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Research Article

Intrinsic Square Function Characterizations of Variable Hardy–Lorentz Spaces

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The aim of this paper is to establish the intrinsic square function characterizations in terms of the intrinsic Littlewood–Paley g-function, the intrinsic Lusin area function, and the intrinsic g_λ^*-function of the variable Hardy–Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$, for $p(\cdot)$ being a measurable function on $\mathbb{R}^n$ satisfying $0 < p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \le \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty$ and the globally log-Hölder continuity condition and $q \in (0,\infty)$, via its atomic and Littlewood–Paley function characterizations.

1. Introduction

In recent years, the theory of function spaces with variable exponents has gained great interest, see, for example, [1–9]. The variable Lebesgue space is one of the generalizations of the classical $L^p(\mathbb{R}^n)$ space, originally introduced by Orlicz [10] via replacing p by the variable exponent function $p(\cdot)\colon \mathbb{R}^n \longrightarrow (0,\infty)$. In 1991's, Kováčik and Rákosník [11] proved some elementary properties for this kind of spaces. This space has been extensively studied by many researchers due to its wide use in different fields such as harmonic analysis and partial differential equations, see, for example, [12–15].

The real variable theory of Hardy spaces $H^p(\mathbb{R}^n)$, introduced by Stein and Weiss in [16], is a well-known generalization of the Lebesgue spaces $L^p(\mathbb{R}^n)$. This theory was also extended to the variable setting. To be more precise, under the assumption that the variable exponent $p(\cdot)$ satisfies the globally log-Hölder condition, Nakai and Sawano [17] introduced the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and established its atomic characterizations which were used to figure out its dual space and to prove the boundedness of singular integrals on $H^{p(\cdot)}(\mathbb{R}^n)$ as well. Sawano [18] extended the atomic characterizations obtained in [17] and gave other applications of these kinds of Hardy spaces. In an independent way and under slightly weaker conditions to that used in [17, 18], Cruz-Uribe and Wang [19] also studied the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and constructed its atomic decomposition and showed the boundedness of several operators. Moreover, Ho [20] extended the atomic decompositions established in [17] to the weighted Hardy spaces with variable exponents and illustrated the relation between the boundedness of the Hardy–Littlewood maximal operators on function spaces and the atomic decompositions of Hardy-type spaces.

The Lorentz space $L^{p,q}(\mathbb{R}^n)$ tracked back to Lorentz [21] is another generalization of the classical $L^p(\mathbb{R}^n)$ space. This space forms a valuable topic in the theory of function spaces and harmonic analysis, see, for example, [5, 22–26]. The theory of Lorentz spaces was generalized to the Hardy–Lorentz space. Particularly, Fefferman et al. [27] investigated the real interpolation of the real Hardy–Lorentz space $H^{p,q}(\mathbb{R}^n)$, Fefferman and Soria [28] studied $H^{1,\infty}(\mathbb{R}^n)$ and established its atomic characterization, Liu [29] established the atomic decomposition for $H^{p,\infty}$ with $0 < p < 1$, and Abu-Shammala and Torchinsky [22] introduced the Hardy–Lorentz space $H^{p,q}(\mathbb{R}^n)$, established its atomic characterizations, and proved the boundedness of singular integrals on $H^{p,q}(\mathbb{R}^n)$ for $p \in (0,1)$ and $q \in (0,\infty)$. The classical Lorentz spaces $L^{p,q}(\mathbb{R}^n)$ were extended to the variable case, more precisely, Kempka and Vybíral [30] introduced the Lorentz spaces $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n)$ and proved that when $q(\cdot)$ is a

constant which equals to q, the space $L^{p(\cdot),q}(\mathbb{R}^n)$ is the real interpolation between $L^\infty(\mathbb{R}^n)$ and the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$, and the authors have also showed that similar to a classical case that $L^{p(\cdot),q(\cdot)}(\mathbb{R}^n) = L^{p(\cdot)}(\mathbb{R}^n)$ when $p(\cdot) = q(\cdot)$ and demonstrated that the Marcinkiewicz interpolation theorem does not work in the variable Lebesgue space setting.

Recently, Yan et al. [8] introduced the variable weak Hardy space $H^{p(\cdot),\infty}(\mathbb{R}^n)$ by means of the radial grand maximal function and proved various characterizations including the atomic and molecular characterizations and investigated the boundedness of convolution δ-type and nonconvolution γ-order Caldéron–Zygmund operators via the atomic characterization established in the same paper. Very recently, Jiao et al. [4] investigated the variable Hardy–Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$, constructed the atomic characterizations for this space, figured out its dual space, and proved the boundedness of singular integrals on $H^{p(\cdot),q}(\mathbb{R}^n)$.

The study of the intrinsic square function on several function spaces has attracted steadily increasing interest. More precisely, in order to settle a conjecture proposed on the boundedness of the Lusin area function $S(f)$ from the weighted Lebesgue space $L^2_{\mathscr{M}(v)}(\mathbb{R}^n)$ to $L^2_v(\mathbb{R}^n)$, where $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathscr{M}(v)$ denotes the Hardy–Littlewood maximal function of v, Wilson [31] introduced the intrinsic square functions and proved that they are bounded on the weighted Lebesgue $L^p_w(\mathbb{R}^n)$ with $p \in (1,\infty)$ and the weight w being in the Muckenhoupt class $A_p(\mathbb{R}^n)$. The square functions of the form $S(f)$ are all dominated by the intrinsic square functions; however, these later are not essentially bigger than any one of them. The generic nature of these functions make them pointwise equivalent to each other and extremely easy to work with, as Fefferman–Stein and Hardy–Littlewood maximal functions. Furthermore, the intrinsic Lusin area function has many advantages of being comparable at various opening cones which is a well-known property that does not hold for the classical Lusin area function, see [31–34]. Recently, Ho [35] broadened the mapping properties for intrinsic square functions to the weighted Hardy spaces with variable exponents studied in [20].

Later, the intrinsic square function characterizations of the weighted Hardy space $H^1_w(\mathbb{R}^n)$ under the additional assumption that $f \in L^1_w(\mathbb{R}^n)$ were established by Huang and Liu [36]. Systematically, Wang and Liu [37] have extended this result to $H^p_w(\mathbb{R}^n)$ for $p \in (n/(n+\alpha), 1)$ and $\alpha \in (0,1]$, under other additional assumptions. Liang and Yang [38] established the s-order intrinsic square function characterization of the Musielak–Orlicz Hardy space $H^\varphi(\mathbb{R}^n)$, which was introduced by Ky [39] via g-function and g^*_λ-function with the best range $\lambda \in (2+2(\alpha+s)/n, \infty)$ which improve the results obtained in [36, 38]. These results were extended to the variable Hardy space setting by [40] with $\lambda \in (3+2(\alpha+s), \infty)$. Recently, Yan [41] extended these results to the weak Musielak–Orlicz Hardy space $\mathrm{WH}^\varphi(\mathbb{R}^n)$ with the same range as in [38], and the same author also established these characterizations for the variable weak Hardy space $H^{p(\cdot),\infty}(\mathbb{R}^n)$ in [42].

Motivated by the series of papers [4, 8, 42], in this work, we aim to prove that the variable Hardy–Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$ can be characterized by means of the intrinsic square functions via using the atomic and the Littlewood–Paley function characterizations established in [4].

We end this introduction by describing the sectionwise treatment of this article. In Section 2, we recall the definitions and existing results related to our work. Section 3 is devoted to establish the intrinsic square function characterization of the variable Hardy–Lorentz space.

As usually, throughout the paper, we denote by $\mathbb{N}$ and $\mathbb{Z}$ the set of nonnegative integers and the set of integers, respectively. We use C and c to denote positive constants that are independent of the essential parameters involved but may differ from line to line. The symbol $A \lesssim B$ means $A \le CB$, and the symbol $A \sim B$ means $A \lesssim B$ and $B \lesssim A$.

2. Preliminaries

In this section, we recall the definition and some properties of the variable Lebesgue space, Lorentz space with variable exponents, and the variable Hardy-Lorentz space, some lemmas, and existing results used in this work.

2.1. Variable Exponent Lebesgue Spaces. A variable exponent is a measurable function $p(\cdot)\colon \mathbb{R}^n \longrightarrow (0,\infty)$. We denote by $\mathscr{P}(\mathbb{R}^n)$ the collection of variable exponents satisfying $0 < p_- \le p_+ < \infty$, where $p_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x)$ and $p_+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x)$. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$. The variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ is the set of measurable functions f such that $\int_{\mathbb{R}^n} |f(x)|^{p(x)} dx < \infty$, equipped with the Luxemburg quasi-norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \inf\left\{\lambda>0\colon \int_{\mathbb{R}^n}\left[\frac{|f(x)|}{\lambda}\right]^{p(x)} dx \le 1\right\}. \quad (1)$$

In the next remark, we collect some basic properties of the variable Lebesgue spaces. For the proofs, see [12, 17].

Remark 1. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $f \in L^{p(\cdot)}(\mathbb{R}^n)$.

(i) For $\lambda \in \mathbb{C}$, we have $\|\lambda f\|_{L^{p(\cdot)}(\mathbb{R}^n)} = |\lambda| \|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

(ii) For $s>0$, we have $\||f|^s\|_{L^{p(\cdot)}(\mathbb{R}^n)} = \|f\|^s_{L^{sp(\cdot)}(\mathbb{R}^n)}$.

(iii) If $g \in L^{p(\cdot)}(\mathbb{R}^n)$, then

$$\|f+g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} \le \|f\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)} + \|g\|^{\underline{p}}_{L^{p(\cdot)}(\mathbb{R}^n)}, \quad (2)$$

here and hereafter $\underline{p} = \min\{p_-, 1\}$.

We recall the Fatou Lemma of $L^{p(\cdot)}(\mathbb{R}^n)$ obtained in [12], Theorem 2.61.

Lemma 1. *Let $p(\cdot)\colon \mathbb{R}^n \longrightarrow [1,\infty)$ and $\{f_k\}_{k\in\mathbb{N}} \subset L^{p(\cdot)}(\mathbb{R}^n)$. If $f_k \longrightarrow f$ as $k \longrightarrow \infty$ pointwise almost everywhere in $\mathbb{R}^n$ and $\liminf_{k\longrightarrow\infty} \|f_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ is finite, then $f \in L^{p(\cdot)}(\mathbb{R}^n)$ and*

$$\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \liminf_{k\longrightarrow\infty} \|f_k\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \quad (3)$$

We say that the variable exponent $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ satisfies the globally log-Hölder continuity condition, and we write

$p(\cdot) \in C^{\log}(\mathbb{R}^n)$ if there exist constants $C_{p(\cdot)}$, C_∞, and p_∞ such that, for any $x, y \in \mathbb{R}^n$,

$$|p(x) - p(y)| \le \frac{C_{p(\cdot)}}{\log(e + 1/|x-y|)},$$
$$|p(x) - p_\infty| \le \frac{C_\infty}{\log(e + |x|)}. \tag{4}$$

Let f be a locally integrable function and $x \in \mathbb{R}^n$. We recall that the Hardy–Littlewood maximal operator $\mathscr{M}$ is defined by

$$\mathscr{M}(f)(x) = \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)| \mathrm{d}y, \tag{5}$$

where the supremum is taken over all balls B of $\mathbb{R}^n$ containing x.

Lemma 2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $1 < p_- \le p_+ < \infty$. Then, there exists a positive constant C such that, for all $f \in L^{p(\cdot)}(\mathbb{R}^n)$, $\|\mathscr{M}(f)\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ (see ([13], Theorem 4.3.8)).*

The vector-valued inequality for the Hardy–Littlewood maximal function on $L^{p(\cdot)}(\mathbb{R}^n)$ is given in the next lemma, (see [2], Corollary 2.1).

Lemma 3. *Let $r \in (1, \infty)$ and $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ with $p_- \in (1, \infty)$. Then, there exists a positive constant C such that, for any sequence $\{f_j\}_{j\in\mathbb{N}}$ of measurable functions,*

$$\left\| \left(\sum_{j=1}^{\infty} |\mathscr{M}(f_j)|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left(\sum_{j=1}^{\infty} |f_j|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{6}$$

The next result can be easily proved using the fact that $\chi_{\kappa B} \le \kappa^{n/r} (\mathscr{M}(\chi_B))^{1/r}$ for any $\kappa \in [1, \infty)$ and the above lemma.

Corollary 1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $\tau \in (0, \min\{1, p_-\})$, and $\kappa \in [1, \infty)$. Then, for any sequence $\{B_j\}_{j\in\mathbb{N}}$, there exists a positive constant C such that*

$$\left\| \sum_{j\in\mathbb{N}} \chi_{\kappa B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C\kappa^{n/\tau} \left\| \sum_{j\in\mathbb{N}} \chi_{B_j} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{7}$$

The following technical lemma is a variant of [18], Theorem 4.1.

Lemma 4. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $r \in (0, \min\{1, p_-\})$, and $q \in [1, \infty)$. Then, there exists a positive constant C such that, for all sequences $\{B_j\}_{j\in\mathbb{N}}$ of balls, numbers $\{\lambda_j\}_{j\in\mathbb{N}} \subset \mathbb{C}$, and measurable functions $\{a_j\}_{j\in\mathbb{N}}$ satisfying that, for each $j \in \mathbb{N}$, $\operatorname{supp} a_j \subset B_j$, and $a_{jL^q(\mathbb{R}^n)} \le |B_j|^{1/q}$, it holds true that*

$$\left\| \left(\sum_{j\in\mathbb{N}} |\lambda_j a_j|^r \right)^{1/r} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le C \left\| \left(\sum_{j\in\mathbb{N}} |\lambda_j \chi_{B_j}|^r \right) \right\|^{1/r}_{L^{p(\cdot)}(\mathbb{R}^n)}. \tag{8}$$

2.2. Lorentz Spaces with Variable Exponents. We now recall the definition of Lorentz spaces with variable exponents considered in this paper.

Definition 1. Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $0 < q \le \infty$. Then, $L^{p(\cdot),q}(\mathbb{R}^n)$ is defined to be the set of all measurable functions f such that $\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} < \infty$, where

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} = \begin{cases} \left(\int_0^\infty \lambda^{q-1} \left\| \chi_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}} \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \mathrm{d}\lambda \right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup_{\lambda>0} \lambda \left\| \chi_{\{x\in\mathbb{R}^n : |f(x)|>\lambda\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, & \text{if } q = \infty. \end{cases} \tag{9}$$

Lemma 5 presents an equivalent discrete characterization of the quasi-norm $\|\cdot\|_{L^{p(\cdot),q}}$. For the proof, we refer to [30], Lemma 2.4.

Lemma 5. *Let $p(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ and $0 < q \le \infty$. If $f \in L^{p(\cdot),q}(\mathbb{R}^n)$, then*

$$\|f\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \sim \begin{cases} \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \chi_{\{x\in\mathbb{R}^n : |f(x)|>2^k\}} \right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \right)^{1/q}, & \text{if } 0 < q < \infty, \\ \sup_{k\in\mathbb{Z}} 2^k \left\| \chi_{\{x\in\mathbb{R}^n : |f(x)|>2^k\}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}, & \text{if } q = \infty. \end{cases} \tag{10}$$

2.3. Variable Hardy–Lorentz Space and Existing Results. Denote by $\mathscr{S}(\mathbb{R}^n)$, the space of all Schwartz functions, and let $\mathscr{S}'(\mathbb{R}^n)$ denote its topological dual space. For $N \in \mathbb{N}$, let

$$\mathscr{F}_N(\mathbb{R}^n) := \left\{ \psi \in \mathscr{S}(\mathbb{R}^n) : \sum_{\beta\in\mathbb{Z}^n_+, |\beta|\le N} \sup_{x\in\mathbb{R}^n} \left[(1+|x|)^N \left| D^\beta \psi(x) \right| \right] \le 1 \right\}, \tag{11}$$

where for any $\{\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{Z}_+^n, |\beta| = \beta_1 + \cdots + \beta_n\}$ and $D^\beta := (\partial/\partial x_1)^{\beta_1}, \ldots, (\partial/\partial x_n)^{\beta_n}$.

For all $f \in \mathcal{S}'(\mathbb{R}^n)$, define radial grand maximal function $f_{N,+}^*$ of f by

$$f_{N,+}^* := \sup\left\{ \begin{array}{c} |f * \psi_t(x)|: t \in (0,\infty), \\ \psi \in \mathcal{F}_N(\mathbb{R}^n) \end{array} \right\}, \tag{12}$$

where for all $t \in (0,\infty)$ and $\xi \in \mathbb{R}^n$, $\psi_t := t^{-n}\psi(\xi/t)$.

Definition 2. Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $N \in ((n/p_-) + n + 1, \infty)$ be a positive integer. The Hardy–Lorentz space $H^{p(\cdot),q}(\mathbb{R}^n)$ is defined to be the set of all $f \in \mathcal{S}'(\mathbb{R}^n)$ such that $f_{N,+}^* \in L^{p(\cdot),q}(\mathbb{R}^n)$, equipped with the quasi-norm

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} = \left\|f_{N,+}^*\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{13}$$

Next, we recall the definition of the atomic Hardy–Lorentz space $H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)$ given in [4], and before this, we recall the definition of $(p(\cdot), r, s)$-atom.

Definition 3. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$ and $r>1$. Fix an integer $s \in ((n/p_-) - n - 1, \infty) \cap \mathbb{Z}_+$. A measurable function a on $\mathbb{R}^n$ is called a $(p(\cdot), r, s)$-atom if there exists a ball B such that

(1) $\operatorname{supp} a \subset B$

(2) $\|a\|_{L^r(\mathbb{R}^n)} \le |B|^{1/r}/\|\chi_B\|_{L^{p(\cdot)}(\mathbb{R}^n)}$

(3) $\int_{\mathbb{R}^n} a(x)x^\alpha \mathrm{d}x = 0$ for all $\alpha \in \mathbb{Z}_+$ with $|\alpha| \le s$

Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $\{\lambda_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$ be a sequence of numbers in $\mathbb{C}$, and $\{B_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$ be a sequence of balls in $\mathbb{R}^n$. Define

$$\mathcal{A}\left(\{\lambda_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}, \{B_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}\right) := \left(\sum_{k\in\mathbb{Z}} \left\| \left\{ \sum_{j\in\mathbb{N}} \left[\frac{|\lambda_{k,j}|\chi_{B_{k,j}}}{\|\chi_{B_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}} \right]^{\underline{p}} \right\}^{1/\underline{p}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}. \tag{14}$$

Definition 4. Let $p(\cdot) \in \mathcal{P}(\mathbb{R}^n)$, $r \in (1,\infty]$, and s as in Definition 3. The variable atomic Hardy–Lorentz space $H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)$ is defined to be the space of all functions $f \in \mathcal{S}'(\mathbb{R}^n)$ which can be decomposed as

$$f = \sum_{k\in\mathbb{Z}} \sum_{j\in\mathbb{N}} \lambda_{k,j} a_{k,j} \text{ in } \mathcal{S}'(\mathbb{R}^n), \tag{15}$$

where $\{a_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$ is a sequence of $(p(\cdot), r, s)$-atoms, associated with balls $\{B_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$, satisfying that, for all $x \in \mathbb{R}^n$ and $k \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{k,j}}(x) \le A$ with A being a positive constant independent of x and k and for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, $\lambda_{k,j} = C2^k \|\chi_{B_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with C being a positive constant independent of k and j. Moreover, for any $f \in H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)$, we define

$$\|f\|_{H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)} := \inf \mathcal{A}\left(\{\lambda_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}, \{B_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}\right), \tag{16}$$

where the infimum is taken over all the decompositions of f as (15).

The following result is Theorem 5.4 of [4].

Lemma 6. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n), r \in (\max\{p_+, 1\}, \infty]$, and s as in Definition 3. Then, $H^{p(\cdot),q}(\mathbb{R}^n) = H_{atom,r,s}^{p(\cdot),q}(\mathbb{R}^n)$ with equivalent quasi-norms.*

Remark 2. Note that if $f \in H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)$, then there exists a sequence $\{a_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$ of $(p(\cdot), r, s)$-atoms, associated with balls $\{B_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$, respectively, satisfying that, for any $k \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{k,j}}(x) \le A$, such that f can be decomposed as in (15), where for all $(k \in \mathbb{Z})$ and $j \in \mathbb{N}$, $\lambda_{k,j} = C2^k \|\chi_{B_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with C being a positive constant independent of k and j, and

$$\|f\|_{H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)} \sim \mathcal{A}\left(\{\lambda_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}, \{B_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}\right). \tag{17}$$

Moreover, by the definition of $\{\lambda_{k,j}\}_{k\in\mathbb{Z}, j\in\mathbb{N}}$, the fact that for any $k \in \mathbb{Z}$, $\sum_{j\in\mathbb{N}} \chi_{B_{k,j}}(x) \le A$, we have

$$\|f\|_{H_{\text{atom},r,s}^{p(\cdot),q}(\mathbb{R}^n)} \sim \left(\sum_{k\in\mathbb{Z}} 2^{kq} \left\| \sum_{j\in\mathbb{N}} \chi_{B_{k,j}} \right\|_{L^{p(\cdot)}(\mathbb{R}^n)}^q \right)^{1/q}. \tag{18}$$

Let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be a radial function satisfying

$$\begin{aligned} &\operatorname{supp} \phi \subset \{x \in \mathbb{R}^n: |x| \le 1\}, \\ &\int_{\mathbb{R}^n} \phi(x)x^\gamma \mathrm{d}x = 0 \quad \text{for all } \gamma \in \mathbb{Z}_+ \text{ with } |\gamma| \\ &\quad \le \max\left\{ \left\lfloor \frac{n}{p_-} - n - 1 \right\rfloor, 0 \right\}, \\ &\int_0^\infty |\widehat{\phi}(\xi t)|^2 \frac{\mathrm{d}t}{t} = 1 \quad \text{for all } \xi \in \mathbb{R}^n \backslash \left\{\overrightarrow{0}_n\right\}. \end{aligned} \tag{19}$$

Recall that the Littlewood–Paley g-function and the g_λ^*-function with $\lambda>0$ are defined for all $f \in \mathcal{S}'(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$ by

$$\begin{aligned} g(f)(x) &:= \left(\int_0^\infty |f * \phi_t(x)|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}, \\ g_\lambda^*(f)(x) &:= \left(\int_0^\infty \int_{\mathbb{R}^n} \left[\frac{t}{t + |x-y|} \right]^{\lambda n} |f * \phi_t(y)|^2 \frac{\mathrm{d}t}{t} \right)^{1/2}. \end{aligned} \tag{20}$$

We recall that a function $f \in \mathcal{S}'(\mathbb{R}^n)$ is vanishing weakly at infinity if for any $\phi \in \mathcal{S}(\mathbb{R}^n)$, $f * \phi_t \longrightarrow 0$ in $\mathcal{S}'(\mathbb{R}^n)$ as

$t \longrightarrow \infty$. The following two propositions are just Theorem 8.2 and 8.3 in [4].

Proposition 1. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$ and $q \in (0,\infty)$. Then, $f \in H^{p(\cdot),q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, for all $f \in H^{p(\cdot),q}(\mathbb{R}^n)$,*

$$C^{-1}\|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \le C\|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}, \tag{21}$$

where C is a positive constant independent of f.

Proposition 2. *Let $p(\cdot) \in C^{\log}(\mathbb{R}^n)$, $q \in (0,\infty)$, and $\lambda \in (1 + (2/\min\{p_-, 2\}), \infty)$. Then, $f \in H^{p(\cdot),q}(\mathbb{R}^n)$ if and only if $f \in \mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g_\lambda^*(f) \in L^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, for all $f \in H^{p(\cdot),q}(\mathbb{R}^n)$,*

$$C^{-1}\|g_\lambda^*(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \le \|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \le C\|g_\lambda^*(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}, \tag{22}$$

where C is a positive constant independent of f.

3. Intrinsic Square Function Characterizations

In this section, we prove that the variable Hardy–Lorentz space can be characterized by means of the intrinsic square functions. We firstly begin by introducing some terminologies.

Let $s \in \mathbb{Z}_+$ denote the set of all functions with continuous classical derivatives up to an order less or equal to s by $C^s(\mathbb{R}^n)$. For $\alpha \in (0,1]$ and $s \in \mathbb{Z}_+$, let $C_{\alpha,s}(\mathbb{R}^n)$ denote the set of all functions $\phi \in C^s(\mathbb{R}^n)$ satisfying

$$\begin{gathered}\operatorname{supp}\phi \subset \{x \in \mathbb{R}^n : |x| \le 1\},\\ \int_{\mathbb{R}^n} \phi(x)x^\gamma \mathrm{d}x = 0 \quad \text{for all } \gamma \in \mathbb{Z}_+^n \text{ with } |\gamma| \le s,\end{gathered} \tag{23}$$

and for all $x_1, x_2 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}_+^n$ with $|\nu| = s$,

$$\left|D^\nu\phi(x_1) - D^\nu\phi(x_2)\right| \le \left|x_1 - x_2\right|^\alpha. \tag{24}$$

For all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $(y,t) \in \mathbb{R}^n \times (0,\infty)$, let

$$A_{\alpha,s}(f)(y,t) := \sup_{\phi \in C_{\alpha,s}(\mathbb{R}^n)} \left|f * \phi_t(y)\right|. \tag{25}$$

The intrinsic g-function, the intrinsic Lusin area integral, and the intrinsic g_λ^*-function of f are defined, respectively, for all $x \in \mathbb{R}^n$ and $\lambda>0$ by

$$\begin{gathered}g_{\alpha,s}(f)(x) := \left(\int_0^\infty \left|A_{\alpha,s}(f)(x,t)\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2},\\ S_{\alpha,s}(f)(x) := \left(\int_{\Gamma(x)} \left|A_{\alpha,s}(f)(x,t)\right|^2 \frac{\mathrm{d}t}{t^{n+1}}\right)^{1/2},\\ g_{\lambda,\alpha,s}^*(f)(x) := \left(\int_0^\infty \int_{\mathbb{R}^n} \left[\frac{t}{t+|x-y|}\right]^{\lambda n} \left|A_{\alpha,s}(f)(x,t)\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}.\end{gathered} \tag{26}$$

For $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\epsilon \in (0,\infty)$, let $C_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$ denote the set of all functions $\phi \in C^s(\mathbb{R}^n)$ satisfying for all $x \in \mathbb{R}^n$, $\gamma \in \mathbb{Z}_+^n$ with $|\gamma| \le s$, $|D^\nu\phi(x)| \le (1+|x|)^{-n-\varepsilon}$, $\int_{\mathbb{R}^n}\phi(x)x^\gamma \mathrm{d}x = 0$, and for all $x_1, x_2 \in \mathbb{R}^n$ and $\nu \in \mathbb{Z}_+^n$ with $|\nu| = s$,

$$\begin{aligned}\left|D^\nu\phi(x_1) - D^\nu\phi(x_2)\right| \le \left|x_1 - x_2\right|^\alpha \Big[&\left(1+\left|x_1\right|\right)^{-n-\varepsilon}\\ &+\left(1+\left|x_2\right|\right)^{-n-\varepsilon}\Big].\end{aligned} \tag{27}$$

It is worth noting that the parameter ϵ has to be chosen large enough.

For all functions f such that

$$|f(\cdot)|(1+|\cdot|)^{-n-\varepsilon} \in L^1(\mathbb{R}^n) \tag{28}$$

and $(y,t) \in \mathbb{R}^n \times (0,\infty)$, define

$$\widetilde{A}_{(\alpha,\varepsilon),s}(y,t) := \sup_{\phi \in C_{(\alpha,\varepsilon),s}(\mathbb{R}^n)} \left|f * \phi_t(y)\right|. \tag{29}$$

For all $x \in \mathbb{R}^n$ and $\lambda>0$, define

$$\begin{gathered}\widetilde{g}_{(\alpha,\varepsilon),s}(f)(x) := \left(\int_0^\infty \left|\widetilde{A}_{(\alpha,\varepsilon),s}(f)(x,t)\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2},\\ \widetilde{S}_{(\alpha,\varepsilon),s}(f)(x) := \left(\int_{\Gamma(x)} \left|\widetilde{A}_{(\alpha,\varepsilon),s}(f)(x,t)\right|^2 \frac{\mathrm{d}t}{t^{n+1}}\right)^{1/2},\\ \widetilde{g}_{\lambda,\alpha,s}^*(f)(x) := \left(\int_0^\infty \int_{\mathbb{R}^n} \left[\frac{t}{t+|x-y|}\right]^{\lambda n} \left|\widetilde{A}_{(\alpha,\varepsilon),s}(f)(x,t)\right|^2 \frac{\mathrm{d}t}{t}\right)^{1/2}.\end{gathered} \tag{30}$$

Remark 3. For $s = 0$, the intrinsic square functions were introduced by Wilson [31] and for $s \in \mathbb{Z}_+$ by Liang and Yang in [38].

The next lemma is just [38], Proposition 2.4. For the case $s = 0$, it was proved in [31], p. 784.

Lemma 7. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\varepsilon \in (0,\infty)$. Then, for all f satisfying (28) and $x \in \mathbb{R}^n$,*

$$\begin{gathered}g_{\alpha,s}(f)(x) \sim S_{\alpha,s}(f)(x),\\ \widetilde{g}_{(\alpha,\varepsilon),s}(f)(x) \sim \widetilde{S}_{(\alpha,\varepsilon),s}(f)(x),\end{gathered} \tag{31}$$

where the implicit positive constants are independent of f.

The next lemma is just [38], Theorem 2.6. For the case $s = 0$, it was proved in [31], p. 775.

Lemma 8. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $\varepsilon \in (\max\{\alpha, s\}, \infty)$. Then, there exists a positive constant C such that, for all f satisfying (28) and $x \in \mathbb{R}^n$,*

$$C^{-1}g_{\alpha,s}(f)(x) \le \widetilde{g}_{(\alpha,\varepsilon),s}(f)(x) \le Cg_{\alpha,s}(f)(x), \tag{32}$$

where the implicit positive constants are independent of f.

The following lemma is just [38], Proposition 3.2.

Lemma 9. *Let $\alpha \in (0,1]$, $s \in \mathbb{Z}_+$, and $q \in (1,\infty)$. Then, there exists a positive constant C such that, for all measurable function f,*

$$\int_{\mathbb{R}^n}\left[g_{\alpha,s}(f)(x)\right]^q \mathrm{d}x \le C\int_{\mathbb{R}^n}|f(x)|^q \mathrm{d}x. \tag{33}$$

The main results of this work are stated as follows.

Theorem 1. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $p_+\in(0,1]$. Assume that $\alpha\in(0,1]$, $s\in\mathbb{Z}_+$, and $p_-\in(n/(n+\alpha+s),1]$.*

(1) If $f\in(\mathscr{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n))^*$, with $(\mathscr{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n))^*$ denoting the dual space of $\mathscr{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$, then there exists a positive constant C such that, for all $f\in H^{p(\cdot),q}(\mathbb{R}^n)$, it holds true that

$$\left\|g_{\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}. \tag{34}$$

(2) If $f\in\mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g_{\alpha,s}(f)\in L^{p(\cdot),q}(\mathbb{R}^n)$, then $f\in H^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \le C\left\|g_{\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{35}$$

Theorem 2. *Let $p(\cdot)\in C^{\log}(\mathbb{R}^n)$ with $p_+\in(0,1]$. Assume that $\alpha\in(0,1]$, $s\in\mathbb{Z}_+$, $p_-\in((n/(n+\alpha+s)),1]$, and $\lambda\in(3+((2(\alpha+s))/n),\infty)$.*

(1) If $f\in(\mathscr{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n))^*$, with $(\mathscr{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n))^*$ denoting the dual space of $\mathscr{C}_{(\alpha,\varepsilon),s}(\mathbb{R}^n)$, then there exists a positive constant C such that, for all $f\in H^{p(\cdot),q}(\mathbb{R}^n)$, it holds true that

$$\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)} \le C\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}. \tag{36}$$

(2) If $f\in\mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g_{\alpha,s}(f)\in L^{p(\cdot),q}(\mathbb{R}^n)$, then $f\in H^{p(\cdot),q}(\mathbb{R}^n)$. Moreover, there exists a positive constant C such that

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)} \le C\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{37}$$

Remark 4

(i) The same results in Theorem 1 hold true if we replace $g_{\alpha,s}$ by $S_{\alpha,s}$, $\widetilde{g}_{\alpha,s}$, and $\widetilde{S}_{\alpha,s}$ with $\varepsilon\in(\max\{\alpha,s\},\infty)$.

(ii) The same results in Theorem 2 hold true if we replace $g^*_{\lambda,\alpha,s}$ by $\widetilde{g}_{\lambda,(\alpha,\varepsilon),s}$ with $\varepsilon\in(\max\{\alpha,s\},\infty)$.

Proof of Theorem 1. By Lemmas 7 and 8, we know that, for $\varepsilon\in(\max\{\alpha,s\},\infty)$, the functions $g_{\alpha,s}$, $S_{\alpha,s}$, $\widetilde{g}_{\alpha,s}$, and $\widetilde{S}_{\alpha,s}$ are pointwise comparable, and we will prove the theorem just for $g_{\alpha,s}$.

We begin by proving (i). Let $f\in H^{p(\cdot),q}(\mathbb{R}^n)$ and $r\in(1,\infty)$. Note that since $\alpha\in(0,1]$, $s\in\mathbb{Z}_+$, and $p_-\in(n/(n+\alpha+s),1]$, it follows that $s\in((n/p_-)-n-1,\infty)\cap\mathbb{Z}_+$. Then, by Lemma 6 and Remark 2, there exists a sequence $\{a_{k,j}\}_{k\in\mathbb{Z},j\in\mathbb{N}}$ of $(p(\cdot),r,s)$-atoms, associated with balls $\{B_{k,j}\}_{k\in\mathbb{Z},j\in\mathbb{N}}$, respectively, satisfying that, for any $k\in\mathbb{Z}$, $\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}(x)\le A$, and $\{\lambda_{k,j}\}_{k\in\mathbb{Z},j\in\mathbb{N}}\subset\mathbb{C}$ such that, for all $k\in\mathbb{Z}$ and $j\in\mathbb{N}$, $\lambda_{k,j}=C2^k\|\chi_{B_{k,j}}\|_{L^{p(\cdot)}(\mathbb{R}^n)}$ with C being a positive constant independent of k and j such that f can be decomposed as

$$f=\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{N}}\lambda_{k,j}a_{k,j}\quad\text{in }\mathcal{S}'(\mathbb{R}^n),$$

$$\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}\sim\|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,s}(\mathbb{R}^n)}\sim\left(\sum_{k\in\mathbb{Z}}2^{kq}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}. \tag{38}$$

Fix $k_0\in\mathbb{Z}$. We write f as

$$f=\sum_{k=-\infty}^{k_0-1}\sum_{j\in\mathbb{N}}\lambda_{k,j}a_{k,j}+\sum_{k=k_0}^{\infty}\sum_{j\in\mathbb{N}}\lambda_{k,j}a_{k,j}=:f_1+f_2, \tag{39}$$

and let $A_{k_0}=\cup_{k=k_0}^{\infty}\cup_{j\in\mathbb{N}}(2B_{k,j})$. Thus, we get

$$\begin{aligned}\left\|\chi_{\{x\in\mathbb{R}^n:g_{\alpha,s}(f)(x)>2^{k_0}\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}&\lesssim\left\|\chi_{\{x\in\mathbb{R}^n:g_{\alpha,s}(f_1)(x)>2^{k_0-1}\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\left\|\chi_{\left\{x\in A_{k_0}:g_{\alpha,s}(f_2)(x)>2^{k_0-1}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\&\quad+\left\|\chi_{\left\{x\in A^c_{k_0}:g_{\alpha,s}(f_2)(x)>2^{k_0-1}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}=:I_1+I_2+I_3.\end{aligned} \tag{40}$$

Note that

$$I_1\lesssim\left\|\chi_{\left\{x\in\mathbb{R}^n:\sum_{k=-\infty}^{k_0-1}\sum_{j\in\mathbb{N}}\lambda_{k,j}g_{\alpha,s}(a_{k,j})\chi_{2B_{k,j}}(x)>2^{k_0-2}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}+\left\|\chi_{\left\{x\in\mathbb{R}^n:\sum_{k=-\infty}^{k_0-1}\sum_{j\in\mathbb{N}}\lambda_{k,j}g_{\alpha,s}(a_{k,j})\chi_{(2B_{k,j})^c}(x)>2^{k_0-2}\right\}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}:=J_1+J_2. \tag{41}$$

We firstly deal with J_1. Let $\beta_1 \in (0, \min\{p_-, q\})$, $\eta_1 \in (1, \min\{r, (1/\beta_1)\})$, and $\sigma_1 \in (0, 1-(1/\eta_1))$, and by the Hölder inequality, we have

$$\sum_{k=-\infty}^{k_0-1}\sum_{j\in\mathbb{N}}\lambda_{k,j}g_{\alpha,s}\left(a_{k,j}\right) \le \left(\sum_{k=-\infty}^{k_0-1}2^{k\sigma_1\eta_1'}\right)^{1/\eta_1'}\left(\sum_{k=-\infty}^{k_0-1}2^{-k\sigma_1\eta_1}\left[\sum_{j\in\mathbb{N}}\lambda_{k,j}g_{\alpha,s}\left(a_{k,j}\right)\right]^{\eta_1}\right)^{1/\eta_1}. \tag{42}$$

Let $v = r/\eta_1$. Then, by Lemma 9, we know that, for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}$,

$$\left\|\left[\left\|\chi_{B_{k,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}g_{\alpha,s}\left(a_{k,j}\right)\right]^{\eta_1}\chi_{2B_{k,j}}\right\|_{L^v(\mathbb{R}^n)} \lesssim \left|B_{k,j}\right|^{1/v}. \tag{43}$$

Thus, following the argument used for (5.8) in [4], we can prove that

$$\left(\sum_{k_0\in\mathbb{Z}}2^{k_0q}J_1^q\right)^{1/q} \lesssim \|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,s}(\mathbb{R}^n)}. \tag{44}$$

To estimate J_2, by [42], (2.7), we know that, for all $k \in \mathbb{Z}$ and $j \in \mathbb{N}$, we have

$$g_{\alpha,s}\left(a_{k,j}\right)(x) \lesssim \left\|\chi_{B_{k,j}}\right\|^{-1}_{L^{p(\cdot)}(\mathbb{R}^n)}\left[M\left(\chi_{B_{k,j}}\right)\right]^{(n+\alpha+s)/n}. \tag{45}$$

Thus, for $\beta_2 \in (0, \min\{n/(n+\alpha+s), q\})$, $\eta_2 \in (n/(n+\alpha+s)\beta_2, 1/\beta_2)$, and $\sigma_2 \in (0, 1-1/\eta_2)$, it was proved in [42] that

$$\begin{aligned} J_2 &\lesssim 2^{-k_0\eta_2(1-\sigma_2)}\left(\sum_{k=-\infty}^{k_0-1}2^{(1-\sigma_2)k\eta_2\beta_2}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|_{L^{p(\cdot)/\beta_2}(\mathbb{R}^n)}\right)^{1/\beta_2} \\ &= 2^{-k_0\eta_2(1-\sigma_2)}\left(\sum_{k=-\infty}^{k_0-1}2^{(1-\sigma_2)k\eta_2\beta_2}\left\|\left\{\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\}^{1/\beta_2}\right\|^{\beta_2}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta_2}. \end{aligned} \tag{46}$$

Then, by taking $\gamma_2 \in (1, (1-\sigma_2)\eta_2)$, the Hölder inequality for $((q-\beta_2)/q) + (\beta_2/q) = 1$ yields to

$$\begin{aligned} J_2 &\lesssim 2^{-k_0\eta_2(1-\sigma_2)}\left(\sum_{k=-\infty}^{k_0-1}2^{(1-\sigma_2)k\eta_2\beta_2-k\gamma_2\beta_2+k\gamma_2\beta_2}\right. \\ &\quad\left.\cdot\left\|\left\{\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\}^{1/\beta_2}\right\|^{\beta_2}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta_2} \\ &\lesssim 2^{-k_0\gamma_2}\left(\sum_{k=-\infty}^{k_0-1}2^{kq\gamma_2}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}. \end{aligned} \tag{47}$$

Therefore,

$$\begin{aligned} \sum_{k_0\in\mathbb{Z}}2^{k_0q}J_2^q &\lesssim \sum_{k_0\in\mathbb{Z}}2^{k_0q}2^{-k_0\gamma_2q}\sum_{k=-\infty}^{k_0-1}2^{kq\gamma_2}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &= \sum_{k=-\infty}^{\infty}2^{kq\gamma_2}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k_0=k+1}^{\infty}2^{k_0q(1-\gamma_2)} \\ &\lesssim \sum_{k\in\mathbb{Z}}2^{kq}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}. \end{aligned} \tag{48}$$

Thus, by (38), we have

$$\left(\sum_{k_0\in\mathbb{Z}}2^{k_0q}J_2^q\right)^{1/q} \lesssim \|f\|_{H^{p(\cdot),q}_{atom,r,s}(\mathbb{R}^n)}. \tag{49}$$

Now, we pass to estimate I_2. For this end, let $\beta_3 \in (0, \min\{p_-, q\})$ and $\gamma_3 \in (0, 1)$. Then, by Remark 1(ii), Lemmas 1 and 2.5, Remark 2, and the Hölder inequality, we obtain

$$\begin{aligned} I_2 &\le \left\|\chi_{A_{k_0}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \le \sum_{k=k_0}^{\infty}\left\|\sum_{j\in\mathbb{N}}\chi_{2B_{k,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le \left\|\sum_{k=k_0}^{\infty}\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)} \\ &\le \left(\sum_{k=k_0}^{\infty}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^{\beta_3}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta_3} \\ &= \left(\sum_{k=k_0}^{\infty}2^{k\gamma_3\beta_3}2^{-k\gamma_3\beta_3}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^{\beta_3}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta_3} \\ &\le \left(\sum_{k=k_0}^{\infty}2^{k\gamma_3q}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q} \\ &\quad\cdot\left(\sum_{k=k_0}^{\infty}2^{-k\gamma_3\beta_3(q/(q-\beta_3))}\right)^{(q-\beta_3)/q\beta_3} \\ &\le 2^{-k_0\gamma_3}\left(\sum_{k=k_0}^{\infty}2^{k\gamma_3q}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}. \end{aligned} \tag{50}$$

Then,

$$\begin{aligned}\sum_{k_0=-\infty}^{\infty}\left(2^{k_0}I_2\right)^q &\lesssim \sum_{k_0=-\infty}^{\infty}2^{k_0q}2^{-k_0\gamma_3q}\sum_{k=k_0}^{\infty}2^{kq\gamma_3}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim \sum_{k=-\infty}^{\infty}2^{kq\gamma_3}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\sum_{k_0=-\infty}^{k}2^{k_0q(1-\gamma_3)}\\ &\lesssim \sum_{k=-\infty}^{\infty}2^{kq}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}.\end{aligned} \tag{51}$$

Thus, by (38), we have

$$\left(\sum_{k_0\in\mathbb{Z}}2^{k_0q}I_2^q\right)^{1/q}\lesssim f_{H^{p(\cdot),q}_{\mathrm{atom},r,s}(\mathbb{R}^n)}. \tag{52}$$

We turn to estimate I_3. Let $\sigma_4\in(0,\infty)$ such that $\sigma_4\in((n/p_-(n+\alpha+s)),1)$ (such that σ_4 exists since $p_-\in(n/(n+\alpha+s),1)$) and $\beta_4\in(0,\min\{n/\sigma_4(n+\alpha+s),q\})$. Thus, by Remark 1(ii), Lemmas 1 and 2, and Remark 2, we get

$$\begin{aligned}I_3&\lesssim 2^{-k_0\sigma_4}\left\|\sum_{k=k_0}^{\infty}\sum_{j\in\mathbb{N}}\left(\lambda_{k,j}g_{\alpha,s}\left(a_{k,j}\right)\right)^{\sigma_4}\chi_{A^c_{k_0}}\right\|_{L^{p(\cdot)}(\mathbb{R}^n)}\\ &\lesssim 2^{-k_0\sigma_4}\left(\sum_{k=k_0}^{\infty}\left\|\left(\sum_{j\in\mathbb{N}}\left(\lambda_{k,j}g_{\alpha,s}\left(a_{k,j}\right)\right)^{\sigma_4}\chi_{A^c_{k_0}}\right)^{\beta_4}\right\|_{L^{p(\cdot)/\beta_4}(\mathbb{R}^n)}\right)^{1/\beta_4}.\end{aligned} \tag{53}$$

Let $\gamma_4\in(\sigma_4,1)$. Then, by (45), Lemma 3, and the Hölder inequality, we have

$$\begin{aligned}I_3&\lesssim 2^{-k_0\sigma_4}\left(\sum_{k=k_0}^{\infty}2^{k\sigma_4\beta_4}\left\|\sum_{j\in\mathbb{N}}\left(M\left(\chi_{B_{k,j}}\right)^{1/\beta_4}\right)^{\beta_4}\right\|_{L^{p(\cdot)/\beta_4}(\mathbb{R}^n)}\right)^{1/\beta_4}\\ &\lesssim 2^{-k_0\sigma_4}\left(\sum_{k=k_0}^{\infty}2^{k\sigma_4\beta_4}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^{\beta_4}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta_4}\\ &\lesssim 2^{-k_0\sigma_4}\left(\sum_{k=k_0}^{\infty}2^{k\sigma_4\beta_4}2^{-k\gamma_4\beta_4}2^{k\gamma_4\beta_4}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^{\beta_4}_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/\beta_4}\\ &\lesssim 2^{-{}_k\sigma_4}\left(\sum_{k=k_0}^{\infty}2^{k\beta_4(\sigma_4-\gamma_4)(q/q-\beta_4)}\right)^{(q-\beta_4)/q\beta_4}\\ &\quad\cdot\left\|\left(\sum_{k=k_0}^{\infty}2^{k\gamma_4q}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}\right.\\ &\lesssim 2^{-k_0\gamma_4}\left(\sum_{k=k_0}^{\infty}2^{k\gamma_4q}\left\|\sum_{j\in\mathbb{N}}\chi_{B_{k,j}}\right\|^q_{L^{p(\cdot)}(\mathbb{R}^n)}\right)^{1/q}.\end{aligned} \tag{54}$$

Then, following the argument used for (52), we can prove that

$$\left(\sum_{k_0\in\mathbb{Z}}2^{k_0q}I_3^q\right)^{1/q}\lesssim\|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,s}(\mathbb{R}^n)}. \tag{55}$$

Combining the estimates (44), (49), (52), and (55), we get

$$\left\|g_{\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\le C\|f\|_{H^{p(\cdot),q}_{\mathrm{atom},r,s}(\mathbb{R}^n)}\sim\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}. \tag{56}$$

To prove (ii), note that, for any $x\in\mathbb{R}^n$, we have

$$g(f)(x)\lesssim g_{\alpha,s}(f)(x). \tag{57}$$

By this and the fact that $f\in\mathcal{S}'(\mathbb{R}^n)$ vanishes weakly at infinity, $g_{\alpha,s}(f)\in L^{p(\cdot),q}(\mathbb{R}^n)$, and Proposition 1, we deduce that $f\in H^{p(\cdot),q}(\mathbb{R}^n)$ and

$$f_{H^{p(\cdot),q}(\mathbb{R}^n)}\lesssim\|g(f)\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\lesssim\left\|g_{\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}. \tag{58}$$

The proof is complete. □

Proof of Theorem 2. We begin by proving (i). Let $f\in H^{p(\cdot),q}(\mathbb{R}^n)$. Following the argument used in [40], p. 1566, we can prove that, for any $\alpha\in(0,1], s\in\mathbb{Z}_+,\varepsilon\in(\max\{\alpha,s\},\infty),\lambda\in(3+2(\alpha+s)/n,\infty)$ and $x\in\mathbb{R}^n$,

$$\widetilde{g}^*_{\lambda,(\alpha,\varepsilon),s}(f)(x)\lesssim\widetilde{S}_{(\alpha,\varepsilon),s}(f)(x). \tag{59}$$

Thus, by Theorem 1, we conclude that

$$\begin{aligned}\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}&\lesssim\left\|\widetilde{g}^*_{\lambda,(\alpha,\varepsilon),s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\\ &\lesssim\left\|\widetilde{S}_{(\alpha,\varepsilon),s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\lesssim\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned} \tag{60}$$

We turn to prove (ii). Let $f\in\mathcal{S}'(\mathbb{R}^n)$, f vanishes weakly at infinity, and $g^*_{\lambda,\alpha,s}(f)\in L^{p(\cdot),q}(\mathbb{R}^n)$. Note that since $\alpha\in(0,1], s\in\mathbb{Z}_+, p_-\in(n/(n+\alpha+s),1]$ and $\lambda\in(3+2(\alpha+s)/n,\infty)$, we have

$$\lambda>1+\frac{2}{\min\{p_-,2\}}. \tag{61}$$

By this, the fact that, for all $x\in\mathbb{R}^n$,

$$g^*_\lambda(f)(x)\lesssim g^*_{\lambda,\alpha,s}(f)(x)\lesssim\widetilde{g}^*_{\lambda,(\alpha,\varepsilon),s}(f)(x), \tag{62}$$

and Proposition 2, we deduce that $f\in H^{p(\cdot),q}(\mathbb{R}^n)$ and

$$\begin{aligned}\|f\|_{H^{p(\cdot),q}(\mathbb{R}^n)}&\lesssim\left\|g^*(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\lesssim\left\|g^*_{\lambda,\alpha,s}(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}\\ &\lesssim\left\|g^*\lambda,(\alpha,\varepsilon),s(f)\right\|_{L^{p(\cdot),q}(\mathbb{R}^n)}.\end{aligned} \tag{63}$$

This ends the proof. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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