < \|(x_1, y_1)\|_E$.

Theorem 8. *Let $g_0 = \bar{g}_0 = 0$ and $g_\infty = \bar{g}_\infty = 0$. In addition, suppose that S_2 holds. Then, system (2) has two positive solutions.*

Proof. By $g_0 = \bar{g}_0 = 0$, we know that there exist $c_1 > 0$ and $h_1 \in (0, q)$ such that

$$\begin{gathered} f_1(t,x,y) \le c_1(x+y), \\ f_2(t,x,y) \le c_1(x+y), \\ \forall 0 \le x, y \le h_1, t \in [0,1], \end{gathered} \tag{53}$$

and $c_1 h < (1/2)$. Set $P_{h_1} = \{(x,y) \in P : \|(x,y)\|_E < h_1\}$.

By Lemma 3 and (53), for $(x,y) \in \partial P_{h_1}$ and $t \in [0,1]$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s, x(s), y(s))\,ds \\ &\le \frac{c_1}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1 - 1}(x(s)+y(s))\,ds \\ &\le \frac{c_1(\|x\|+\|y\|)}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1}\,ds \\ &= c_1 k_2 \|(x,y)\|_E < c_1 h \|(x,y)\|_E < \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{54}$$

So

$$\|T_1(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_1}. \tag{55}$$

Similarly,

$$\|T_2(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_1}. \tag{56}$$

So by (55) and (56), we have

$$\begin{aligned}\|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &< \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_1}.\end{aligned} \tag{57}$$

By Lemma 4 and (57), we have

$$i(T, P_{h_1}, P) = 1. \tag{58}$$

From $g_\infty = \bar{g}_\infty = 0$, we know that there exist $c_2 > 0$ and $H > h_1$ such that

$$\begin{gathered} f_1(t,x,y) \le c_2(x+y), \\ f_2(t,x,y) \le c_2(x+y), \\ \forall x + y \ge H, t \in [0,1], \end{gathered} \tag{59}$$

and $c_2 h < (1/4)$. In this part, we divide two cases. One case is that f_1 and f_2 are bounded. Namely, there exists $W > 0$ such that

$$\begin{gathered} f_1(t,x,y) \le W, \\ f_2(t,x,y) \le W, \\ \forall x, y \in [0, +\infty), t \in [0,1]. \end{gathered} \tag{60}$$

Take $h_2 > \max\{2Wh, q\}$. Set $P_{h_2} = \{(x,y) \in P : \|(x,y)\|_E < h_2\}$. For $(x,y) \in \partial P_{h_2}$ and $t \in [0,1]$, we get

$$\begin{aligned} T_1(x,y)(t) &= \int_0^1 K_1(t,s) f_1(s,x(s),y(s))\,ds \le W \int_0^1 K_1(t,s)\,ds \\ &\le \frac{W}{\Gamma(\theta_1)(1-B_1)} \int_0^1 (1-s)^{\theta_1-1}\,ds \le Wh < \frac{h_2}{2} \\ &= \frac{1}{2}\|(x,y)\|_E. \end{aligned} \tag{61}$$

Another case is that f_1 and f_2 are unbounded. By the continuity of f_1, there exists $h_2 > \max\{h_1, q, H\}$ such that

$$f_1(t,x,y) \le f_1(t, h_2, h_2), \quad \text{for } 0 < x, y \le h_2. \tag{62}$$

Hence, set $P_{h_2} = \{(x,y) \in P : \|(x,y)\|_E < h_2\}$. For $(x,y) \in \partial P_{h_2}$, we get

$$\begin{aligned}T_1(x,y)(t) &= \int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds \\ &\le \frac{1}{\Gamma(\theta_1)(1-B_1)}\int_0^1 (1-s)^{\theta_1-1}f_1(s,h_2,h_2)ds \\ &\le \frac{1}{\Gamma(\theta_1)(1-B_1)}\int_0^1 (1-s)^{\theta_1-1}c_2(h_2+h_2)ds \\ &= 2c_2k_2h_2 = 2c_2k_2\|(x,y)\|_E < 2c_2h\|(x,y)\|_E \\ &< \frac{1}{2}\|(x,y)\|_E.\end{aligned} \tag{63}$$

So in either case, there always exists $h_2 > \max\{h_1, q\}$ such that

$$\|T_1(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_2}. \tag{64}$$

Similarly, we have

$$\|T_2(x,y)\| < \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_2}. \tag{65}$$

Thus

$$\begin{aligned}\|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &< \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_{h_2}.\end{aligned} \tag{66}$$

So by Lemma 4 and (66), we get

$$i(T, P_{h_2}, P) = 1. \tag{67}$$

Set $P_q = \{(x,y) \in P : \|(x,y)\|_E < q\}$. For $(x,y) \in \partial P_q$

$$\min_{t\in[(1/4),(3/4)]}(x(t)+y(t)) \ge K(\|x\| + \|y\|) = Kq. \tag{68}$$

Then, by S_2 and Lemma 3, we get

$$\begin{aligned}T_1(x,y)(t) &= \int_0^1 K_1(t,s)f_1(s,x(s),y(s))ds \\ &\ge \frac{N_1}{\Gamma(\theta_1)}\int_{1/4}^{3/4}(1-s)^{\theta_1-1}f_1(s,x(s),y(s))ds \\ &> \frac{N_1Kq}{2k\Gamma(\theta_1)}\int_{1/4}^{3/4}(1-s)^{\theta_1-1}ds > \frac{q}{2} = \frac{1}{2}\|(x,y)\|_E.\end{aligned} \tag{69}$$

So

$$\|T_1(x,y)\| > \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_q. \tag{70}$$

Similarly,

$$\|T_2(x,y)\| > \frac{1}{2}\|(x,y)\|_E, \quad \forall (x,y) \in \partial P_q. \tag{71}$$

So by (70) and (71), we have

$$\begin{aligned}\|T(x,y)\|_E &= \|T_1(x,y)\| + \|T_2(x,y)\| \\ &> \|(x,y)\|_E, \quad \forall (x,y) \in \partial P_q.\end{aligned} \tag{72}$$

By Lemma 4 and (72), we have

$$i(T, P_q, q) = 0. \tag{73}$$

Since $0 < h_1 < q < h_2$, by (58), (67), and (73), we get $i(T, P_q \setminus \bar{P}_{h_1}, P) = i(T, P_q, P) - i(T, P_{h_1}, P) = -1$. So T has a fixed point $(x_1, y_1) \in (P_q \setminus \bar{P}_{h_1})$.$i(T, P_{h_2} \setminus \bar{P}_q, P) = i(T, P_{h_2}, P) - i(T, P_q, P) = 1$. So T has another fixed point $(x_2, y_2) \in P_{h_2} \setminus \bar{P}_q$. Therefore, system (2) has at least two positive solutions.

4. Applications

Example 9. We study the following Caputo-type fractional system:

$$\begin{cases} -{}^cD_{0+}^{(5/2)}x(t) = (2+3t)\sqrt{x(t)+y(t)}, & t \in [0,1], \\ -{}^cD_{0+}^{(5/2)}y(t) = \left(1+t^2+t^4\right)\sqrt[5]{x(t)+y(t)}, & t \in [0,1], \\ x(0) = x''(0) = 0, \quad x(1) = \dfrac{1}{2}\displaystyle\int_0^1 x(t)dt, \\ y(0) = y''(0) = 0, \quad y(1) = \dfrac{1}{2}\displaystyle\int_0^1 y(t)dt, \end{cases} \tag{74}$$

where $\theta_1 = \theta_2 = 5/2$, $A_1(t) = A_2(t) = (1/2)t$, $B_1 = B_2 = 1/4$, $f_1(t,x,y) = (2+3t)\sqrt{x+y}$, and $f_2(t,x,y) = (1+t^2+t^4)\sqrt[5]{x+y}$. Obviously, $g_0 = \bar{g}_0 = \infty$ and $g_\infty = \bar{g}_\infty = 0$. From Theorem 6, system (74) has a positive solution.

Example 10. We study the following Caputo-type fractional system:

$$\begin{cases} -{}^cD_{0+}^{5/2}x(t) = f_1(t,x(t),y(t)), & t \in [0,1], \\ -{}^cD_{0+}^{5/2}y(t) = f_2(t,x(t),y(t)), & t \in [0,1], \\ x(0) = x''(0) = 0, \quad x(1) = \dfrac{1}{2}\displaystyle\int_0^1 x(t)dt, \\ y(0) = y''(0) = 0, \quad y(1) = \dfrac{1}{2}\displaystyle\int_0^1 y(t)dt, \end{cases} \tag{75}$$

where $\theta_1 = \theta_2 = 5/2$, $A_1(t) = A_2(t) = (1/2)t$, and $B_1 = B_2 = 1/4$.

Take

$$f_1(t,x,y)=\begin{cases}10^4(t+1), & t\in[0,1],0\le x,y\le \dfrac{1}{2},\\ 10^4(t+1)(x+y), & t\in[0,1],\dfrac{1}{2}\le x,y\le 1,\\ 2\times 10^4(t+1), & t\in[0,1],1\le x,y\le 10^6,\\ 10^{-8}\left(\dfrac{t+1}{2}\right)(x+y)^2, & t\in[0,1],x,y\ge 10^6,\end{cases} \tag{76}$$

and

$$f_2(t,x,y)=\begin{cases}10^4(t+1), & t\in[0,1],0\le x,y\le \dfrac{1}{2},\\ 10^4(t+1)(x+y), & t\in[0,1],\dfrac{1}{2}\le x,y\le 1,\\ 2\times 10^4(t+1), & t\in[0,1],1\le x,y\le 10^6,\\ 10^{-14}\left(\dfrac{t+1}{4}\right)(x+y)^3, & t\in[0,1],x,y\ge 10^6.\end{cases} \tag{77}$$

By simple computation, we get $K=9/256$, $k=(9(9\sqrt{3-1}))/(327680\sqrt{\pi})$, and $h=32/(45\sqrt{\pi})$.

Choose $a=7000$, $b_2=10^6$, and $b_1=1$, then $(b_2/2h)=((45\sqrt{\pi})/64)\cdot 10^6$.

When $0\le x,y\le b_1=1$,

(1) $0\le x,y\le(1/2), f_1(t,x,y)=10^4(t+1)\ge 10^4\ge 7000(x+y)$; $f_2(t,x,y)=10^4(t+1)\ge 10^4\ge 7000(x+y)$

(2) $(1/2)\le x,y\le 1, f_1(t,x,y)=10^4(t+1)(x+y)\ge 10^4(x+y)\ge 7000(x+y)$; $f_2(t,x,y)=10^4(t+1)(x+y)\ge 10^4(x+y)\ge 7000(x+y)$

When $0\le x,y\le b_2=10^6$,

(1) $0\le x,\ \ y\le(1/2),\ \ f_1(t,x,y)=10^4(t+1)\le 2\times 10^4<((45\sqrt{\pi})/64)\cdot 10^6$; $f_2(t,x,y)=10^4(t+1)\le 2\times 10^4<((45\sqrt{\pi})/64)\cdot 10^6$

(2) $(1/2)\le x,\ \ y\le 1,\ f_1(t,x,y)=10^4(t+1)(x+y)\le 4\times 10^4<((45\sqrt{\pi})/64)\cdot 10^6$; $f_2(t,x,y)=10^4(t+1)(x+y)\le 4\times 10^4<((45\sqrt{\pi})/64)\cdot 10^6$

(3) $1\le x,\ \ y\le 10^6, f_1(t,x,y)=2\times 10^4(t+1)\le 4\times 10^4<((45\sqrt{\pi})/64)\cdot 10^6$; $f_2(t,x,y)=2\times 10^4(t+1)\le 4\times 10^4<((45\sqrt{\pi})/64)\cdot 10^6$

Obviously, $g_\infty=\bar{g}_\infty=\infty$. So by Theorem 7, system (74) has at least two positive solutions.

Example 11. We study the following Caputo-type fractional system:

$$\begin{cases}-{}^cD_{0+}^{5/2}x(t)=f_1(t,x(t),y(t)), & t\in[0,1],\\ -{}^cD_{0+}^{5/2}y(t)=f_2(t,x(t),y(t)), & t\in[0,1],\\ x(0)=x''(0)=0, & x(1)=\dfrac{1}{2}\displaystyle\int_0^1 x(t)dt,\\ y(0)=y''(0)=0, & y(1)=\dfrac{1}{2}\displaystyle\int_0^1 y(t)dt,\end{cases} \tag{78}$$

where $\theta_1=\theta_2=(5/2)$, $A_1(t)=A_2(t)=(1/2)t$, and $B_1=B_2=(1/4)$.

Take

$$f_1(t,x,y)=\begin{cases}\dfrac{1}{2}\times 10^8(t+1)(x+y)^2, & t\in[0,1],0\le x,y\le 10^{-4},\\ 10^4(t+1)(x+y), & t\in[0,1],10^{-4}\le x,y\le 10^{-2},\\ \dfrac{4(t+1)}{x+y}, & t\in[0,1],x,y\ge 10^{-2},\end{cases} \tag{79}$$

and

$$f_2(t,x,y)=\begin{cases}\dfrac{1}{2}\times 10^8(t+1)(x+y)^2, & t\in[0,1],0\le x,y\le 10^{-4},\\ 10^4(t+1)(x+y), & t\in[0,1],10^{-4}\le x,y\le 10^{-2},\\ \dfrac{8\times 10^{-2}(t+1)}{(x+y)^2}, & t\in[0,1],x,y\ge 10^{-2}.\end{cases} \tag{80}$$

Take $q=0.001$, then $Kq=(9/25600)$, $(Kq/2k)\approx 0.7778$. When $x+y\in[Kq,q]$, $f_1(t,x,y)=f_2(t,x,y)=10^4(t+1)(x+y)\ge 2\ge(Kq/2k)$. Obviously, $g_0=\bar{g}_0=0$ and $g_\infty=\bar{g}_\infty=0$. So, by Theorem 8, system (78) has at least two positive solutions.

Data Availability

The data set supporting the conclusions is included within this article.

Conflicts of Interest

The authors declare that they have no competing interests regarding the publication of this paper.

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References

[1] R. P. Agarwal, M. Benchohra, and S. Hamani, "A survey on existence results for boundary value problems of nonlinear fractional differential equations and inclusions," *Acta Applicandae Mathematicae*, vol. 109, no. 3, pp. 973–1033, 2010.

[2] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Diferential Equations*, Elsevier, New York, NY, USA, 2006.

[3] Z. B. Bai, W. C. Sun, and W. H. Zhang, "Positive solutions for boundary value problems of singular fractional differential equations," *Abstract and Applied Analysis*, vol. 2013, Article ID 129640, 7 pages, 2013.

[4] Y. M. Zou, L. S. Liu, and Y. J. Cui, "The existence of solutions for four-point coupled boundary value problems of fractional differential equations at resonance," *Abstract and Applied Analysis*, vol. 2014, Article ID 314083, 8 pages, 2014.

[5] A. Cabada and Z. Hamdi, "Nonlinear fractional differential equations with integral boundary value conditions," *Applied Mathematics and Computation*, vol. 228, pp. 251–257, 2014.

[6] Y. J. Cui, "Uniqueness of solution for boundary value problems for fractional differential equations," *Applied Mathematics Letters*, vol. 51, pp. 48–54, 2016.

[7] X. G. Zhang, L. S. Liu, Y. H. Wu, and Y. J. Cui, "New result on the critical exponent for solution of an ordinary fractional differential problem," *Journal of Function Spaces*, vol. 2017, Article ID 3976469, 4 pages, 2017.

[8] X. Zhang and Q. Zhong, "Uniqueness of solution for higher-order fractional differential equations with conjugate type integral conditions," *Fractional Calculus and Applied Analysis*, vol. 20, no. 6, pp. 1471–1484, 2017.

[9] Y. M. Zou and G. P. He, "On the uniqueness of solutions for a class of fractional differential equations," *Applied Mathematics Letters*, vol. 74, pp. 68–73, 2017.

[10] Y. J. Cui, Q. Sun, and X. W. Su, "Monotone iterative technique for nonlinear boundary value problems of fractional order $p \in (2, 3)$," *Advances in Difference Equations*, vol. 2017, Article ID 248, 2017.

[11] Y. M. Zou and G. P. He, "The existence of solutions to integral boundary value problems of fractional differential equations at resonance," *Journal of Function Spaces*, vol. 2017, Article ID 2785937, 7 pages, 2017.

[12] M. Y. Zuo, X. Hao, L. S. Liu, and Y. J. Cui, "Existence results for impulsive fractional integro-differential equation of mixed type with constant coefficient and antiperiodic boundary conditions," *Boundary Value Problems*, vol. 2017, Article ID 161, 2017.

[13] Q. L. Song and Z. B. Bai, "Positive solutions of fractional differential equations involving the Riemann-Stieltjes integral boundary condition," *Advances in Difference Equations*, vol. 2018, Article ID 183, 2018.

[14] W. J. Ma and Y. J. Cui, "The eigenvalue problem for Caputo type fractional differential equation with Riemann-Stieltjes integral boundary conditions," *Journal of Function Spaces*, vol. 2018, Article ID 2176809, 9 pages, 2018.

[15] X. Hao, H. Sun, L. Liu, and D. Wang, "Positive solutions for semipositone fractional integral boundary value problem on the half-line," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales Serie A Matemáticas*, vol. 113, no. 4, pp. 3055–3067, 2019.

[16] Y. Wei, Q. Song, and Z. Bai, "Existence and iterative method for some fourth order nonlinear boundary value problems," *Applied Mathematics Letters*, vol. 87, pp. 101–107, 2019.

[17] J. Wu, X. G. Zhang, L. S. Liu, Y. H. Wu, and Y. J. Cui, "The convergence analysis and error estimation for unique solution of a p-Laplacian fractional differential equation with singular decreasing nonlinearity," *Boundary Value Problems*, vol. 2018, Article ID 82, 2018.

[18] X. Hao and L. Zhang, "Positive solutions of a fractional thermostat model with a parameter," *Symmetry*, vol. 11, no. 1, p. 122, 2019.

[19] Y. J. Cui, W. J. Ma, Q. Sun, and X. W. Su, "New uniqueness results for boundary value problem of fractional differential equation," *Nonlinear Analysis: Modelling and Control*, vol. 23, pp. 31–39, 2018.

[20] Q. Sun, H. W. Ji, and Y. J. Cui, "Positive solutions for boundary value problems of fractional differential equation with integral boundary conditions," *Journal of Function Spaces*, vol. 2018, Article ID 6461930, 6 pages, 2018.

[21] X. Hao, L. Zhang, and L. Liu, "Positive solutions of higher order fractional integral boundary value problem with a parameter," *Nonlinear Analysis: Modelling and Control*, vol. 24, no. 2, pp. 210–223, 2019.

[22] S. Y. Song and Y. J. Cui, "Existence of solutions for integral boundary value problems of mixed fractional differential equations under resonance," *Boundary Value Problems*, vol. 2020, Article ID 23, 2020.

[23] F. L. Wang and Y. J. Cui, "Positive solutions for an infinite system of fractional order boundary value problems," *Advances in Difference Equations*, vol. 2019, Article ID 169, 2019.

[24] X. Q. Zhang, L. S. Liu, Y. H. Wu, and Y. M. Zou, "Existence and uniqueness of solutions for systems of fractional differential equations with Riemann-Stieltjes integral boundary condition," *Advances in Difference Equations*, vol. 2018, Article ID 204, 2018.

[25] W. Cheng, J. F. Xu, Y. J. Cui, and Q. Ge, "Positive solutions for a class of fractional difference systems with coupled boundary conditions," *Advances in Difference Equations*, vol. 2019, Article ID 249, 2019.

[26] C. B. Zhai and R. T. Jiang, "Unique solutions for a new coupled system of fractional differential equations," *Advances in Difference Equations*, vol. 2018, Article ID 1, 2018.

[27] X. Qiu, J. Xu, D. O'Regan, and Y. Cui, "Positive solutions for a system of nonlinear semipositone boundary value problems with Riemann-Liouville fractional derivatives," *Journal of Function Spaces*, vol. 2018, Article ID 7351653, 10 pages, 2018.

[28] T. Qi, Y. Liu, and Y. Cui, "Existence of solutions for a class of coupled fractional differential systems with nonlocal boundary conditions," *Journal of Function Spaces*, vol. 2017, Article ID 6703860, 9 pages, 2017.

[29] T. Qi, Y. Liu, and Y. Zou, "Existence result for a class of coupled fractional differential systems with integral boundary value conditions," *Journal of Nonlinear Sciences and Applications*, vol. 10, no. 7, pp. 4034–4045, 2017.

[30] Y. Cui and Y. Zou, "Existence results and the monotone iterative technique for nonlinear fractional differential systems with coupled four-point boundary value problems," *Abstract and Applied Analysis*, vol. 2014, Article ID 242591, 6 pages, 2014.

[31] H. Li and J. Zhang, "Positive solutions for a system of fractional differential equations with two parameters," *Journal of Function Spaces*, vol. 2018, Article ID 1462505, 9 pages, 2018.

[32] X. Hao, H. Wang, L. Liu, and Y. J. Cui, "Positive solutions for a system of nonlinear fractional nonlocal boundary value problems with parameters and p-Laplacian operator," *Boundary Value Problems*, vol. 2017, Article ID 182, 2017.

[33] C. B. Zhai, L. Zhao, S. Y. Li, and H. R. Marasi, "New existing results for a system of nonlinear thirdorder differential equation via fixed point index," *Journal of Applied Analysis and Computation*, vol. 8, no. 6, pp. 1694–1706, 2018.

[34] Y. Cui, W. Ma, X. Wang, and X. Su, "Uniqueness theorem of differential system with coupled integral boundary conditions," *Electronic Journal of Qualitative Teory of Diferential Equations*, vol. 9, pp. 1–10, 2018.

[35] S. Song and Y. Cui, "Multiplicity solutions for integral boundary value problem of fractional differential systems," *Discrete Dynamics in Nature and Society*, vol. 2020, Article ID 2651845, 10 pages, 2020.

[36] D. Guo and V. Lakshmikantham, *Nonlinear Problems in Abstract Cones*, Academic Press, New York, NY, USA, 1988.