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Research Article

Further Results about a Special Fermat-Type Difference Equation

Hongwei Ma, Jianming Qi, and Zhenjie Zhang

School of Business, Shanghai Dianji University, Shanghai 201306, China

Correspondence should be addressed to Zhenjie Zhang; zhangzj12345678@sina.com

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In this paper, we prove the difference equation $F(z)^3 + \Delta_c F(z+c)^3 = 1$ does not have meromorphic solution of finite order over the complex plane $\mathbb{C}$. We also discuss an application to the unique range set problem.

1. Introduction and Main Result

This kind of questions is from the analogues to the Fermat diophantine equations

$$f^n + g^n = 1 \tag{1}$$

for a positive integer n. When $n > 3$, Gross [1] and Baker [2] proved that (1) does not nonconstant meromorphic solution in the complex plane $\mathbb{C}$.

Gross [1] and Baker [2] also showed for $n > 2$ there are no entire solutions. For the case $n = 3$, Gross [1] and Baker [2] also got the meromorphic solutions such as

$$f(z) = 4^{-1/6} \mathscr{P}'(z)^{-1}\left\{1 + 3^{-1/2} 4^{1/8} \mathscr{P}\right\} = \frac{\left\{1/2 + \mathscr{P}'(z)/(12)^{1/2}\right\}}{\mathscr{P}} \tag{2}$$

and

$$g(z) = 4^{-1/6} \mathscr{P}(z)^{-1}\left\{1 - 3^{-1/2} 4^{1/8} \mathscr{P}\right\} = \frac{\left\{1/2 - \mathscr{P}'(z)/(12)^{1/2}\right\}}{\mathscr{P}}, \tag{3}$$

where $\mathscr{P}$ is a Weierstrass $\mathscr{P}$-function.

Now the equation $w^3 + z^3 = l$ defines an algebraic function whose Riemann surface has genus 1, and there is accordingly a uniformization by Weierstrass elliptic functions. Weierstrass elliptic function $\mathscr{P}(z) := \mathscr{P}(z, g_2, g_3)$ is a meromorphic function with double periods ω_1, ω_2 and defined as

$$\mathscr{P}(z; w_1, w_2) := \frac{1}{z^2} + \sum_{\mu,\nu\in Z, \mu^2+\nu^2\neq 0} \left\{\frac{1}{(z + \mu w_1 + \nu w_2)^2} - \frac{1}{(\mu w_1 + \nu w_2)^2}\right\}, \tag{4}$$

which is even and satisfies, after appropriately choosing w_1 and w_2, to the form

$$\left(\mathscr{P}'(z)\right)^2 = 4\mathscr{P}^3(z) - 1. \tag{5}$$

In the same paper, Gross [1] conjectured all meromorphic solutions of $f^3 + g^3 = 1$ are necessarily elliptic functions of entire functions. The conjecture was proved by Baker [2]. He proved.

Theorem 1. *Any function F G, which are meromophic in the complex plane and satisfy*

$$F^3 + G^3 = 1 \tag{6}$$

have the form

$$F(z) = f(h(z)),\ \ G(z) = \eta f(h(z)) = \eta f(-h(z)) = f\left(-\eta^2 h(z)\right), \tag{7}$$

where h is an entire function and η is a cube-root of unity.

We assume that the reader is familiar with the standard notations and results such as the proximity function $m(r, f)$, counting function $N(r, f)$, characteristic function $T(r, f)$, the first and second main theorems, lemma on the logarithmic derivatives etc. of Nevanlinna theory, (see [3, 4]). Given a meromorphic function f, we shall call a meromorphic function $a(z)$ a small function of $f(z)$ if $T(r, a) = S(r, f)$, where $S(r, f)$ is used to denote any quantity that satisfies $S(r, f) = o(T(r, f))$ as $r \to \infty$, possibly outside of a set r of finite logarithmic measure. Here, the order $\rho(f)$ is defined by

$$\rho(f) = \limsup_{r\to\infty} \frac{\log T(r,f)}{\log r}. \tag{8}$$

Now, the estimate the order of $\mathscr{P}(z)$, Bank and Langley [5] indicates that

$$T(r,\mathscr{P}) = \frac{\pi}{A} r^2 (1+o(1)), \quad \text{and} \quad \rho(\mathscr{P}) = 2. \tag{9}$$

Nevanlinna value distribution theory of meromorphic functions has been extensively applied to resolve growth and solvability of meromorphic solutions of linear and nonlinear differential and difference equations (see [4, 6–10]).

Below, we list some well-known facts concerning the order of composite meromorphic functions that can be found in Edrei and Fuchs [11], and Bergweiler [12].

Theorem 2. *Let f be meromorphic and h be entire in $\mathbb{C}$. When $0 < \rho(f), \rho(h) < \infty$, then $\rho(f\circ h) < \infty$ and h is transcendental, then $\rho(f) = 0$.*

In recent years, Nevanlinna characteristic of $f(z+c)$ ($c \in \mathbb{C}\backslash\{0\}$), the value distribution theory for difference polynomials, Nevanlinna theory of the difference operator and the difference analogue of the lemma on the logarithmic derivative had been established, see e.g., [13–22]. Due to these theories, there has been a recent study on whether the derivative f' or f can be replaced by the difference operator $f(z+c)$ or $\Delta_c f(z) = f(z+c) - f(z)$ in the above question.

We may want to study all meromorphic solutions f of the following difference equation

$$f^n(z) + f^m(z+c) = 1 \tag{10}$$

for a fixed nonzero constant c [23, 24]. Shimomura [25] proved that the (10) has an entire solution of infinite order for the $n > m = 1$. Later, Liu et al. [26] proved that Eq. (10) has no transcendental entire solutions of finite order when $n \neq m$. If $n = m = 1$, Liu et al. [27] proved that the solutions of (10) are periodic functions of period $2c$, and proved that (10) has transcendental entire solutions of finite order for $n = m = 2$.

For a meromorphic functions $f(z)$, we define its difference operators by

$$\begin{aligned} \Delta_c f(z) &= f(z+c) - f(z), \\ \Delta_c^n f(z) &= \Delta_c^{n-1}(\Delta_c f(z)), \quad n \in \mathbb{N}, n \geq 2. \end{aligned} \tag{11}$$

Recently, Lü and Han [28] described a property of meromorphic solutions to equation in (6) with $G(z) := F(z+c)$, for $c \in \mathbb{C}\backslash\{0\}$, as the following result.

Theorem 3. *The difference equation $F(z)^3 + F(z+c)^3 = 1$ does not have meromorphic solution of finite order.*

It is natural to ask whether the shift $F(z+c)$ can be replaced by $\Delta_c F(z)$ in above Theorem 3. In this paper, based on the ideas of [28], we mainly consider this problem and prove the following results.

Theorem 4. *The difference equation $F(z)^3 + \Delta_c F(z+c)^3 = 1$ does not have meromorphic solution of finite order.*

Proof. It follows from (2) and (3) that

$$\begin{aligned} F(z) &= \frac{\left\{(1/2) + (\mathscr{P}'(h(z)))/(12)^{1/2}\right\}}{\mathscr{P}(h(z))}, \\ F(z+c) - F(z) &= \eta \frac{\left\{(1/2) - (\mathscr{P}'(h(z)))/(12)^{1/2}\right\}}{\mathscr{P}(h(z))}. \end{aligned} \tag{12}$$

Then we have

$$\begin{aligned} F(z+c) &= \eta \frac{\left\{1/2 - \mathscr{P}'(h(z))/(12)^{1/2}\right\}}{\mathscr{P}(h(z))} + F(z) \\ &= \frac{\left\{(\eta+1)/2 + (1-\eta)\mathscr{P}'(h(z))/(12)^{1/2}\right\}}{\mathscr{P}(h(z))}. \end{aligned} \tag{13}$$

Rewrite it as

$$\begin{aligned} &\frac{\left\{(\eta+1)/2 + (1-\eta)\mathscr{P}'(h(z))/(12)^{1/2}\right\}}{\mathscr{P}(h(z))} \\ &= \frac{\left\{1/2 + \mathscr{P}'(h(z+c))/(12)^{1/2}\right\}}{\mathscr{P}(h(z+c))}. \end{aligned} \tag{14}$$

We know

$$\frac{\mathscr{P}'(h(z))}{\sqrt{3}} = 2F(z)\mathscr{P}'(h) - 1. \tag{15}$$

Assume $\rho(F) < \infty$. Then, combining (13) and (14) with (5), one has

$$4\mathscr{P}^3(h(z)) - 1 = 12F^2(z)\mathscr{P}^2(h(z)) - 12F(z)\mathscr{P}(h(z)) + 3. \tag{16}$$

Finally, we get

$$\mathscr{P}^3(h(z)) = 3F^2(z)\mathscr{P}^2(h(z)) - 3F(z)\mathscr{P}(h(z)) + 1. \tag{17}$$

So

$$3T(r,\mathscr{P}(h(z))) \leq 2T(r,F) + 2T(r,\mathscr{P}(h)) + O(1), \tag{18}$$

and $\rho(\mathscr{P}(h)) < \infty$. Note that f is a of finite order and (9) and Theorem 2 combined force h to be a polynomial.

Notice when $\mathscr{P}(z_0) = 0$, then $(\mathscr{P}')^2(z_0) = -1$. By (5) Now, write all the zeros of $\mathscr{P}(z)$ by $\{z_n\}_{n=1}^{\infty}$ and $|z_n| \to \infty$ as $n \to \infty$. Note that h is a polynomial, then that all the zeros of $h(z) = z_n$ are simple zeros of $h(z) - z_n$ for $n > N$, where N is a positive integer, and $h(a_n) = z_n$ for $n > N$. Now, we divide two steps to prove our result.

Step 1. If there exists a subsequence of $\{a_n\}$ (with $a_n \to a_0$, $n \to \infty$, which we still denote it by $\{a_n\}$) such that $\mathscr{P}(h(a_0+c)) = 0$. Then we have $\mathscr{P}'(h(a_0+c)) = \pm i$ and $\mathscr{P}'(h(a_0)) = \pm i$, so we also have a_0 is a simple zero of $\mathscr{P}(h(z))$.

Differential (14), we have

$$\begin{aligned} &\left\{(1-\eta)\frac{\mathscr{P}''(h(z))}{\sqrt{3}} h'(z)\right\} \mathscr{P}(h(z+c)) \\ &\quad + \left\{(\eta+1) + (1-\eta)\frac{\mathscr{P}'(h(z))}{\sqrt{3}}\right\} \\ &\cdot \mathscr{P}'(h(z+c))h'(z+c) = \left\{\frac{\mathscr{P}''(h(z+c))}{\sqrt{3}} h'(z+c)\right\} \mathscr{P}(h(z)) \\ &\quad + \left\{1 + \frac{\mathscr{P}'(h(z+c))}{\sqrt{3}}\right\} \mathscr{P}'(h(z))h'(z). \end{aligned} \tag{19}$$

Substitute a_n (for enough large n) into the above equation and by $\mathscr{P}(h(a_n+c))=0$, $\mathscr{P}(h(a_n))=0$, we have

$$\begin{aligned} & h'(a_n+c)\mathscr{P}'(h(a_n+c))\left\{\frac{\eta+1}{2}+\frac{1-\eta}{2}\frac{\mathscr{P}'(h(a_n))}{(12)^{1/2}}\right\} \\ &= h'(a_n)\mathscr{P}'(h(a_n))\left\{\frac{1}{2}+\frac{\mathscr{P}'(h(a_n+c))}{(12)^{1/2}}\right\}. \end{aligned} \tag{20}$$

Noting that $\mathscr{P}'(h(a_n))=\pm i$ and $\mathscr{P}'(h(a_n+c))=\pm i$. Without of loss generality, together with (14), we also assume there exists a sub-sequence of $\{a_n\}$ (here we still denote it by $\{a_n\}$) such that four cases:

Case 1. If $\mathscr{P}'(h(a_n))=i$ and $\mathscr{P}'(h(a_n+c))=i$, we get

$$\left\{(\eta+1)+(1-\eta)\frac{\sqrt{3}}{3}i\right\}h'(a_n+c)=h'(a_n)\left\{1+\frac{\sqrt{3}i}{3}\right\}. \tag{21}$$

Case 2. If $\mathscr{P}'(h(a_n))=-i$ and $\mathscr{P}'(h(a_n+c))=i$, we get

$$\left\{(\eta+1)-(1-\eta)\frac{\sqrt{3}}{3}i\right\}h'(a_n+c)=-h'(a_n)\left\{1+\frac{\sqrt{3}i}{3}\right\}. \tag{22}$$

Case 3. If $\mathscr{P}'(h(a_n))=i$ and $\mathscr{P}'(h(a_n+c))=-i$, we get

$$\left\{(\eta+1)+(1-\eta)\frac{\sqrt{3}}{3}i\right\}(-1)h'(a_n+c)=h'(a_n)\left\{1-\frac{\sqrt{3}i}{3}\right\}. \tag{23}$$

Case 4. If $\mathscr{P}'(h(a_n))=-i$ and $\mathscr{P}'(h(a_n+c))=-i$, we get

$$\left\{(\eta+1)-(1-\eta)\frac{\sqrt{3}}{3}i\right\}h'(a_n+c)=h'(a_n)\left\{1-\frac{\sqrt{3}i}{3}\right\}. \tag{24}$$

Noting that $h(z), h(z+c)$ are polynomials and infinite many a_n (with $|a_n|\to\infty$ when $n\to\infty$), we would have to get

$$\begin{aligned} &\left\{(\eta+1)+(1-\eta)\tfrac{\sqrt{3}}{3}i\right\}h'(z+c)=h'(z)\left\{1+\tfrac{\sqrt{3}i}{3}\right\}, \\ &\left\{(\eta+1)-(1-\eta)\tfrac{\sqrt{3}}{3}i\right\}h'(z+c)=-h'(z)\left\{1+\tfrac{\sqrt{3}i}{3}\right\}(-1), \\ &\left\{(\eta+1)+(1-\eta)\tfrac{\sqrt{3}}{3}i\right\}(-1)h'(z)=h'(z)\left\{1-\tfrac{\sqrt{3}i}{3}\right\}, \\ &\left\{(\eta+1)-(1-\eta)\tfrac{\sqrt{3}}{3}i\right\}h'(z+c)=h'(z)\left\{1-\tfrac{\sqrt{3}i}{3}\right\}. \end{aligned} \tag{25}$$

Either $\eta=1$ or $\eta=-1/2\pm(\sqrt{3}/2)i$, and by comparing the highest coefficient of the both polynomials of the above equations, These are simply impossible.

So there exists a positive integer N satisfying $\mathscr{P}(h(a_n+c))\neq 0$ for $n>N$.

Step 2. In view of $\mathscr{P}(h(a_k))=0$ and $(\mathscr{P}')^2(h(a_k))=-1$, we have that $\mathscr{P}(h(a_n+c))=\infty$ for $n>N$.

Thus, we get the proposition that the zeros of $\mathscr{P}(h(z))$ are the poles of $\mathscr{P}(h(z+c))$ except for finitely many points. This is to say

$$\begin{aligned} N\left(r,\frac{1}{\mathscr{P}(h(z))}\right) &= \overline{N}\left(r,\frac{1}{\mathscr{P}(h(z))}\right)+M\log r \\ &\le \overline{N}(r,\mathscr{P}(h(z+c)))+M\log r, \end{aligned} \tag{26}$$

where M is a positive number. Noting that F is transcendental function, so $M\log r=S(r,F)$ and

$$N\left(r,\frac{1}{\mathscr{P}(h(z))}\right)\le\overline{N}(r,\mathscr{P}(h(z+c)))+S(r,F). \tag{27}$$

By $F(z)=\left\{1/2+\mathscr{P}'(h(z))/(12)^{1/2}\right\}/\mathscr{P}(h(z))$ and $\rho(\mathscr{P}(h))<\infty$, we have

$$T(r,F)\le T(r,\mathscr{P}(h(z)))+T(r,\mathscr{P}'(h))+O(1)\le O(T(r,\mathscr{P}(h))), \tag{28}$$

which implies that $\rho(F)=\rho(\mathscr{P}(h))$ and $S(r,F)=S(r,\mathscr{P}(h(z)))$. By the equation $-G^3=F^3-1=(F-1)(F-\eta)(F-\eta^2)$, we deduce that all zeros of $F-1$, $F-\eta$ and $F-\eta^2$ are with multiplicities at least 3.

Then, we have

$$\begin{aligned} 2T(r,F) &\le \overline{N}\left(r,\frac{1}{F-1}\right)+\overline{N}\left(r,\frac{1}{F-\eta}\right) \\ &\quad+\overline{N}\left(r,\frac{1}{F-\eta^2}\right)+\overline{N}(r,F)+S(r,F) \\ &\le \frac{1}{3}N\left(r,\frac{1}{F-1}\right)+\frac{1}{3}N\left(r,\frac{1}{F-\eta}\right)+\frac{1}{3}N\left(r,\frac{1}{F-\eta^2}\right) \\ &\quad+N(r,F)+S(r,F)\le T(r,F)+N(r,F)+S(r,F), \end{aligned} \tag{29}$$

which implies that $T(r,F)=N(r,F)+S(r,F)$. It leads to $m(r,F)=S(r,F)=S(r,\mathscr{P}(h(z)))$.

Note the form of F, we have $(1/2)/\mathscr{P}(h(z))=F(z)-(\mathscr{P}'(h(z))/(12)^{1/2}\mathscr{P}(h(z)))$. Furthermore, we have

$$\begin{aligned} m\left(r,\frac{1}{\mathscr{P}(h(z))}\right) &= m\left(r,\frac{(1/2)}{\mathscr{P}(h(z))}\right)+O(1) \\ &\le m(r,F)+m\left(r,\frac{h'(z)\mathscr{P}'(h(z))}{\mathscr{P}(h(z))}\right) \\ &\quad+m\left(r,\frac{1}{h'(z)}\right)+O(1)=S(r,\mathscr{P}(h(z))). \end{aligned} \tag{30}$$

Then, all the above discussions yield that

$$\begin{aligned} T(r,\mathscr{P}(h(z))) &= T\left(r,\frac{1}{\mathscr{P}(h(z))}\right)+O(1)=m\left(r,\frac{1}{\mathscr{P}(h(z))}\right) \\ &\quad+N\left(r,\frac{1}{\mathscr{P}(h(z))}\right)+O(1)\le\overline{N}(r,\mathscr{P}(h(z+c))) \\ &\quad+S(r,\mathscr{P}(h(z)))\le\frac{1}{2}N(r,\mathscr{P}(h(z+c))) \\ &\quad+S(r,\mathscr{P}(h(z)))\le\frac{1}{2}T(r,\mathscr{P}(h(z+c)))+S(r,\mathscr{P}(h(z))) \\ &\le\frac{1}{2}T(r,\mathscr{P}(h(z)))+S(r,\mathscr{P}(h(z)))+O\left(r^{\rho(\mathscr{P}(h))-1+\epsilon}\right), \end{aligned} \tag{31}$$

a contradiction.

Thus, we finish the proof of Theorem 4. □

Remark 1. It is easy to get the solution of the equation $F(z)+\Delta_c F(z)=1$ is only for constant. For $n=2$, the solution of the equation $F^2(z) + \Delta_c^2 F(z) = 1$, Liu (see Proposition 5.3 in [19]) proved there is no nonconstant finite order entire solution.

2. An Application

For a meromorphic function f and a set $S \in \mathbb{C}$, we define

$$E_f(S) = \bigcup_{a\in S}\{z \mid f(z) - a = 0, \text{ counting multiplicities}\}. \quad (32)$$

We say that f and g share a set S CM, provided that $E_f(S) = E_g(S)$.

In 1976, Gross [29] posed the following question:

Question 1 (see [29, Question 6]). Can one find two (or possibly even one) finite sets S_j $(j = 1, 2)$ such that any two entire functions f and g satisfying $E_f(S_j) = E_g(S_j)$ $(j = 1, 2)$ must be identical?

If the answer to Question 1 is affirmative, it would be interesting to know how large both sets would have to be. Many authors have been considering about it, and got a lot of related results. Some of them are due to Yi [30], Mues and Reinders [31], and Frank and Reinders [32].

Recently, value distribution in difference analogues of meromorphic functions has become a subject of great interest [33, 34]. Zhang [35] obtained the following results.

Theorem B. *Let $S_1 = \{\infty\}$ and $S_2 = \{1, \omega, \ldots, \omega^{n-1}\}$, where $w = e^{2\pi/ni}$ and n is a positive integer, let $c \in \mathbb{C}$. Suppose that f is a nonconstant meromorphic function of finite order such that $E_{f(z)}(S_i) = E_{f(z+c)}(S_i)$ $(i = 1, 2)$. If $n \geq 4$, then $f(z) = tf(z + c)$, where $t^n = 1$.*

Recently Lü and Han [28] proved that

Theorem C. *Under the conditions of Theorem B and if $n \geq 3$, then the conclusion of Theorem B still holds.*

In this paper, we mainly consider the $f(z)$ and $\Delta_c f(z)$ share the set which has three elements and get the following result.

Theorem 5. *Let $S_1 = \{1, \eta, \eta^2\}$, for $\eta = -1/2 \pm (\sqrt{3}/2)i$, and $S_2 = \{\infty\}$. Take $c \neq 0$. Let f be a meromorphic function of finite order satisfying $E_{f(z)}(S_l) = E_{\Delta_c f(z)}(S_l)$ for $l = 1, 2$. Then, either $f(z) = \sqrt[3]{2} f(z + c)$ or $f(z) = \sqrt[3]{2}\eta f(z + c)$.*

Proof. By Theorem C, we set $F = f^3$. Obviously, $F(z)$ and $\Delta_c F(z)$ share $1, \infty$ CM. Then we can assume that

$$\frac{\Delta_c F(z) - 1}{F(z) - 1} = e^{Q(z)}, \quad (33)$$

where Q is an entire function. Note that F is of finite order, we have Q is a polynomial. Obviously, $T(r, e^{Q(z)}) = m(r, e^{Q(z)}) = S(r, F)$.

Rewrite (33) as

$$F(z + c) = \left(e^{Q(z)} + 1\right)\left\{F(z) - \frac{e^{Q(z)} - 1}{e^{Q(z)} + 1}\right\}, \quad (34)$$

$$F(z) = \frac{1}{e^{Q(z)} + 1}\left\{F(z + c) - \left(1 - e^{Q(z)}\right)\right\}. \quad (35)$$

Similarly as above, we get $F(z) - (e^{Q(z)} - 1)/(e^{Q(z)} + 1)$ just has zeros with multiplicities at least 3. Furthermore, we have $F(z) - \left(1 - e^{Q(z-c)}\right)$ just has zeros with multiplicities at least 3. Suppose that $1 - e^{Q(z-c)}, 0, (e^{Q(z)} - 1)/(e^{Q(z)} + 1)$ are distinct from each other. Then, by the second main theorem, we have

$$\begin{aligned}
2T(r, F) &\leq \overline{N}(r, F) + \overline{N}\left(r, \frac{1}{F}\right) + \overline{N}\left(r, \frac{1}{F - \left(1 - e^{Q(z-c)}\right)}\right) \\
&\quad + \overline{N}\left(r, \frac{1}{F - \left((e^{Q(z)} - 1)/(e^{Q(z)} + 1)\right)}\right) \\
&\quad + S(r, F) \leq \frac{1}{3}N(r, F) + \frac{1}{3}\overline{N}\left(r, \frac{1}{F}\right) \\
&\quad + \frac{1}{3}N\left(r, \frac{1}{F - \left(1 - e^{Q(z-c))}\right)}\right) \\
&\quad + \frac{1}{3}N\left(r, \frac{1}{F - \left((e^{Q(z)} - 1)/(e^{Q(z)} + 1)\right)}\right) \\
&\quad + S(r, F) \leq \frac{4}{3}T(r, F) + S(r, F),
\end{aligned} \quad (36)$$

a contradiction. Thus, there are at lest two functions equal in the set

$$\left\{1 - e^{Q(z-c)}, 0, \left(\frac{e^{Q(z)} - 1}{e^{Q(z)} + 1}\right)\right\}. \quad (37)$$

We consider two cases.

Case 1. $1 - e^{Q(z-c)} \equiv 0$ or $\left(e^{Q(z)} - 1\right)/\left(e^{Q(z)} + 1\right) \equiv 0$ and then $e^{Q(z)} - 1 \equiv 0$, then Q must be a constant with $e^{Q(z)} \equiv 1$, so $F(z + c) - F(z) \equiv F(z)$, and then $F(z + c) = 2F(z)$, at last $f^3(z + c) = 2f^3(z)$.

Then, we have $e^Q = 1$. It follows from (33) that the assertion holds.

Case 2. $1 - e^{Q(z-c)} = 1 - e^{-Q(z)}$. That is $e^{Q(z-c)+Q(z)} = 1$.

Noting that Q is a polynomial, we deduce that $Q(z)$ is a constant by the assumption of Case 2. Furthermore, $e^{2Q} = 1$ and $e^Q = -1$. By (33), we get $f^3(z) + f^3(z + c) = 2$. Then, it follows from Theorem 1 that the case cannot occur.

Thus, we finish the proof of Theorem 5. □

Example 1. Consider $f(z) = \sqrt{2}\sin z$. Then $f(z + c) = \sqrt{2}\cos z$ for $c = \pi/2$, and $\Delta_c f(z) = f(z + c) - f(z) = 2\sin(\pi/4 - z)$. Notice that $f(z)$, $\Delta_c f(z)$ share $\{\sqrt{2}, -\sqrt{2}\}$, $\{\infty\}$ CM, but $f(z) \neq \pm f(z + c)$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors typed, read, and approved the final manuscript.

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