Hindawi
Journal of Function Spaces
Volume 2020, Article ID 3075390, 7 pages
https://doi.org/10.1155/2020/3075390

Research Article

Petrović-Type Inequalities for Harmonic h-convex Functions

Imran Abbas Baloch [1,2] and Yu-Ming Chu [3,4]

[1] *Abdus Salam School of Mathematical Sciences, GC University, Lahore, Pakistan*
[2] *Department of Higher Education, Goverment College for Boys Gulberg, Punjab, Pakistan*
[3] *Department of Mathematics, Huzhou University, Huzhou 313000, China*
[4] *School of Mathematics and Statistics, Changsha University of Science & Technology, Changsha 410114, China*

Correspondence should be addressed to Yu-Ming Chu; chuyuming@zjhu.edu.cn

Received 29 June 2019; Accepted 13 August 2019; Published 8 January 2020

Academic Editor: Kehe Zhu

In the article, we establish several Petrović-type inequalities for the harmonic h-convex (concave) function if h is a submultiplicative (super-multiplicative) function, provide some new majorizaton type inequalities for harmonic convex function, and prove the superadditivity, subadditivity, linearity, and monotonicity properties for the functionals derived from the Petrović type inequalities.

1. Introduction

Let $I \subseteq \mathbb{R}$ be a nonempty interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be convex (concave) on I if the inequality

$$f(tx + (1-t)y) \le (\ge) tf(x) + (1-t)f(y), \tag{1}$$

holds for all $x, y \in I$ and $t \in [0, 1]$.

It is well known that the convex (concave) functions have wide applications in pure and applied mathematics [1–21], many remarkable properties and inequalities can be found in the literature [22–48] via the theory of convexity. Recently, a great deal of generalizations, extensions, and variants have been made for convexity, for example, GA-convexity [49], GG-convexity [50], s-convexity [51, 52], preinvex convexity [53], strong convexity [54–57], Schur convexity [58–60], and others [61–67].

The following well-known Petrović inequality was established by Petrović [68] in 1932.

Theorem 1. *Let $a > 0$ and $f : [0, a] \to \mathbb{R}$ be a convex function. Then the inequality*

$$\sum_{k=1}^{n} p_k f(x_k) \le f\left(\sum_{k=1}^{n} p_k x_k\right) + \left(\sum_{k=1}^{n} p_k - 1\right) f(0), \tag{2}$$

holds for all nonnegative n-tuples $(x_1, x_2, \cdots, x_n)$ and $(p_1, p_2, \cdots, p_n)$ such that $x_i \le \sum_{k=1}^{n} p_k x_k \le a$ $(i = 1, 2, \cdots, n)$.

Now, we recall the definitions of various convexities and the majorization relation between two n-tuples.

Definition 2. Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be submultiplicative (supermultiplicative) on I if the inequality

$$f(x, y) \le (\ge) f(x) f(y), \tag{3}$$

holds for all $x, y \in I$.

Definition 3. Let $n \ge 2$, $\mathbf{a} = (a_1, a_2, \cdots, a_n)$ and $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ be two n-tuples. Then $\mathbf{b}$ is said to be majorized by $\mathbf{a}$ or $\mathbf{a}$ majorizes $\mathbf{b}$ (in symbol $\mathbf{a} \succ \mathbf{b}$) if

$$\sum_{i=1}^{k} a_{[i]} \ge \sum_{i=1}^{k} b_{[i]}, \tag{4}$$

for $i = 1, 2, \cdots, n-1$ and

$$\sum_{i=1}^{n} a_{[i]} = \sum_{i=1}^{n} b_{[i]}, \tag{5}$$

where $a_{[i]}$ denotes the ith largest component in $\mathbf{a}$.

Definition 4. Let $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be harmonic convex (concave) on I if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (\ge) t f(y) + (1-t) f(x), \quad (6)$$

holds for all $x, y \in I$ with $(x, y) \neq (0, 0)$ and $t \in [0, 1]$.

Definition 5. Let $\alpha \in [0,1]$, $m \in (0,1]$, and $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be harmonic (α, m)-convex (concave) on I if the inequality

$$f\left(\frac{mxy}{(1-t)x+mty}\right) \le (\ge) t^{\alpha} f(x) + m(1-t^{\alpha}) f(y), \quad (7)$$

holds for all $x, y \in I$ with $(x, y) \neq (0, 0)$ and $t \in [0, 1]$.

Definition 6. Let $s, m \in (0, 1]$ and $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be harmonic (s, m)-convex (concave) in the second sense if the inequality

$$f\left(\frac{mxy}{(1-t)x+mty}\right) \le (\ge) t^{s} f(x) + m(1-t)^{s} f(y), \quad (8)$$

holds for all $x, y \in I$ with $(x, y) \neq (0, 0)$ and $t \in [0, 1]$.

Definition 7. Let $s, m \in (0, 1]$, $p \in \mathbb{R}\backslash\{0\}$, and $I \subseteq \mathbb{R}$ be an interval. Then a real-valued function $f : I \to \mathbb{R}$ is said to be harmonic $(p, (s, m))$-convex (concave) on I if the inequality

$$f\left(\frac{mxy}{\left[(1-t)x^{p}+t(my)^{p}\right]^{1/p}}\right) \le (\ge) t^{s} f(x) + m(1-t)^{s} f(y), \quad (9)$$

holds for all $t \in [0, 1]$ and $x, y \in I$ with $(x, y) \neq (0, 0)$ and $my \in I$.

Definition 8. Let $I \in \mathbb{R}$ be an interval and $h : [0, 1] \to \mathbb{R}$ be a nonnegative real-valued function. Then a real-valued function $f : I \to \mathbb{R}$ is said to be harmonic h-convex (concave) if the inequality

$$f\left(\frac{xy}{tx+(1-t)y}\right) \le (\ge) h(t) f(y) + h(1-t) f(x), \quad (10)$$

holds for all $t \in [0, 1]$ and $x, y \in I$ with $(x, y) \neq (0, 0)$.

Abbas Baloch [69] proved that.

Lemma 9. *Let $I \subseteq (0, \infty)$ and $f : I \to \mathbb{R}$ be a real-valued function. Then f is harmonic convex on I if and only if the function $xf(x)$ is convex on I.*

The following three examples also can be found in the literature [67].

Example 1. Let $f(x) = x$. Then f is harmonic convex on $(0, \infty)$ and harmonic concave on $(-\infty, 0)$.

Example 2. Let $f : (0, \infty) \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x, & 0 < x < 2, \\ 4 - \frac{4}{x}, & x \ge 2. \end{cases} \quad (11)$$

Then f is harmonic convex on $(0, \infty)$ due to $xf(x)$ is convex on $(0, \infty)$.

Example 3. The function

$$g(x) = \frac{(x-1)^{2}+1}{x}, \quad (12)$$

is harmonic convex on $(0, \infty)$ due to $xg(x)$ is convex on $(0, \infty)$.

Lemma 10 (see [70]). *Let $I \subseteq \mathbb{R}$ be an interval and $f : I \to \mathbb{R}$ be a real-valued function. Then f is convex on I if and only if the bivariate function*

$$h_{f}(x, y) = \frac{f(y) - f(x)}{y - x}, \quad (13)$$

is increasing with respect to its variables x and y.

2. Main Results

Lemma 11. *Let $c, a \in \mathbb{R}$ with $c < a$, $(x_1, x_2, \cdots, x_n)$ and $(p_1, p_2, \cdots, p_n)$ be two n-tuples such that*

$$c < x_i \le \sum_{k=1}^{n} p_k x_k \le a, \quad (14)$$

for $i = 1, 2, \cdots, n$, and $h : [c, a] \to (0, \infty)$ and $f : [c, a] \to \mathbb{R}$ be two real-valued functions. Then one has

$$\sum_{k=1}^{n} p_k f(x_k) \le \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} f\left(\sum_{k=1}^{n} p_k x_k\right), \quad (15)$$

if the function $f(x)/h(x - c)$ is increasing on $(c, a]$.

Proof. It follows from (14) and the monotonicity of the function $f(x)/h(x - c)$ on the interval $(c, a]$ that

$$\begin{gathered} \frac{f(\sum_{k=1}^{n} p_k x_k)}{h(\sum_{k=1}^{n} p_k x_k - c)} \ge \frac{f(x_j)}{h(x_j - c)}, \\ h\left(\sum_{k=1}^{n} p_k x_k - c\right) f(x_j) \le h(x_j - c) f\left(\sum_{k=1}^{n} p_k x_k\right), \end{gathered} \quad (16)$$

for all $j = 1, 2, \cdots, n$.

From above inequality we clearly see that

$$\begin{gathered} h\left(\sum_{k=1}^{n} p_k x_k - c\right) \sum_{j=1}^{n} p_j f(x_j) \le f\left(\sum_{k=1}^{n} p_k x_k\right) \sum_{j=1}^{n} p_j h(x_j - c), \\ \sum_{k=1}^{n} p_k f(x_k) \le \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} f\left(\sum_{k=1}^{n} p_k x_k\right). \end{gathered} \quad (17)$$

Theorem 12. *Let $c, a \in \mathbb{R}$ with $c < a$, $(x_1, x_2, \cdots, x_n)$, and $(p_1, p_2, \cdots, p_n)$ be two n-tuples such that*

$$c < x_i \le \sum_{k=1}^{n} p_k x_k \le a, \quad (18)$$

for $i = 1, 2, \cdots, n$, and $h : [c, a] \to (0, \infty)$ be a supermultiplicative function such that

$$h(\alpha) + h(1 - \alpha) \le 1, \quad (19)$$

for all $\alpha \in (0, 1)$. Then one has

$$\sum_{k=1}^{n} p_k x_k f(x_k) \le \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} \left(\sum_{k=1}^{n} p_k x_k\right) f\left(\sum_{k=1}^{n} p_k x_k\right) + c\left[\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)}\right] f(c), \tag{20}$$

if $f : [c, a] \to \mathbb{R}$ is a harmonic h-convex function.

Proof. It follows from Lemma 9 and the harmonic h-convexity of the function f on $[c, a]$ that the function $g(z) = zf(z)$ is h-convex on $[c, a]$. Let $x, y \in (c, a]$ such that $x = \alpha y + (1 - \alpha)c$ and

$$S_h(x) = \frac{g(x) - g(c)}{h(x - c)}. \tag{21}$$

Then we clearly see that $y > x$ and

$$S_h(x) = \frac{g[\alpha y + (1 - \alpha)c - g(c)]}{h[\alpha(y - c)]} \le \frac{h(\alpha)g(y) + [h(1 - \alpha) - 1]g(c)}{h[\alpha(y - c)]}. \tag{22}$$

Since h is a supermultiplicative function, we have

$$S_h(x) \le \frac{h(\alpha)g(y) + [h(1 - \alpha) - 1]g(c)}{h(\alpha)h(y - c)}. \tag{23}$$

Making use of (19) we get

$$S_h(x) \le \frac{g(y) - g(c)}{h(y - c)} = S_h(y), \tag{24}$$

which implies that the function $S_h(x) = [g(x) - g(c)]/h(x - c)$ is increasing on $(c, a]$. Then from Lemma 11 one has

$$\sum_{k=1}^{n} p_k (g(x_k) - g(c)) \le \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} \left[g\left(\sum_{k=1}^{n} p_k x_k\right) - g(c)\right], \tag{25}$$

which is equivalent to

$$\sum_{k=1}^{n} p_k g(x_k) \le \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} g\left(\sum_{k=1}^{n} p_k x_k\right) + \left[\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)}\right] g(c). \tag{26}$$

Therefore, inequality (20) can be obtained immediately by replacing $g(x) = xf(x)$ in the last inequality above.

Let $h(x) = x$. Then h is a supermultiplicative function and inequality (19) becomes an identity, and Theorem 12 leads to Theorem 13 immediately.

Theorem 13. *Let $c, a \in \mathbb{R}$ with $c < a$, $(x_1, x_2, \cdots, x_n)$, and $(p_1, p_2, \cdots, p_n)$ be two n-tuples such that*

$$c < x_i \le \sum_{k=1}^{n} p_k x_k \le a, \tag{27}$$

for $i = 1, 2, \cdots, n$. Then the inequality

$$\sum_{k=1}^{n} p_k x_k f(x_k) \le \frac{\sum_{k=1}^{n} p_k (x_k - c)}{(\sum_{k=1}^{n} p_k x_k - c)} \left(\sum_{k=1}^{n} p_k x_k\right) f\left(\sum_{k=1}^{n} p_k x_k\right) + c\left[\sum_{k=1}^{n} p_k - \frac{\sum_{k=1}^{n} p_k (x_k - c)}{(\sum_{k=1}^{n} p_k x_k - c)}\right] f(c), \tag{28}$$

holds if $f : [c, a] \to \mathbb{R}$ is a harmonic convex function.

Remark 14. Let $c, a \in \mathbb{R}$ with $c < a$, $(x_1, x_2, \cdots, x_n)$, and $(p_1, p_2, \cdots, p_n)$ be two n-tuples such that

$$c < x_i \le \sum_{k=1}^{n} p_k x_k \le a, \tag{29}$$

for $i = 1, 2, \cdots, n$. Then from the proof of Theorem 12 we clearly see that the following statements are true:

(1) If $h : [c, a] \to (0, \infty)$ and $f : [c, a] \to \mathbb{R}$ are two functions such that $f(x)/h(x - c)$ is decreasing on $(c, a]$, then the reverse inequality of (15) holds.

(2) If $h : [c, a] \to (0, \infty)$ is a submultiplicative function such that $h(\alpha) + h(1 - \alpha) \ge 1$ for all $\alpha \in (0, 1)$ and $f : [c, a] \to \mathbb{R}$ is a harmonic h-concave function, then the reverse inequality of (20) holds.

(3) If $f : [c, a] \to \mathbb{R}$ is a harmonic concave function, then the reverse inequality of (28) holds.

Theorem 15. *Let $I \in \mathbb{R} \backslash \{0\}$, $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in I$, $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ such that $\mathbf{a} \succ \mathbf{b}$, and $f : I \to \mathbb{R}$ be a harmonic convex function. Then the inequality*

$$\sum_{i=1}^{n} a_i f(a_i) \ge \sum_{i=1}^{n} b_i f(b_i), \tag{30}$$

holds.

Proof. Let $g(z) = zf(z)$ and

$$\lambda_i = \frac{g(b_i) - g(a_i)}{b_i - a_i}. \tag{31}$$

Then from Lemmas 9 and 10 together with the harmonic convexity of f that g is convex on I and the sequence $\{\lambda_i\}_{i=1}^{n}$ is decreasing. Let

$$A_k = \sum_{i=1}^{k} a_i, B_k = \sum_{i=1}^{k} b_i, A_0 = B_0. \tag{32}$$

Then $A_n = B_n$ due to $\mathbf{a} \succ \mathbf{b}$ and

$$\begin{aligned}
\sum_{i=1}^{n} a_i f(a_i) - \sum_{i=1}^{n} b_i f(b_i) &= \sum_{i=1}^{n} g(a_i) - \sum_{i=1}^{n} g(b_i) = \sum_{i=1}^{n} \lambda_i (a_i - b_i) \\
&= \sum_{i=1}^{n} \lambda_i (A_i - A_{i-1} - B_i + B_{i-1}) \\
&= \sum_{i=1}^{n} \lambda_i (A_i - B_i) - \sum_{i=1}^{n} \lambda_i (A_{i-1} - B_{i-1}) \\
&= \sum_{i=1}^{n-1} \lambda_i (A_i - B_i) - \sum_{i=1}^{n-1} \lambda_{i+1} (A_i - B_i) \\
&= \sum_{i=1}^{n-1} (\lambda_i - \lambda_{i+1})(A_i - B_i).
\end{aligned} \tag{33}$$

Therefore, inequality (30) follows from the monotonicity of the sequence $\{\lambda_i\}_{i=1}^{n-1}$ and $A_i \geq B_i$ for all $i = 1, 2, \cdots, n-1$ together with the last identity above.

The weighted version of Theorem 15 can be stated as follows.

Theorem 16. *Let $I \in \mathbb{R}\backslash\{0\}$, $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n \in I$, $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\mathbf{b} = (b_1, b_2, \cdots, b_n)$ such that $\mathbf{a} \succ \mathbf{b}$, $\mathbf{p} = (p_1, p_2, \cdots, p_n) \in (0, \infty)^n$ and $f : I \rightarrow \mathbb{R}$ be a harmonic convex function. Then the inequality*

$$\sum_{i=1}^{n} p_i a_i f(a_i) \geq \sum_{i=1}^{n} p_i b_i f(b_i), \tag{34}$$

holds.

3. Related Results

In this section, we present several interesting properties for the functionals derived from the Petrović-type inequalities given in Section 2.

Let $c, a \in \mathbb{R}$ with $c < a$, $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in (c, a]^n$, and $\mathbf{p} = (p_1, p_2, \cdots, p_n) \in (0, \infty)^n$ be two n-tuples such that

$$x_i \leq \sum_{k=1}^{n} p_k x_k \leq a, \tag{35}$$

for $i = 1, 2, \cdots, n$, and $h : [c, a] \rightarrow (0, \infty)$ and $f : [c, a] \rightarrow \mathbb{R}$ be two real-valued functions. Then the Petrović-type functional $\mathcal{F}_1(\mathbf{x}, \mathbf{p}; f)$ is defined by

$$\mathcal{F}_1(\mathbf{x}, \mathbf{p}; f) = \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} f\left(\sum_{k=1}^{n} p_k x_k\right) - \sum_{k=1}^{n} p_k f(x_k). \tag{36}$$

Remark 17. From Lemma 11 we clearly see that

$$\mathcal{F}_1(\mathbf{x}, \mathbf{p}; f) \geq 0, \tag{37}$$

if $f(x)/h(x-c)$ is increasing on $(c, a]$ and

$$\mathcal{F}_1(\mathbf{x}, \mathbf{p}; f) \leq 0, \tag{38}$$

if $f(x)/h(x-c)$ is decreasing on $(c, a]$.

Let $I \in \mathbb{R}\backslash\{0\}$, $\mathbf{a} = (a_1, a_2, \cdots, a_n)$, $\mathbf{b} = (b_1, b_2, \cdots, b_n) \in I^n$, $\mathbf{p} = (p_1, p_2, \cdots, p_n) \in (0, \infty)^n$, and $f : I \rightarrow \mathbb{R}$ be a real-valued function. Then the Petrović-type functional $\mathcal{F}_2(\mathbf{a}, \mathbf{b}, \mathbf{p}; f)$ is defined by

$$\mathcal{F}_2(\mathbf{a}, \mathbf{b}, \mathbf{p}; f) = \sum_{i=1}^{n} p_i a_i f(a_i) - \sum_{i=1}^{n} p_i b_i f(b_i). \tag{39}$$

Remark 18. Theorem 16 leads to the conclusion that

$$\mathcal{F}_2(\mathbf{a}, \mathbf{b}, \mathbf{p}; f) \geq 0, \tag{40}$$

if $\mathbf{a} \succ \mathbf{b}$ and f is harmonic convex on I and

$$\mathcal{F}_2(\mathbf{a}, \mathbf{b}, \mathbf{p}; f) \leq 0, \tag{41}$$

if $\mathbf{a} \succ \mathbf{b}$ and f is harmonic concave on I.

Theorem 19. *Let $c, a \in \mathbb{R}$ with $c < a$, $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in (c, a]^n$ and $\mathbf{p} = (p_1, p_2, \cdots, p_n)$, $\mathbf{q} = (q_1, q_2, \cdots, q_n) \in (0, \infty)^n$ be two n-tuples such that*

$$x_i \leq \sum_{k=1}^{n} p_k x_k \leq a, x_i \leq \sum_{k=1}^{n} q_k x_k \leq a, \tag{42}$$

for $i = 1, 2, \cdots, n$, and $h : [c, a] \rightarrow [1, \infty)$ and $f : [c, a] \rightarrow \mathbb{R}$ be two real-valued functions, and $\mathcal{F}_1$ be defined by (36). Then the following statements are true:

(1) *If $\sum_{k=1}^{n} (p_k + q_k) x_k \in (c, a]$ and $f(x)/h(x-c)$ is increasing on $(c, a]$, then*

$$\mathcal{F}_1(\mathbf{x}, \mathbf{p} + \mathbf{q}; f) \geq \mathcal{F}_1(\mathbf{x}, \mathbf{p}; f) + \mathcal{F}_1(\mathbf{x}, \mathbf{q}; f), \tag{43}$$

that is $\mathcal{F}_1(p) = \mathcal{F}_1(x, p; f)$ is superadditive on $(0, \infty)^n$.

(2) *If $p \geq q$ (namely, $p_j \geq q_j$ for all $j = 1, 2, \cdots, n$) and $\sum_{k=1}^{n} (p_k - q_k) x_k \geq x_j$ for all $j = 1, 2, \cdots, n$, and $f(x)/h(x-c)$ is increasing on $(c, a]$, then*

$$\mathcal{F}_1(\mathbf{x}, \mathbf{p}; f) \geq \mathcal{F}_1(\mathbf{x}, \mathbf{q}; f) \geq 0, \tag{44}$$

that is $\mathcal{F}_1(p) = \mathcal{F}_1(\mathbf{x}, \mathbf{p}; f)$ is increasing with respect to positive n-tuples.

(3) *If $\sum_{k=1}^{n} (p_k + q_k) x_k \in (c, a]$ and $f(x)/h(x-c)$ is decreasing on $(c, a]$, then the reverse inequality of (43) holds, that is $\mathcal{F}_1(\mathbf{p}) = \mathcal{F}_1(\mathbf{x}, \mathbf{p}; f)$ is subadditive on $(0, \infty)^n$. If $\mathbf{p} \geq \mathbf{q}$ and $\sum_{k=1}^{n} (p_k - q_k) x_k \geq x_j$ for all $j = 1, 2, \cdots, n$, and $f(x)/h(x-c)$ is decreasing on $(c, a]$, then the reverse inequality of (44) holds, that is $\mathcal{F}_1(\mathbf{p}) = \mathcal{F}_1(\mathbf{x}, \mathbf{p}; f)$ is decreasing with respect to positive n-tuples.*

Proof. (1) It follows from (36) that

$$\begin{aligned}
\mathcal{F}_1(\mathrm{x}, \mathbf{p} + \mathbf{q}; f) &= \frac{\sum_{k=1}^{n} (p_k + q_k) h(x_k - c)}{h(\sum_{k=1}^{n} (p_k + q_k)(x_k - c))} \\
&\quad \cdot f\left(\sum_{k=1}^{n} (p_k + q_k) x_k\right) - \sum_{k=1}^{n} (p_k + q_k) f(x_k) \\
&= \frac{\sum_{k=1}^{n} p_k h(x_k - c) + \sum_{k=1}^{n} q_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k + \sum_{k=1}^{n} q_k x_k - c)} \\
&\quad \cdot \left(\sum_{k=1}^{n} p_k x_k + \sum_{k=1}^{n} q_k x_k\right) \\
&\quad - \sum_{k=1}^{n} p_k f(x_k) - \sum_{k=1}^{n} q_k f(x_k).
\end{aligned} \tag{45}$$

Making use of the monotonicity of the function $f(x)/h(x-c)$ and Lemma 11 one has

$$\begin{aligned}
\sum_{k=1}^{n} p_k f(x_k) + \sum_{k=1}^{n} q_k f(x_k) &\leq \frac{\sum_{k=1}^{n} p_k h(x_k - c)}{h(\sum_{k=1}^{n} p_k x_k - c)} f\left(\sum_{k=1}^{n} p_k x_k\right) \\
&\quad + \frac{\sum_{k=1}^{n} q_k h(x_k - c)}{h(\sum_{k=1}^{n} q_k x_k - c)} f\left(\sum_{k=1}^{n} q_k x_k\right) \\
&\leq \frac{h(\sum_{k=1}^{n} p_k x_k - c) + h(\sum_{k=1}^{n} q_k x_k - c)}{h(\sum_{k=1}^{n} p_k x_k + \sum_{k=1}^{n} q_k x_k - c)} \\
&\quad \cdot f\left(\sum_{k=1}^{n} p_k x_k + \sum_{k=1}^{n} q_k x_k\right).
\end{aligned} \tag{46}$$

Combining (45) and (46) gives

$$\begin{aligned}
\mathscr{F}_1(x,\mathbf{p}+\mathbf{q};f) \geq{}& \frac{\sum_{k=1}^{n} p_k h(x_k-c)+\sum_{k=1}^{n} q_k(x_k-c)}{h(\sum_{k=1}^{n} p_k x_k-c)+h(\sum_{k=1}^{n} q_k x_k-c)} \\
&\cdot\left[f\left(\sum_{k=1}^{n} p_k x_k\right)+f\left(\sum_{k=1}^{n} q_k x_k\right)\right] \\
&-\sum_{k=1}^{n} p_k x_k-\sum_{k=1}^{n} q_k x_k \\
={}& \frac{\sum_{k=1}^{n} p_k h(x_k-c)+\sum_{k=1}^{n} q_k(x_k-c)}{h(\sum_{k=1}^{n} p_k x_k-c)+h(\sum_{k=1}^{n} q_k x_k-c)} f\left(\sum_{k=1}^{n} p_k x_k\right) \\
&+\frac{\sum_{k=1}^{n} p_k h(x_k-c)+\sum_{k=1}^{n} q_k(x_k-c)}{h(\sum_{k=1}^{n} p_k x_k-c)+h(\sum_{k=1}^{n} q_k x_k-c)} f\left(\sum_{k=1}^{n} q_k x_k\right) \\
&-\sum_{k=1}^{n} p_k x_k-\sum_{k=1}^{n} q_k x_k \\
\geq{}& \frac{\sum_{k=1}^{n} p_k h(x_k-c)}{h(\sum_{k=1}^{n} p_k x_k-c)+h(\sum_{k=1}^{n} q_k x_k-c)} f\left(\sum_{k=1}^{n} p_k x_k\right) \\
&+\frac{\sum_{k=1}^{n} p_k h(x_k-c)}{h(\sum_{k=1}^{n} p_k x_k-c)+h(\sum_{k=1}^{n} q_k x_k-c)} f\left(\sum_{k=1}^{n} q_k x_k\right) \\
&-\sum_{k=1}^{n} p_k f(x_k)-\sum_{k=1}^{n} q_k f(x_k) \\
\geq{}& \frac{\sum_{k=1}^{n} p_k h(x_k-c)}{h(\sum_{k=1}^{n} p_k x_k-c)} f\left(\sum_{k=1}^{n} p_k x_k\right) \\
&+\frac{\sum_{k=1}^{n} q_k h(x_k-c)}{h(\sum_{k=1}^{n} q_k x_k-c)} f\left(\sum_{k=1}^{n} q_k x_k\right) \\
&-\sum_{k=1}^{n} p_k f(x_k)-\sum_{k=1}^{n} q_k f(x_k).
\end{aligned} \tag{47}$$

Therefore,

$$\mathscr{F}_1(x,\mathbf{p}+\mathbf{q};f) \geq \mathscr{F}_1(x,\mathbf{p};f)+\mathscr{F}_1(x,\mathbf{q};f), \tag{48}$$

follows from (47).

(2) It follows from part (1) and Remark 17 that

$$\begin{aligned}
\mathscr{F}_1(\mathrm{x},\mathbf{p};f) &= \mathscr{F}_1(\mathrm{x},(\mathbf{p}-\mathbf{q})+\mathbf{q};f) \\
&\geq \mathscr{F}_1(\mathrm{x},\mathbf{q};f)+\mathscr{F}_1(\mathrm{x},\mathbf{p}-\mathbf{q};f) \\
&\geq \mathscr{F}_1(\mathrm{x},\mathbf{q};f) \geq 0.
\end{aligned} \tag{49}$$

(3) Part (3) can be proved by similar methods as in proving parts (1) and (2).

Theorem 20. *Let $I \in \mathbb{R}\backslash\{0\}$, $\mathbf{a}=(a_1,a_2,\cdots,a_n)$, $\mathbf{b}=(b_1,b_2,\cdots,b_n) \in I^n$ with $\mathbf{a} \succ \mathbf{b}$, $\mathbf{p}=(p_1,p_2,\cdots,p_n)$, $\mathbf{q}=(q_1,q_2,\cdots,q_n) \in (0,\infty)^n$, $f: I \to \mathbb{R}$ be a harmonic convex function, and $\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f)$ be defined by (39). Then the following statements are true:*

(1) *$\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f)$ is linear with respect to $\mathbf{p}$, that is*

$$\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p}+\mathbf{q};f)=\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f)+\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{q};f). \tag{50}$$

(2) *If $\mathbf{p} \geq \mathbf{q}$, then*

$$\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f) \geq \mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{q};f) \geq 0, \tag{51}$$

that is $\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f)$ is increasing with respect to $\mathbf{p}$.

(3) *If f is harmonic concave on I, then inequalities* (50) *and* (51) *also hold true.*

Proof. (1) It follows from (39) that

$$\begin{aligned}
\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p}+\mathbf{q};f) &= \sum_{i=1}^{n}(p_i+q_i)a_i f(a_i)-\sum_{i=1}^{n}(p_i+q_i)b_i f(b_i) \\
&= \sum_{i=1}^{n} p_i a_i f(a_i)-\sum_{i=1}^{n} p_i b_i f(b_i) \\
&\quad+\sum_{i=1}^{n} q_i a_i f(a_i)-\sum_{i=1}^{n} q_i b_i f(b_i) \\
&= \mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f)+\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{q};f).
\end{aligned} \tag{52}$$

(2) From part (1) and Remark 18 we get

$$\begin{aligned}
\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p};f) &= \mathscr{F}_2(\mathbf{a},\mathbf{b},(\mathbf{p}-\mathbf{q})+\mathbf{q};f) \\
&= \mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{p}-\mathbf{q};f)+\mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{q};f) \\
&\geq \mathscr{F}_2(\mathbf{a},\mathbf{b},\mathbf{q};f) \geq 0.
\end{aligned} \tag{53}$$

(3) Part (3) can be proved by similar methods in proving parts (1) and (2). □

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

Acknowledgments

This work was supported by the National Natural Science Foundation of China (Grant nos. 61673169, 11301127, 11701176, 11626101, 11601485) and the Higher Education Commission of Pakistan.

References

[1] J. Wang, C. Huang, and L. Huang, "Discontinuity-induced limit cycles in a general planar piecewise linear system of saddle–focus type," *Nonlinear Analysis: Hybrid Systems*, vol. 33, pp. 162–178, 2019.

[2] Y. Jiang and X. Xu, "A monotone finite volume method for time fractional Fokker-Planck equations," *Science China Mathematics*, vol. 62, no. 4, pp. 783–794, 2019.

[3] T.-H. Zhao, Y.-M. Chu, and H. Wang, "Logarithmically complete monotonicity properties relating to the gamma function," *Abstract and Applied Analysis*, vol. 2011, Article ID 896483, 13 pages, 2011.

[4] J. Li, J. Ying, and D. Xie, "On the analysis and application of an ion size-modified Poisson–Boltzmann equation," *Nonlinear Analysis: Real World Applications*, vol. 47, pp. 188–203, 2019.

[5] J. Wang, X. Chen, and L. Huang, "The number and stability of limit cycles for planar piecewise linear systems of node–saddle

type," *Journal of Mathematical Analysis and Applications*, vol. 469, no. 1, pp. 405–427, 2019.

[6] Z. Cai, J. Huang, and L. Huang, "Periodic orbit analysis for the delayed Filippov system," *Proceedings of the American Mathematical Society*, vol. 146, no. 11, pp. 4667–4682, 2018.

[7] K. X. Zhu, Y. Q. Xie, and F. Zhou, "Pullback attractors for a damped semilinear wave equation with delays," *Acta Mathematica Sinica, English Series*, vol. 34, no. 7, pp. 1131–1150, 2018.

[8] Z. Liu, N. Wu, X. Qin, and Y. Zhang, "Trigonometric transform splitting methods for real symmetric Toeplitz systems," *Computers & Mathematics with Applications*, vol. 75, no. 8, pp. 2782–2794, 2018.

[9] L. Duan, X. Fang, and C. Huang, "Global exponential convergence in a delayed almost periodic Nicholson's blowflies model with discontinuous harvesting," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 5, pp. 1954–1965, 2018.

[10] W. Tang and J. Zhang, "Symplecticity-preserving continuous-stage Runge–Kutta–Nyström methods," *Applied Mathematics and Computation*, vol. 323, pp. 204–219, 2018.

[11] Y. Tan, C. Huang, B. Sun, and T. Wang, "Dynamics of a class of delayed reaction–diffusion systems with Neumann boundary condition," *Journal of Mathematical Analysis and Applications*, vol. 458, no. 2, pp. 1115–1130, 2018.

[12] S. Zaheer Ullah, M. Adil Khan, and Y.-M. Chu, "A note on generalized convex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 10, 2019.

[13] Z. Liu, X. Qin, N. Wu, and Y. Zhang, "The shifted classical circulant and skew circulant splitting iterative methods for Toeplitz matrices," *Canadian Mathematical Bulletin*, vol. 60, no. 4, pp. 807–815, 2017.

[14] H. Hu and X. Zou, "Existence of an extinction wave in the fisher equation with a shifting habitat," *Proceedings of the American Mathematical Society*, vol. 145, no. 11, pp. 4763–4771, 2017.

[15] W. Wang, "On A -stable one-leg methods for solving nonlinear Volterra functional differential equations," *Applied Mathematics and Computation*, vol. 314, pp. 380–390, 2017.

[16] Z.-W. Cai, J.-H. Huang, and L.-H. Huang, "Generalized Lyapunov-Razumikhin method for retarded differential inclusions: applications to discontinuous neural networks," *Discrete & Continuous Dynamical Systems - B*, vol. 22B, no. 9, pp. 3591–3614, 2017.

[17] H. Hu and L. Liu, "Weighted inequalities for a general commutator associated to a singular integral operator satisfying a variant of Hörmander's condition," *Mathematical Notes*, vol. 101, no. 5–6, pp. 830–840, 2017.

[18] W.-M. Qian, Y.-Y. Yang, H.-W. Zhang, and Y.-M. Chu, "Optimal two-parameter geometric and arithmetic mean bounds for the Sándor-Yang mean," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 12, 2019.

[19] W. Wang and Y. Chen, "Fast numerical valuation of options with jump under Merton's model," *Journal of Computational and Applied Mathematics*, vol. 318, pp. 79–92, 2017.

[20] L. Duan, L. Huang, Z. Guo, and X. Fang, "Periodic attractor for reaction–diffusion high-order Hopfield neural networks with time-varying delays," *Computers & Mathematics with Applications*, vol. 73, no. 2, pp. 233–245, 2017.

[21] L. Duan and C. Huang, "Existence and global attractivity of almost periodic solutions for a delayed differential neoclassical growth model," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 3, pp. 814–822, 2017.

[22] M.-K. Wang, W. Zhang, and Y.-M. Chu, "Monotonicity, convexity and inequalities involving the generalized elliptic integrals," *Acta Mathematica Scientia*, vol. 39B, no. 5, pp. 1440–1450, 2019.

[23] M.-K. Wang, Y.-M. Chu, and W. Zhang, "Precise estimates for the solution of Ramanujan's generalized modular equation," *The Ramanujan Journal*, vol. 49, no. 3, pp. 653–668, 2019.

[24] W.-M. Qian, Z.-Y. He, H.-W. Zhang, and Y.-M. Chu, "Sharp bounds for Neuman means in terms of two-parameter contraharmonic and arithmetic mean," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 13, 2019.

[25] X.-H. He, W.-M. Qian, H.-Z. Xu, and Y.-M. Chu, "Sharp power mean bounds for two Sándor-Yang means," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 113, no. 3, pp. 2627–2638, 2019.

[26] M.-K. Wang, Y.-M. Chu, and W. Zhang, "Monotonicity and inequalities involving zero-balanced hypergeometric function," *Mathematical Inequalities & Applications*, vol. 22, no. 2, pp. 601–617, 2019.

[27] Y. Tan and K. Jing, "Existence and global exponential stability of almost periodic solution for delayed competitive neural networks with discontinuous activations," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 11, pp. 2821–2839, 2016.

[28] S.-L. Qiu, X.-Y. Ma, and Y.-M. Chu, "Sharp Landen transformation inequalities for hypergeometric functions, with applications," *Journal of Mathematical Analysis and Applications*, vol. 474, no. 2, pp. 1306–1337, 2019.

[29] Z.-F. Dai, D.-H. Li, and F.-H. Wen, "Worse-case conditional value-at-risk for asymmetrically distributed asset scenarios returns," *Journal of Computational Analysis & Applications*, vol. 20, no. 2, pp. 237–251, 2016.

[30] T.-H. Zhao, B.-C. Zhou, M.-K. Wang, and Y.-M. Chu, "On approximating the quasi-arithmetic mean," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 12, 2019.

[31] Z.-H. Yang, Y.-M. Chu, and W. Zhang, "High accuracy asymptotic bounds for the complete elliptic integral of the second kind," *Applied Mathematics and Computation*, vol. 348, pp. 552–564, 2019.

[32] Z. Dai, "Comments on a new class of nonlinear conjugate gradient coefficients with global convergence properties," *Applied Mathematics and Computation*, vol. 276, pp. 297–300, 2016.

[33] X. Fang, Y. Deng, and J. Li, "Plasmon resonance and heat generation in nanostructures," *Mathematical Methods in the Applied Sciences*, vol. 38, no. 18, pp. 4663–4672, 2015.

[34] Z. Dai, X. Chen, and F. Wen, "A modified perry's conjugate gradient method-based derivative-free method for solving large-scale nonlinear monotone equations," *Applied Mathematics and Computation*, vol. 270, pp. 378–386, 2015.

[35] T.-R. Huang, B.-W. Han, X.-Y. Ma, and Y.-M. Chu, "Optimal bounds for the generalized Euler-Mascheroni constant," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 9, 2018.

[36] M. Adil Khan, S. Begum, Y. Khurshid, and Y.-M. Chu, "Ostrowski type inequalities involving conformable fractional integrals," *Journal of Inequalities and Applications*, vol. 2018, no. 1, p. 14, 2018.

[37] I. Abbas Baloch and İ İşcan, "Some Hermite-Hadamard type integral inequalities for harmonically $(p, (s, m))$-convex functions," *Journal of Inequalities and Special Functions*, vol. 8, no. 4, pp. 65–84, 2017.

[38] I. Abbas Baloch, İ İşcan, and S. S. Dragomir, "Fejér type inequalities for harmonically (s, m)-convex functions,"

International Journal of Analysis and Applications, vol. 12, no. 2, pp. 188–197, 2016.

[39] W. Zhou and F. Wang, "A PRP-based residual method for large-scale monotone nonlinear equations," *Applied Mathematics and Computation*, vol. 261, pp. 1–7, 2015.

[40] X.-S. Zhou, "Weighted sharp function estimate and boundedness for commutator associated with singular integral operator satisfying a variant of Hörmander's condition," *Journal of Mathematical Inequalities*, vol. 9, no. 2, pp. 587–596, 2015.

[41] C. Huang, S. Guo, and L. Liu, "Boundedness on Morrey space for Toeplitz type operator associated to singular integral operator with variable Calderón-Zygmund kernel," *Journal of Mathematical Inequalities*, vol. 8, no. 3, pp. 453–464, 2014.

[42] C. Huang, Z. Yang, T. Yi, and X. Zou, "On the basins of attraction for a class of delay differential equations with non-monotone bistable nonlinearities," *Journal of Differential Equations*, vol. 256, no. 7, pp. 2101–2114, 2014.

[43] C. Huang, C. Peng, X. Chen, and F. Wen, "Dynamics analysis of a class of delayed economic model," *Abstract and Applied Analysis*, vol. 2013, Article ID 962738, 12 pages, 2013.

[44] L. Zhang and J. Li, "A new globalization technique for nonlinear conjugate gradient methods for nonconvex minimization," *Applied Mathematics and Computation*, vol. 217, no. 24, pp. 10295–10304, 2011.

[45] C. Huang and J. Cao, "Stochastic dynamics of nonautonomous Cohen-Grossberg neural networks," *Abstract and Applied Analysis*, vol. 2011, Article ID 297147, 17 pages, 2011.

[46] Q. Zhu, C. Huang, and X. Yang, "Exponential stability for stochastic jumping BAM neural networks with time-varying and distributed delays," *Nonlinear Analysis: Hybrid Systems*, vol. 5, no. 1, pp. 52–77, 2011.

[47] X. Yang, Q. Zhu, and C. Huang, "Generalized lag-synchronization of chaotic mix-delayed systems with uncertain parameters and unknown perturbations," *Nonlinear Analysis: Real World Applications*, vol. 12, no. 1, pp. 93–105, 2011.

[48] X. Yang, J. Cao, C. Huang, and Y. Long, "Existence and global exponential stability of almost periodic solutions for SICNNs with nonlinear behaved functions and mixed delays," *Abstract and Applied Analysis*, vol. 2010, Article ID 915451, 20 pages, 2010.

[49] X.-M. Zhang, Y.-M. Chu, and X.-H. Zhang, "The Hermite-Hadamard type inequality of GA-convex functions and its applications," *Journal of Inequalities and Applications*, vol. 2010, no. 1, p. 11, 2010.

[50] Y. Khurshid, M. Adil Khan, and Y.-M. Chu, "Conformable integral inequalities of the Hermite-Hadamard type in terms of GG- and GA-convexities," *Journal of Function Spaces*, vol. 2019, Article ID 6926107, 8 pages, 2019.

[51] M. Adil Khan, M. Hanif, Z. Abdul Hameed Khan, K. Ahmad, and Y.-M. Chu, "Association of Jensen's inequality for s-convex function with Csiszár divergence," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 14, 2019.

[52] M. Adil Khan, Y.-M. Chu, T. U. Khan, and J. Khan, "Some new inequalities of Hermite-Hadamard type for s-convex functions with applications," *Open Mathematics*, vol. 15, no. 1, pp. 1414–1430, 2017.

[53] Y. Khurshid, M. Adil Khan, Y.-M. Chu, and Z. A. Khan, "Hermite-Hadamard-Fejér inequalities for conformable fractional integrals via preinvex functions," *Journal of Function Spaces*, vol. 2019, Article ID 3146210, 9 pages, 2019.

[54] S. Zaheer Ullah, M. Adil Khan, and Y.-M. Chu, "Majorization theorem for strongly convex functions," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 13, 2019.

[55] D. Xie and J. Li, "A new analysis of electrostatic free energy minimization and Poisson–Boltzmann equation for protein in ionic solvent," *Nonlinear Analysis: Real World*, vol. 21, pp. 185–196, 2015.

[56] S. Zaheer Ullah, M. Adil Khan, Z. A. Khan, and Y.-M. Chu, "Integral majorization type inequalities for the functions in the sense of strong convexity," *Journal of Function Spaces*, vol. 2019, Article ID 9487823, 11 pages, 2019.

[57] W. Tang and Y. Sun, "Construction of Runge–Kutta type methods for solving ordinary differential equations," *Applied Mathematics and Computation*, vol. 234, pp. 179–191, 2014.

[58] Y.-M. Chu, G.-D. Wang, and X.-H. Zhang, "The Schur multiplicative and harmonic convexities of the complete symmetric function," *Mathematische Nachrichten*, vol. 284, no. 5–6, pp. 653–663, 2011.

[59] B. Wang, C.-L. Luo, S.-H. Li, and Y.-M. Chu, "Sharp one-parameter geometric and quadratic means bounds for the Sándor-Yang means," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales. Serie A. Matemáticas*, vol. 114, no. 1, 2020.

[60] S.-H. Wu and Y.-M. Chu, "Schur m-power convexity of generalized geometric Bonferroni mean involving three parameters," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 11, 2019.

[61] L. Zhang and S. Jian, "Further studies on the Wei–Yao–Liu nonlinear conjugate gradient method," *Applied Mathematics and Computation*, vol. 219, no. 14, pp. 7616–7621, 2013.

[62] Y.-C. Liu and J. Wu, "Fixed point theorems in piecewise continuous function spaces and applications to some nonlinear problems," *Mathematical Methods in the Applied Sciences*, vol. 37, no. 4, pp. 508–517, 2014.

[63] Y. Jiang and J. Ma, "Spectral collocation methods for Volterra-integro differential equations with noncompact kernels," *Journal of Computational and Applied Mathematics*, vol. 244, pp. 115–124, 2013.

[64] W. Zhou and X. Chen, "On the convergence of a modified regularized Newton method for convex optimization with singular solutions," *Journal of Computational and Applied Mathematics*, vol. 239, pp. 179–188, 2013.

[65] W. Zhou, "On the convergence of the modified Levenberg-Marquardt method with a nonmonotone second order Armijo type line search," *Journal of Computational and Applied Mathematics*, vol. 239, pp. 152–161, 2013.

[66] Y.-M. Chu, M.-K. Wang, and S.-L. Qiu, "Optimal combinations bounds of root-square and arithmetic means for Toader mean," *Proceedings - Mathematical Sciences*, vol. 122, no. 1, pp. 41–51, 2012.

[67] M.-K. Wang, H.-H. Chu, and Y.-M. Chu, "Precise bounds for the weighted Hölder mean of the complete p-elliptic integrals," *Journal of Mathematical Analysis and Applications*, vol. 480, no. 2, Article ID 123388, 2019.

[68] M. Petrović, "Sur une fonctionnelle," *Publications de l'Institut Mathématique Belgrade*, vol. 1, no. 1, pp. 149–156, 1932.

[69] I. Abbas Baloch, "Characterizations of classes of harmonic convex functions and applications," 2019, https://rgmia.org/papers/v22/v22a59.pdf.

[70] A. W. Roberts and D. E. Varberg, "Convex functions," Academic Press, NY, USA, 1973.