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Research Article

Equivalent Conditions of a Hilbert-Type Multiple Integral Inequality Holding

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Let $\sum_{i=1}^{n} 1/p_i = 1$ $(p_i > 1)$, in this paper, by using the method of weight functions and technique of real analysis; it is proved that the equivalent parameter condition for the validity of multiple integral Hilbert-type inequality $\int_{R_+^n} K(x_1,\cdots,x_n)\prod_{i=1}^{n} f_i(x_i)\, dx_1 \cdots dx_n \le M\prod_{i=1}^{n}\|f_i\|_{p_i,\alpha_i}$ with homogeneous kernel $K(x_1,\cdots,x_n)$ of order λ is $\sum_{i=1}^{n}\alpha_i/p_i = \lambda + n - 1$, and the calculation formula of its optimal constant factor is obtained. The basic theory and method of constructing a Hilbert-type multiple integral inequality with the homogeneous kernel and optimal constant factor are solved.

1. Preliminary

Assuming that $r > 1$, $f(x) \ge 0$, and $\alpha \in \mathbf{R}$, define

$$L_{r,\alpha}(0,+\infty) = \left\{ f(x) \ge 0 : \|f\|_{r,\alpha} = \left(\int_0^{+\infty} x^{\alpha} f^{r}(x)\, dx \right)^{1/r} < +\infty \right\}. \quad (1)$$

Particularly, denote $L_r(0,+\infty) = L_{r,0}(0,+\infty)$ and $\|f\|_r = \|f\|_{r,0}$. If $(1/p) + (1/q) = 1$ $(p > 1)$, $f \in L_p(0,+\infty)$, and $g \in L_q(0,+\infty)$, then there holds the well-known Hilbert's integral inequality [1]

$$\int_0^{+\infty}\int_0^{+\infty} \frac{f(x)g(y)}{x+y}\, dxdy < \frac{\pi}{\sin(\pi/p)}\|f\|_p\|g\|_q, \quad (2)$$

where the constant factor $\pi/(\sin(\pi/p))$ is optimal.

In general, let $\sum_{i=1}^{n} 1/p_i = 1$ $(p_i > 1)$, $\alpha_i \in \mathbf{R}$ $(i = 1, 2,\cdots,n)$, $f_i(x_i) \in L_{p_i,\alpha_i}(0,+\infty)$, $K(x_1,\cdots,x_n)$ be a nonnegative measurable function, and M be a constant; we call

$$\int_{R_+^n} K(x_1,\cdots,x_n)\prod_{i=1}^{n} f_i(x_i)dx_1 \cdots dx_n \le M\prod_{i=1}^{n}\|f_i\|_{p_i,\alpha_i}, \quad (3)$$

the Hilbert-type multiple integral inequality.

Fruitful results have been obtained for the Hilbert-type inequality [2–12], but the results of the multi-integral form are much less, especially for the parameters' conditions and the best constant factor for the Hilbert-type multi-integral inequality. The research for these problems is natural and important. However, the related references are less. In this paper, we will discuss the cases for the homogeneous integral kernel.

Lemma 1. *Let $n \ge 2$ be an integer, $\alpha_i \in \mathbf{R}$ $(i = 1, 2,\cdots,n)$, $K(x_1,\cdots,x_n)$ be a homogeneous nonnegative measurable function of order λ, $\sum_{i=1}^{n} 1/p_i = 1$ $(p_i > 1)$, and $\sum_{i=1}^{n}\alpha_i/p_i = \lambda + n - 1$. Denote*

$$\omega_j(x_j) = \int_{R_+^{n-1}} K(x_1,\cdots,x_n) \cdot \prod_{i=1(i\neq j)}^{n} x_i^{-(\alpha_i+1)/p_i}\, dx_1 \cdots dx_{j-1} d_{j+1} \cdots dx_n,$$

$$W_j = \int_{R_+^{n-1}} K\left(u_1,\cdots,u_{j-1}, 1, u_{j+1},\cdots,u_n\right) \cdot \prod_{i=1(i\neq j)}^{n} u_i^{-(\alpha_i+1)/p_i} du_1 \cdots du_{j-1} du_{j+1} \cdots\cdots du_n, \tag{4}$$

where $j = 1, 2, \cdots, n$. Then, we have

$$\omega_j\left(x_j\right) = x_j^{((\alpha_j+1)/p_j)-1} W_j, \tag{5}$$

and $W_1 = W_2 = \cdots = W_n$.

Proof. Since $K(x_1,\cdots,x_n)$ is a homogeneous function of order λ, we have

$$K(x_1,\cdots,x_n) = x_j^{\lambda} K\left(\frac{x_1}{x_j},\cdots,\frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j},\cdots,\frac{x_n}{x_j}\right), \tag{6}$$

and then

$$\omega_j\left(x_j\right) = x_j^{\lambda} \int_{R_+^{n-1}} K\left(\frac{x_1}{x_j},\cdots,\frac{x_{j-1}}{x_j}, 1, \frac{x_{j+1}}{x_j},\cdots,\frac{x_n}{x_j}\right) \times \prod_{i=1(i\neq j)}^{n} x_i^{-(\alpha_i+1)/p_i} dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n. \tag{7}$$

Setting $x_i/x_j = u_i$ $(i = 1,\cdots,j-1, j+1,\cdots,n)$, then we find

$$\frac{\partial\left(x_1,\cdots,x_{j-1}, x_{j+1},\cdots,x_n\right)}{\partial\left(u_1,\cdots,u_{j-1}, u_{j+1},\cdots,u_n\right)} = x_j^{n-1},$$

$$\begin{aligned}\omega_j\left(x_j\right) &= x_j^{\lambda+n-1-\sum_{i=1(i\neq j)}^{n}(\alpha_i+1)/p_i} \int_{R_+^{n-1}} K\left(u_1,\cdots,u_{j-1}, 1, u_{j+1},\cdots,u_n\right) \\ &\quad \times \prod_{i=1(i\neq j)}^{n} u_i^{-(\alpha_i+1)/p_i} du_1 \cdots du_{j-1} du_{j+1} \cdots du_n \\ &= x_j^{((\alpha_j+1)/p_j)-1} W_j.\end{aligned} \tag{8}$$

Thus, (5) holds, because for any j, we have

$$W_j = \int_{R_+^{n-1}} u_1^{\lambda} K\left(1, \frac{u_2}{u_1},\cdots,\frac{u_{j-1}}{u_1}, \frac{1}{u_1}, \frac{u_{j+1}}{u_1},\cdots,\frac{u_n}{u_1}\right) \times \prod_{i=1(i\neq j)}^{n} u_i^{-(\alpha_i+1)/p_i} du_1 \cdots du_{j-1} du_{j+1} \cdots du_n. \tag{9}$$

Setting $u_2/u_1 = v_2, \cdots, u_{j-1}/u_1 = v_{j-1}, 1/u_1 = v_j, u_{j+1}/u_1 = v_{j+1}, \cdots,$ and $u_n/u_1 = v_n$, then it follows that $u_1 = 1/v_j$, $u_2 = (1/v_j)v_2, \cdots, u_{j-1} = (1/v_j)v_{j-1}$, $u_{j+1} = (1/v_j)v_{j+1}, \cdots,$ and $u_n = (1/v_j)v_n$. So we get

$$\begin{aligned}W_j &= \int_{\mathbf{R}_+^{n-1}} \left(\frac{1}{v_j}\right)^{\lambda} K(1, v_2,\cdots,v_n) \left(\frac{1}{v_j}\right)^{-(\alpha_1+1)/p_1} \left(\frac{v_2}{v_j}\right)^{-(\alpha_2+1)/p_2} \cdots \\ &\quad \cdot \left(\frac{v_{j-1}}{v_j}\right)^{-(\alpha_{j-1}+1)/p_{j-1}} \times \left(\frac{v_{j+1}}{v_j}\right)^{-(\alpha_{j+1}+1)/p_{j+1}} \cdots \\ &\quad \cdot \left(\frac{v_n}{v_j}\right)^{-(\alpha_n+1)/p_n} v_j^{-n} dv_2 \cdots dv_n \\ &= \int_{\mathbf{R}_+^{n-1}} K(1, v_2,\cdots,v_n) v_2^{-(\alpha_2+1)/p_2} \cdots v_{j-1}^{-(\alpha_{j-1}+1)/p_{j-1}} \\ &\quad \cdot v_j^{-\lambda-n+\sum_{i=1(i\neq j)}^{n}(\alpha_i+1)/p_i} \times v_{j+1}^{-(\alpha_{j+1}+1)/p_{j+1}} \cdots v_n^{-(\alpha_n+1)/p_n} dv_2 \cdots dv_n \\ &= \int_{R_+^{n-1}} K(1, v_2,\cdots,v_n) \prod_{i=2}^{n} v_i^{-(\alpha_i+1)/p_i} dv_2 \cdots dv_n = W_1.\end{aligned} \tag{10}$$

Hence, we obtain $W_1 = W_2 = \cdots = W_n$.

Lemma 2. *Let $0 < a < 2$, then*

$$\int_0^{+\infty} \frac{\min\{1, t\}}{\max\{1, t\}} t^{-a} dt = \frac{1}{2-a} + \frac{1}{a}. \tag{11}$$

Proof. Since $0 < a < 2$, we have $a - 1 < 1$ and $a + 1 > 1$, so

$$\begin{aligned}\int_0^{+\infty} \frac{\min\{1, t\}}{\max\{1, t\}} t^{-a} dt &= \int_0^1 \frac{t}{1} t^{-a} dt + \int_1^{+\infty} \frac{1}{t} t^{-a} dt \\ &= \int_0^1 \frac{1}{t^{a-1}} dt + \int_1^{+\infty} \frac{1}{t^{a+1}} dt = \frac{1}{2-a} + \frac{1}{a}.\end{aligned} \tag{12}$$

2. The Equivalent Conditions for a Hilbert-Type Multiple Integral Inequality Holding

Theorem 3. *Let $n \geq 2$ be an integer, $\sum_{i=1}^{n} 1/p_i = 1$ $(p_i > 1)$, $\alpha_i \in \mathbf{R}$ $(i = 1, 2,\cdots,n)$, $K(x_1,\cdots,x_n)$ be a homogeneous non-negative measurable function of order λ, and*

$$W_n = \int_{R_+^{n-1}} K(u_1, \cdots, u_{n-1}, 1) \prod_{i=1}^{n-1} u_i^{-(\alpha_i+1)/p_i} du_1 \cdots du_{n-1}, \tag{13}$$

is convergent, such that

$$\begin{aligned}W_n^{(1)} &= \int_0^1 \cdots \int_0^1 K(u_1,\cdots,u_{n-1}, 1) \prod_{i=1}^{n-1} u_i^{-(\alpha_i+1)/p_i} du_1 \cdots du_{n-1} > 0, \\ W_n^{(2)} &= \int_1^{\infty} \cdots \int_1^{\infty} K(u_1,\cdots,u_{n-1}, 1) \prod_{i=1}^{n-1} u_i^{-(\alpha_i+1)/p_i} du_1 \cdots du_{n-1} > 0.\end{aligned} \tag{14}$$

Then

(i) For all $f_i(x_i) \in L_{p_i,\alpha_i}(0,+\infty)$ $(i = 1, 2, \cdots, n)$, there exists a constant M, such that the Hilbert-type multiple integral inequality

$$\int_{R_+^n} K(x_1, \cdots, x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n \le M \prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i}, \quad (15)$$

holds true if and only if $\sum_{i=1}^{n} \alpha_i/p_i = \lambda + n - 1$.

(ii) If (15) holds, then the best constant factor is inf $M = W_n$.

Proof. (i) Suppose that there exists a constant M such that (15) holds. Denote $c = \sum_{i=1}^{n} \alpha_i/p_i - (\lambda + n - 1)$.

If $c > 0$, then for $0 < \varepsilon < c$, we set

$$f_i(x_i) = \begin{cases} x_i^{(-\alpha_i-1+\varepsilon)/p_i}, & 0 < x_i \le 1, \\ 0, & x_i > 1, \end{cases} \quad (16)$$

where $i = 1, 2, \cdots, n$. We find

$$\prod_{i=1}^{n} \left(\int_0^{+\infty} x_i^{\alpha_i} f_i^{p_i}(x_i)\, dx_i \right)^{1/p_i} = \prod_{i=1}^{n} \left(\int_0^1 x_i^{-1+\varepsilon}\, dx_i \right)^{1/p_i} = \frac{1}{\varepsilon},$$

$$\begin{aligned} &\int_{\mathbf{R}_+^n} K(x_1, , \cdots, x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n \\ &= \int_0^1 x_n^{(-\alpha_n-1+\varepsilon)/p_n} \left(\int_0^1 \cdots \int_0^1 K(x_1, \cdots, x_n) \right. \\ &\quad \cdot \left. \prod_{i=1}^{n-1} x_i^{(-\alpha_i-1+\varepsilon)/p_i} dx_1 \cdots dx_{n-1} \right) dx_n \\ &= \int_0^1 x_n^{(-\alpha_n-1+\varepsilon)/p_n+\lambda} \left(\int_0^1 \cdots \int_0^1 K\left(\frac{x_1}{x_n}, \cdots, \frac{x_{n-1}}{x_n}, 1\right) \right. \\ &\quad \cdot \left. \prod_{i=1}^{n-1} x_i^{(-\alpha_i-1+\varepsilon)/p_i} dx_1 \cdots dx_{n-1} \right) dx_n = \int_0^1 x_n^{(-\alpha_n-1+\varepsilon)/p_n+\lambda} \\ &\quad \times \left(\int_0^{1/x_n} \cdots \int_0^{1/x_n} K(t_1, \cdots, t_{n-1}, 1) x_n^{n-1} \prod_{i=1}^{n-1} (x_n t_i)^{(-\alpha_i-1+\varepsilon)/p_i} \right. \\ &\quad \cdot \left. dt_1 \cdots dt_{n-1} \right) dx_n = \int_0^1 x_n^{((-\alpha_n-1+\varepsilon)/p_n)+\lambda+n-1+\sum_{i=1}^{n-1}(-\alpha_i-1+\varepsilon)/p_i} \\ &\quad \times \left(\int_0^{1/x_n} \cdots \int_0^{1/x_n} K(t_1, \cdots, t_{n-1}, 1) \prod_{i=1}^{n-1} t_i^{(-\alpha_i-1+\varepsilon)/p_i} dt_1 \cdots dt_{n-1} \right) dx_n \\ &= \int_0^1 x_n^{-1-c+\varepsilon} \left(\int_0^{1/x_n} \cdots \int_0^{1/x_n} K(t_1, \cdots, t_{n-1}, 1) \right. \\ &\quad \cdot \left. \prod_{i=1}^{n-1} t_i^{(-\alpha_i-1+\varepsilon)/p_i} dt_1 \cdots dt_{n-1} \right) dx_n \ge \int_0^1 x_n^{-1-c+\varepsilon}\, dx_n \\ &\quad \cdot \left(\int_0^1 \cdots \int_0^1 K(t_1, \cdots, t_{n-1}, 1) \prod_{i=1}^{n-1} t_i^{\frac{-\alpha_i-1+\varepsilon}{p_i}} dt_1 \cdots dt_{n-1} \right). \end{aligned} \quad (17)$$

Thus, by (15), we get

$$\int_0^1 x_n^{-1-c+\varepsilon}\, dx_n \left(\int_0^1 \cdots \int_0^1 K(t_1, \cdots, t_{n-1}, 1) \prod_{i=1}^{n-1} t_i^{(-\alpha_i-1+\varepsilon)/p_i} dt_1 \cdots dt_{n-1} \right) \le \frac{M}{\varepsilon}. \quad (18)$$

Since $W_n^{(1)} > 0$ and $\varepsilon - c < 0$, $\int_0^1 x_n^{-1-c+\varepsilon}\, dx_n$ is divergent to $+\infty$. So we get a contradiction that $+\infty \le M/\varepsilon$, namely, $c > 0$, cannot be held.

If $c < 0$, then for $0 < \varepsilon < -c$, we set

$$f_i(x_i) = \begin{cases} x_i^{(-\alpha_i-1-\varepsilon)/p_i}, & x_i \ge 1, \\ 0, & 0 < x_i < 1, \end{cases} \quad (19)$$

where $i = 1, 2, \cdots, n$. Similarly, we get

$$\begin{aligned} &\int_1^{+\infty} x_n^{-1-c-\varepsilon}\, dx_n \left(\int_1^{+\infty} \cdots \int_1^{+\infty} K(t_1, \cdots, t_{n-1}, 1) \right. \\ &\quad \cdot \left. \prod_{i=1}^{n-1} t_i^{(-\alpha_i-1-\varepsilon)/p_i} dt_1 \cdots dt_{n-1} \right) \le \frac{M}{\varepsilon}. \end{aligned} \quad (20)$$

Since $W_n^{(2)} > 0$ and $-c - \varepsilon > 0$, $\int_1^{+\infty} x_n^{-1-c-\varepsilon}\, dx_n$ is divergent to $+\infty$; also, we get a contradiction that $+\infty \le M/\varepsilon$, namely, $c < 0$, cannot be held.

From the above discussions, we get $c = 0$; that is, $\sum_{i=1}^{n} \alpha_i/p_i = \lambda + n - 1$.

Conversely, assume that $\sum_{i=1}^{n} \alpha_i/p_i = \lambda + n - 1$ holds. Note that

$$\begin{aligned} &\prod_{j=1}^{n} x_j^{(\alpha_j+1)/p_j} \left(\prod_{i=1}^{n} x_i^{(\alpha_i+1)/p_i} \right)^{-1/p_j} \\ &= \prod_{j=1}^{n} x_j^{(\alpha_j+1)/p_j} \prod_{i=1}^{n} x_i^{-\alpha_i+1/p_i \sum_{k=1}^{n} 1/p_k} \\ &= \prod_{j=1}^{n} x_j^{(\alpha_j+1)/p_j} \prod_{i=1}^{n} x_i^{-(\alpha_i+1)/p_i} = 1. \end{aligned} \quad (21)$$

By Hölder's inequality and Lemma 1, we find

$$\begin{aligned} &\int_{\mathbf{R}_+^n} K(x_1, , \cdots, x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n \\ &= \int_{R_+^n} K(x_1, , \cdots, x_n) \prod_{j=1}^{n} \left[x_j^{(\alpha_j+1)/p_j} \left(\prod_{i=1}^{n} x_i^{(\alpha_i+1)/p_i} \right)^{-1/p_j} f_j(x_j) \right] \\ &\quad \cdot dx_1 \cdots dx_n \le \prod_{j=1}^{n} \left[\int_{R_+^n} x_j^{\alpha_j+1} \left(\prod_{i=1}^{n} x_i^{-(\alpha_i+1)/p_i} \right) f_j^{p_j}(x_j) K \right. \\ &\quad \left. \cdot (x_1, , \cdots, x_n) dx_1 \cdots dx_n \right]^{1/p_j} = \prod_{j=1}^{n} \left[\int_0^{+\infty} x_j^{\alpha_j+1-((\alpha_j+1)/p_j)} f_j^{p_j} \right. \\ &\quad \cdot (x_j) \left(\int_{R_+^{n-1}} \prod_{i=1 (i \ne j)}^{n} x_i^{-(\alpha_i+1)/p_i} K(x_1, \cdots, x_n) \right. \end{aligned}$$

$$\times dx_1 \cdots dx_{j-1} dx_{j+1} \cdots dx_n) dx_j \Bigg]^{1/p_j}$$
$$= \prod_{j=1}^{n} \left(\int_0^{+\infty} x_j^{\alpha_j+1-((\alpha_j+1)/p_j)} f_j^{p_j}(x_j) \omega_j(x_j) dx_j \right)^{1/p_j}$$
$$= \prod_{j=1}^{n} \left(\int_0^{+\infty} x_j^{\alpha_j+1-((\alpha_j+1)/p_j)} f_j^{p_j}(x_j) x_j^{((\alpha_j+1)/p_j)-1} W_j dx_j \right)^{1/p_j}$$
$$= \left(\prod_{j=1}^{n} W_j^{1/p_j} \right) \prod_{j=1}^{n} \left(\int_0^{+\infty} x_j^{\alpha_j} f_j^{p_j}(x_j) dx_j \right)^{1/p_j}$$
$$= W_n \prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i}. \tag{22}$$

So, for all $M \geq W_n$, (15) holds.

(ii) Next, we prove that when the equality (15) holds, $\inf M = W_n$. Otherwise, there exists a constant $M_0 < W_n$, such that

$$\int_{R_+^n} K(x_1,\cdots,x_n) \prod_{i=1}^{n} f_i(x_i) dx_1 \cdots dx_n \leq M_0 \prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i}. \tag{23}$$

For a sufficient small $\varepsilon > 0$ and $\delta > 0$, let

$$f_1(x_1) = \begin{cases} x_1^{(-\alpha_1-1-\varepsilon)/p_1}, & x_1 \geq 1, \\ 0, & 0 < x_1 < 1, \end{cases} \tag{24}$$

and when $i = 2, 3, \cdots, n$, we let

$$f_i(x_i) = \begin{cases} x_i^{(-\alpha_i-1-\varepsilon)/p_i}, & x_i \geq \delta, \\ 0, & 0 < x_i < \delta. \end{cases} \tag{25}$$

Thus, we get

$$\prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i} = \left(\int_1^{+\infty} x_1^{-1-\varepsilon} dx_1 \right)^{1/p_1} \prod_{i=2}^{n} \left(\int_\delta^{+\infty} x_i^{-1-\varepsilon} dx_i \right)^{1/p_i}$$
$$= \frac{1}{\varepsilon} \prod_{i=2}^{n} \left(\frac{1}{\delta^\varepsilon} \right)^{1/p_i}. \tag{26}$$

We still have

$$\int_{\mathbf{R}_+^n} K(x_1,\cdots,x_n) \prod_{i=1}^{n} f_i(x_i)\, dx_1 \cdots dx_n$$
$$= \int_1^{+\infty} x_1^{-(\alpha_1+1+\varepsilon)/p_1} \left(\int_\delta^{+\infty} \cdots \int_\delta^{+\infty} K(x_1,\cdots,x_n) \right.$$
$$\left. \cdot \prod_{i=2}^{n} x_i^{-(\alpha_i+1+\varepsilon)/p_i} dx_2 \cdots dx_n \right) dx_1 = \int_1^{+\infty} x_1^{\lambda-((\alpha_1+1+\varepsilon)/p_1)}$$
$$\cdot \left(\int_\delta^{+\infty} \cdots \int_\delta^{+\infty} K\left(1, \frac{x_2}{x_1} x_{2/x_1}, \cdots, \frac{x_n}{x_1}\right) \right.$$
$$\left. \cdot \prod_{i=2}^{n} x_i^{-(\alpha_i+1+\varepsilon)/p_i} dx_2 \cdots dx_n \right) dx_1 = \int_1^{+\infty} x_1^{\lambda-((\alpha_1+1+\varepsilon)/p_1)}$$
$$\cdot \left(\int_{\delta/x_1}^{+\infty} \cdots \int_{\delta/x_1}^{+\infty} K(1, u_2,\cdots,u_n) \prod_{i=2}^{n} (x_1 u_i)^{-(\alpha_i+1+\varepsilon)/p_i} x_1^{n-1} \right.$$
$$\left. \cdot du_2 \cdots du_n \right) dx_1 = \int_1^{+\infty} x_1^{-1-\varepsilon} \left(\int_{\delta/x_1}^{+\infty} \cdots \int_{\delta/x_1}^{+\infty} K(1, u_2,\cdots,u_n) \right.$$
$$\left. \cdot \prod_{i=2}^{n} u_i^{-(\alpha_i+1+\varepsilon)/p_i} du_2 \cdots du_n \right) dx_1$$
$$\geq \int_1^{+\infty} x_1^{-1-\varepsilon} \left(\int_\delta^{+\infty} \cdots \int_\delta^{+\infty} K(1, u_2,\cdots,u_n) \right.$$
$$\left. \cdot \prod_{i=2}^{n} u_i^{-(\alpha_i+1+\varepsilon)/p_i} du_2 \cdots du_n \right) dx_1$$
$$= \frac{1}{\varepsilon} \int_\delta^{+\infty} \cdots \int_\delta^{+\infty} K(1, u_2,\cdots,u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1+\varepsilon)/p_i} du_2 \cdots du_n. \tag{27}$$

Combining this with (23) and (26), we obtain

$$\int_\delta^{+\infty} \cdots \int_\delta^{+\infty} K(1, u_2,\cdots,u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1+\varepsilon)/p_i} du_2 \cdots du_n \leq M_0 \prod_{i=2}^{n} \left(\frac{1}{\delta^\varepsilon} \right)^{1/p_i}. \tag{28}$$

Let $\varepsilon \longrightarrow 0^+$, and then by the Lebesgue dominated convergence theorem, we have

$$\int_\delta^{+\infty} \cdots \int_\delta^{+\infty} K(1, u_2,\cdots,u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1)/p_i} du_2 \cdots du_n \leq M_0. \tag{29}$$

Taking $\delta \longrightarrow 0^+$, we get

$$W_n = W_1 = \int_{\mathbf{R}_+^{n-1}} K(1, u_2,\cdots,u_n) \prod_{i=2}^{n} u_i^{-(\alpha_i+1)/p_i} du_2 \cdots du_n \leq M_0. \tag{30}$$

This is a contradiction compared with $M_0 < W_n$, and then $\inf M = W_n$.

3. Applications

Taking some different integral kernels and different parameters, we can get a great deal of Hilbert-type inequalities in former literatures and other some new equalities. Moreover, the necessary and sufficient conditions for the existence of these inequalities are obtained.

Corollary 4. *Let the integer $n \geq 2$, $\sum_{i=1}^{n} 1/p_i = 1$ $(p_i > 1)$, $\alpha_i \in R$, $f_i(x_i) \in L_{p_i,\alpha_i}(0,+\infty)$ $(i = 1, 2, \cdots, n)$, and*

$$W_n = \int_{\mathbf{R}_+^{n-1}} \frac{\min\{u_1,\cdots,u_{n-1},1\}}{\max\{u_1,\cdots,u_{n-1},1\}} \prod_{i=1}^{n-1} u_i^{-(\alpha_i+1)/p_i}\, du_1 \cdots du_{n-1}, \tag{31}$$

convergence. Then, there exists a constant M, such that the necessary and sufficient condition for the equality hold

$$\int_{R_+^n} \frac{\min\{x_1,\cdots,x_n\}}{\max\{x_1,\cdots,x_n\}} \prod_{i=1}^{n} f_i(x_i)\, dx_1 \cdots dx_n \leq M \prod_{i=1}^{n} \|f_i\|_{p_i,\alpha_i}, \tag{32}$$

is $\sum_{i=1}^{n} \alpha_i/p_i = n-1$. And when equation (32) holds, the best constant factor is inf $M = W_n$.

Proof. Let $K(x_1,\cdots,x_n) = \min\{x_1,\cdots,x_n\}/\max\{x_1,\cdots,x_n\}$; then, $K(x_1,\cdots,x_n)$ certainly is a homogeneous nonnegative measurable function of order $\lambda\,(=0)$. By Theorem 3, the corollary holds.

Corollary 5. *Let $(1/p) + (1/q) = 1$ $(p > 1)$, $-1 < \alpha < 2p-1$, $-1 < \beta < 2q-1$, $f(x) \in L_{p,\alpha}(0,+\infty)$, and $g(y) \in L_{q,\beta}(0,+\infty)$. Then, there exists a constant M, such that the necessary and sufficient condition for the equality hold*

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\min\{x,y\}}{\max\{x,y\}} f(x)g(y)dxdy \leq M\,\|f\|_{p,\alpha}\|g\|_{q,\beta}, \tag{33}$$

is $(\alpha/p) + (\beta/q) = 1$. And when equation (33) holds, the best constant factor is inf $M = (p/(\alpha+1)) + (q/(\beta+1))$.

Proof. Since $0 < (\beta+1)/q < 2$, by Lemma 2, we have

$$\int_0^{+\infty} \frac{\min\{1,u_2\}}{\max\{1,u_2\}} u_2^{-(\beta+1)/q}\, du_2 = \left(2 - \frac{\beta+1}{q}\right)^{-1} + \left(\frac{\beta+1}{q}\right)^{-1} = \frac{p}{\alpha+1} + \frac{q}{\beta+1}. \tag{34}$$

Combining this with the case $n = 2$ of Corollary 4, the proof is completed.

Remark 6. (i) Letting $\alpha_i = p_i - 1$ $(i = 1,\cdots,n)$ in Corollary 5, we can get

$$\int_{R_+^n} \frac{\min\{x_1,\cdots,x_n\}}{\max\{x_1,\cdots,x_n\}} \prod_{i=1}^{n} f_i(x_i)\, dx_1 \cdots dx_n \leq n! \prod_{i=1}^{n} \|f_i\|_{p_i,p_i-1}, \tag{35}$$

where the constant factor $n!$ is the best possible. The above equality is the main result in [2].

(ii) Letting $\alpha = \beta = 1$ in Corollary 5, we get

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\min\{x,y\}}{\max\{x,y\}} f(x)g(y)\, dxdy \leq \frac{pq}{2} \|f\|_{p,1}\|g\|_{q,1}, \tag{36}$$

where the constant factor $pq/2$ is optimal.

(iii) Letting $\alpha = p/2$ and $\beta = q/2$ in Corollary 5, then we have

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\min\{x,y\}}{\max\{x,y\}} f(x)g(y)\, dxdy \leq \frac{8pq}{(p+2)(q+2)} \|f\|_{p,p/2}\|g\|_{q,q/2}, \tag{37}$$

where the constant factor $8pq/((p+2)(q+2))$ is the best possible.

(iv) Letting $\alpha = (3/2)p - 2$ and $\beta = (3/2)q - 2$ in Corollary 5, we obtain

$$\int_0^{+\infty}\int_0^{+\infty} \frac{\min\{x,y\}}{\max\{x,y\}} f(x)g(y)\, dxdy \leq \frac{8pq}{(3p-2)(3q-2)} \|f\|_{p,3/2p-2}\|g\|_{q,3/2q-2}, \tag{38}$$

where the constant factor $8pq/((3p-2)(3q-2))$ is optimal.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflict of interests.

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