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Research Article

Bounds of a Unified Integral Operator via Exponentially (s, m)-Convexity and Their Consequences

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Various known fractional and conformable integral operators can be obtained from a unified integral operator. The aim of this paper is to find bounds of this unified integral operator via exponentially (s, m)-convex functions. The resulting bounds provide compact formulas for the bounds of associated fractional and conformable integral operators. Several Hadamard-type inequalities have been produced from a compact version for unified integral operators for exponentially (s, m)-convex functions.

1. Introduction

Fractional integral inequalities are useful in the field of fractional calculus. Many mathematicians have introduced fractional differential, fractional integral, and fractional conformable integral operators in this field (see [1–14]). Recently, several mathematical inequalities have been introduced via (s, m)-convexity (see [15, 16]). The goal of this paper is to obtain the bounds of all integral operators explained in Remarks 5 and 6 in a unified form for exponentially (s, m)-convex functions.

Definition 1 (see [6]). Let $f : [a, b] \longrightarrow \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ where $\Re(\mu) > 0$ are defined by

$$ {}_g^{\mu}I_{a^+}f(x) = \frac{1}{\Gamma(\mu)}\int_a^x (g(x)-g(t))^{\mu-1} g'(t) f(t)dt, \quad x > a, \tag{1}$$

$$ {}_g^{\mu}I_{b^-}f(x) = \frac{1}{\Gamma(\mu)}\int_x^b (g(t)-g(x))^{\mu-1} g'(t) f(t)dt, \quad x < b, \tag{2}$$

where $\Gamma(.)$ is the gamma function.

Definition 2 (see [17]). Let $f : [a, b] \longrightarrow \mathbb{R}$ be an integrable function. Also let g be an increasing and positive function on $(a, b]$, having a continuous derivative g' on (a, b). The left-sided and right-sided fractional integrals of a function f with respect to another function g on $[a, b]$ of order μ *where* $\Re(\mu) > 0$, $k > 0$, are defined by

$$ {}_g^{\mu}I_{a^+}^k f(x) = \frac{1}{k\Gamma_k(\mu)}\int_a^x (g(x)-g(t))^{(\mu/k)-1} g'(t) f(t)dt, \quad x > a, \tag{3}$$

$$ {}_g^{\mu}I_{b^-}^k f(x) = \frac{1}{k\Gamma_k(\mu)}\int_x^b (g(t)-g(x))^{(\mu/k)-1} g'(t) f(t)dt, \quad x < b, \tag{4}$$

where $\Gamma_k(.)$ is the gamma function given as follows [18]:

$$\Gamma_k(x) = \int_0^\infty t^{x-1} e^{-t^k/k} dt, \quad \Re(x) > 0. \tag{5}$$

A fractional integral operator containing an extended generalized Mittag-Leffler function in its kernel is defined as follows:

Definition 3 (see [19]). Let $\omega, \mu, \alpha, l, \gamma, c \in \mathbb{C}$, $\Re(\mu), \Re(\alpha), \Re(l) > 0$, $\Re(c) > \Re(\gamma) > 0$ *with* $p \geq 0$, $\delta > 0$, *and* $0 < k \leq \delta + \Re(\mu)$. *Let* $f \in L_1[a, b]$ *and* $x \in [a, b]$. Then the generalized fractional integral operators $\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f$ *and* $\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f$ are defined by

$$\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,a^+} f\right)(x\,; p) = \int_a^x (x-t)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(x-t)^\mu\,; p) f(t) dt, \tag{6}$$

$$\left(\varepsilon^{\gamma,\delta,k,c}_{\mu,\alpha,l,\omega,b^-} f\right)(x\,; p) = \int_x^b (t-x)^{\alpha-1} E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(t-x)^\mu\,; p) f(t) dt, \tag{7}$$

where

$$E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(t\,; p) = \sum_{n=0}^\infty \frac{\beta_p(\gamma + nk, c-\gamma)}{\beta(\gamma, c-\gamma)} \frac{(c)_{nk}}{\Gamma(\mu n + \alpha)} \frac{t^n}{(l)_{n\delta}} \tag{8}$$

is the extended generalized Mittag-Leffler function.

Recently, a unified integral operator is defined as follows:

Definition 4 (see [20]). Let $f, g : [a, b] \longrightarrow \mathbb{R}$, $0 < a < b$, *be* the functions such that f be positive and $f \in L_1[a, b]$, *and* g be differentiable and strictly increasing. Also let ϕ/x be an increasing function on $[a, \infty)$ *and* $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu, \delta \geq 0$, *and* $0 < k \leq \delta + \mu$. *Then, for* $x \in [a, b]$, the left and right integral operators are defined by

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+} f\right)(x, \omega\,; p) = \int_a^x K_x^y\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g\,; \phi\right) f(y) d(g(y)), \tag{9}$$

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-} f\right)(x, \omega\,; p) = \int_x^b K_y^x\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g\,; \phi\right) f(y) d(g(y)), \tag{10}$$

where $K_x^y(E^{\gamma,\delta,k,c}_{\mu,\alpha,l}, g\,; \phi) = (\phi(g(x) - g(y))/(g(x) - g(y))) E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x) - g(y))^\mu\,; p)$.

For suitable settings of functions ϕ, g, and certain values of parameters included in Mittag-Leffler function (8), some interesting consequences can be obtained which are comprised in the upcoming remarks.

Remark 5.

(i) Let $\phi(x) = x^{\beta/k}\Gamma(\beta)/k\Gamma_k(\beta)$, $k > 0$, $\beta > k$, and $p = \omega = 0$, in unified integral operators (9) and (10). Then, generalized Riemann-Liouville fractional integral operators (3) and (4) are obtained.

(ii) For $k = 1$, (3) and (4) fractional integrals coincide with (1) and (2) fractional integrals, which further produce the following fractional and conformable integrals:

1. By taking g as identity function, (3) and (4) fractional integrals coincide with k-fractional Riemann-Liouville integrals defined by Mubeen et al. in [7]
2. For $k = 1$, along with g as the identity function, (3) and (4) fractional integrals coincide with Riemann-Liouville fractional integrals [6]
3. For $k = 1$ and $g(x) = x^\rho/\rho$, $\rho > 0$, (3) and (4) produce fractional integrals defined by Chen and Katugampola in [1]
4. For $k = 1$ and $g(x) = x^{\tau+s}/(\tau + s)$, (3) and (4) produce generalized conformable fractional integrals defined by Khan and Khan in [5]
5. If we take $g(x) = (x-a)^s/s$, $s > 0$, in (3) and $g(x) = -(b-x)^s/s$, $s > 0$, in (4), then conformable (k, s)-fractional integrals are achieved as defined by Habib et al. in [3]
6. If we take $g(x) = x^{1+s}/(1+s)$, then conformable fractional integrals are achieved as defined by Sarikaya et al. in [11]
7. If we take $g(x) = (x-a)^s/s$, $s > 0$, in (3) and $g(x) = -(b-x)^s/s$, $s > 0$, in (4) with $k = 1$, then conformable fractional integrals are achieved as defined by Jarad et al. in [4]
8. If we take $\omega = p = 0$ and $\phi(t) = \Gamma(\mu) F^{\sigma,k}_{\rho,\lambda}(\omega(t)^\rho)$ in (9) and (10), then generalized k-fractional integral operators are achieved as defined by Tunc et al. in [13]
9. If we take $k = 1$ and $\phi(t) = \Gamma(\mu)(\exp(-At)/\mu)$, $A = (1-\mu)/\mu$, $\mu > 0$, in (3) and (4), then generalized fractional integral operators with an exponential kernel are obtained [2]

Remark 6. Let $\phi(x) = x^\beta$ *and* $g(x) = x$, $\beta > 0$, in unified integral operators (9) and (10). Then, fractional integral operators (6) and (7) are obtained, which along with different settings of k, δ, l, c, γ in generalized Mittag-Leffler function give the following integral operators:

1. By setting $p = 0$, fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Salim and Faraj in [10]

(2) By setting $l=\delta=1$, fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Rahman et al. in [9]

(3) By setting $p=0$ and $l=\delta=1$, fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Srivastava and Tomovski in [12]

(4) By setting $p=0$ and $l=\delta=k=1$, fractional integral operators (6) and (7) reduce to the fractional integral operators defined by Prabhakar in [8]

(5) By setting $p=\omega=0$, fractional integral operators (6) and (7) reduce to the left-sided and right-sided Riemann-Liouville fractional integrals

(6) By setting $p=\omega=0$ in fractional integral operators (9) and (10) we get $(1/\Gamma(\mu))(F_{a^+}^{\phi,g}f)(x)=({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f)(x,0;0)$ and $(1/\Gamma(\mu))(F_{b^-}^{\phi,g}f)(x)=({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f)(x,0;0)$ where $(F_{a^+}^{\phi,g}f)(x)$ and $(F_{b^-}^{\phi,g}f)(x)$ are defined in [21]

Convex functions play a very vital role in the theory of inequalities. Motivated by its analytical interpretation, many other extended and generalized notions have been defined in literature. These notions are used to extend and generalize a lot of classical results (see [15, 16, 22, 23] and references therein). A generalized notion called exponentially (s,m)-convexity is defined in [24].

Definition 7. Let $s\in[0,1]$ and $I\subseteq[0,\infty)$ be an interval. A function $f:I\longrightarrow\mathbb{R}$ is said to be exponentially (s,m)-convex in the second sense, if

$$f(tx+m(1-t)y)\le t^s\frac{f(x)}{e^{\alpha x}}+m(1-t)^s\frac{f(y)}{e^{\alpha y}}, \tag{11}$$

holds for all $x,y\in I$, $m\in[0,1]$ and $\alpha\in\mathbb{R}$.

One can note the deducible definitions in the following remark:

Remark 8.

(i) For $m=1$, (11) produces the definition of exponentially s-convex function

(ii) For $\alpha=0$, (11) produces the definition of (s,m)-convex function

(iii) For $\alpha=0$ and $m=1$, (11) produces the definition of s-convex function in the second sense

(iv) For $\alpha=0$ and $m=1$, (11) produces the definition of the convex function

(v) For $\alpha=0$ and $s=1$, (11) produces the definition of the m-convex function

In the upcoming section, bounds of unified integral operators for exponentially (s,m)-convex functions are given in different forms. Bounds of associated fractional and conformable integral operators which are known in literature are also deduced. The Hadamard inequality is derived for exponentially (s,m)-convex functions. Its diverse conformable and fractional versions are presented. A modulus inequality is established by using exponentially (s,m)-convexity of $|f'|$. In Section 3, boundedness and continuity of these operators are given.

2. Main Results

Bounds of unified integral operators (9) and (10), by using exponentially (s,m)-convexity, are established in the following theorem:

Theorem 9. *Let $f:[a,b]\longrightarrow\mathbb{R}$, $0\le a<b$, be a positive exponentially (s,m)-convex function with $m\in(0,1]$ and $g:[a,b]\longrightarrow\mathbb{R}$ be a differentiable and strictly increasing function. Also let ϕ/x be an increasing function on $[a,b]$. If $\alpha,l,\gamma,c\in\mathbb{C}$, $p,\mu\ge0$, $\delta\ge0$, and $0<k\le\delta+\mu$, then for $x\in(a,b)$, we have*

$$\begin{aligned}&\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)+\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\\&\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\left(m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\right.\\&\left.\cdot\left(m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{f(a)^s}{e^{\alpha a}}I_{a^+}g(a)\right)\right)+K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\&\cdot\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\right.\\&\left.\cdot\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(x)-m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right).\end{aligned} \tag{12}$$

Proof. Under the suppositions of ϕ and g, we can obtain

$$\frac{\phi(g(x)-g(t))}{g(x)-g(t)}\le\frac{\phi(g(x)-g(a))}{g(x)-g(a)};\quad t\in[a,x],\ x\in(a,b). \tag{13}$$

Multiplying with $E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(t))^\mu;p)g'(t)$, we can obtain

$$\begin{aligned}K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)g'(t)&\le\frac{\phi(g(x)-g(a))}{g(x)-g(a)}\\&\cdot E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(t))^\mu;p)g'(t).\end{aligned} \tag{14}$$

By using $E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(t))^\mu;p)\le E_{\mu,\alpha,l}^{\gamma,\delta,k,c}(\omega(g(x)-g(a))^\mu;p)$, the following inequality is obtained:

$$K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)g'(t)\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)g'(t). \tag{15}$$

Using exponentially (s,m)-convexity of f, we have

$$f(t)\le\left(\frac{x-t}{x-a}\right)^s\frac{f(a)}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^s\frac{f(x/m)}{e^{\alpha(x/m)}}. \tag{16}$$

Multiplying (15) and (16) and integrating over $[a, x]$, we can obtain

$$\int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) f(t) d(g(t)) \leq \frac{f(a)}{e^{\alpha a}} K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) \cdot \int_a^x\left(\frac{x-t}{x-a}\right)^s d(g(t))+m \frac{f(x/m)}{e^{\alpha(x/m)}} K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) \cdot \int_a^x\left(\frac{t-a}{x-a}\right)^s d(g(t)). \tag{17}$$

By using Definition 4 and integrating by parts, the following inequality is obtained:

$$\left({}_g F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c} f\right)(x, \omega ; p) \leq K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) \cdot\left(m \frac{f(x/m)}{e^{\alpha(x/m)}} g(x)-\frac{f(a)}{e^{\alpha a}} g(a)-\frac{\Gamma(s+1)}{(x-a)^s} \cdot\left(m \frac{f(x/m)^s}{e^{\alpha(x/m)}} I_{x^-} g(a)-\frac{f(a)^s}{e^{\alpha a}} I_{a^+} g(a)\right)\right). \tag{18}$$

Now on the other hand for $t \in (x, b]$ and $x \in (a, b)$, the following inequality holds true:

$$K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) g'(t) \leq K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) g'(t). \tag{19}$$

Using exponentially (s, m)-convexity of f, we have

$$f(t) \leq\left(\frac{t-x}{b-x}\right)^s \frac{f(b)}{e^{\alpha b}}+m\left(\frac{b-t}{b-x}\right)^s \frac{f(x/m)}{e^{\alpha(x/m)}}. \tag{20}$$

Adopting the same pattern as we did for (15) and (16), we obtained the following inequality from (19) and (20):

$$\left({}_g F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c} f\right)(x, \omega ; p) \leq K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g ; \phi\right) \cdot\left(\frac{f(b)}{e^{\alpha b}} g(b)-m \frac{f(x/m)}{e^{\alpha(x/m)}} g(x)-\frac{\Gamma(s+1)}{(b-x)^s} \cdot\left(\frac{f(b)^s}{e^{\alpha b}} I_{b^-} g(x)-m \frac{f(x/m)^s}{e^{\alpha(x/m)}} I_{x^+} g(b)\right)\right). \tag{21}$$

By adding (18) and (21), (12) can be obtained.

Remark 10.

(1) If we put $p=\omega=0$ in (12), then result for exponentially (s, m)-convex functions can be obtained for the integral operators defined in [23].

(2) If we put $p=\omega=\alpha=0$ in (12), then Theorem 1 in [23] can be obtained

(3) If we put $(s, m)=(1,1)$ and $\alpha=0$ in (12), Theorem 8 in [20] can be obtained

(4) If we put $(s, m)=(1,1)$ and $p=\omega=0$ in (12), then the result for exponentially convex functions can be obtained for the integral operators defined in [23].

(5) If we put $(s, m)=(1,1)$, $p=\omega=0$, and $\alpha=0$ in (12), then the result for convex functions can be obtained for the integral operators defined in [23].

(6) If we put $\phi(t)=t^\mu$, $\mu>0$, $\alpha=0$, $p=\omega=0$, and $(s, m)=(1,1)$ in (12), then Proposition 10 in [20] can be obtained

(7) If we put $\phi(t)=t^\mu$, $\mu>0$, $g(x)=x$, $\alpha=0$, $p=\omega=0$, and $(s, m)=(1,1)$ in (12), then Corollary 1 in [26] can be obtained

Proposition 11. *Let $\phi(t)=t^\mu$, and $p=\omega=0$. Then, (12) gives the following bound for fractional integral operators defined in [6] for $\mu \geq 1$:*

$$\left({}_g^\mu I_{a^+} f\right)(x)+\left({}_g^\mu I_{b^-} f\right)(x) \leq \frac{(g(x)-g(a))^{\mu-1}}{\Gamma(\mu)} \cdot\left(m \frac{f(x/m)}{e^{\alpha(x/m)}} g(x)-\frac{f(a)}{e^{\alpha a}} g(a)-\frac{\Gamma(s+1)}{(x-a)^s} \cdot\left(m \frac{f(x/m)^s}{e^{\alpha(x/m)}} I_{x^-} g(a)-\frac{f(a)^s}{e^{\alpha a}} I_{a^+} g(x)\right)\right) +\frac{(g(b)-g(x))^{\mu-1}}{\Gamma(\mu)}\left(\frac{f(b)}{e^{\alpha b}} g(b)-m \frac{f(x/m)}{e^{\alpha(x/m)}} g(x) -\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)^s}{e^{\alpha b}} I_{b^-} g(x)-m \frac{f(x/m)^s}{e^{\alpha(x/m)}} I_{x^+} g(b)\right)\right). \tag{22}$$

Proposition 12. *Let $g(x)=I(x)=x$ and $p=\omega=0$. Then, (12) gives the following bound for integral operators defined in [27]:*

$$\left({}_{a^+} I_\phi f\right)(x)+\left({}_{b^-} I_\phi f\right)(x) \leq \frac{\phi(x-a)}{(x-a)} \cdot\left(m \frac{f(x/m)}{e^{\alpha(x/m)}} g(x)-\frac{f(a)}{e^{\alpha a}} g(a)-\frac{\Gamma(s+1)}{(x-a)^s} \cdot\left(m \frac{f(x/m)^s}{e^{\alpha(x/m)}} I_{x^-} g(a)-\frac{f(a)^s}{e^{\alpha a}} I_{a^+} g(x)\right)\right) +\frac{\phi(b-x))}{(b-x)}\left(\frac{f(b)}{e^{\alpha b}} g(b)-m \frac{f(x/m)}{e^{\alpha(x/m)}} \cdot g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)^s}{e^{\alpha b}} I_{b^-} g(x)-m \frac{f(x/m)^s}{e^{\alpha(x/m)}} I_{x^+} g(b)\right)\right). \tag{23}$$

Corollary 13. *If we take $\phi(t)=\Gamma(\mu) t^{\mu/k}/k\Gamma_k(\mu)$ and $p=\omega=0$, then (12) gives the following bound for fractional integral*

operators defined in [17] for $\mu \geq k$:

$$\left({}_{g}^{\mu}I_{a^+}^{k}f\right)(x)+\left({}_{g}^{\mu}I_{b^-}^{k}f\right)(x)\leq \frac{(g(x)-g(a))^{(\mu/k)-1}}{k\Gamma_k(\mu)}\cdot\left(m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{f(a)^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right)+\frac{(g(b)-g(x))^{(\mu/k)-1}}{k\Gamma_k(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f(x/m)}{e^{\alpha(x/m)}}\cdot g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(x)-m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right). \tag{24}$$

Corollary 14. *If we take $\phi(t)=t^\mu$, $p=\omega=0$, and $g(x)=I(x)=x$, then (12) gives the following bound for Riemann-Liouvelle fractional integrals defined in [6] for $\mu \geq 1$:*

$$({}^{\mu}I_{a^+}f)(x)+({}^{\mu}I_{b^-}f)(x)\leq \frac{(x-a)^{\mu-1}}{\Gamma(\mu)}\cdot\left(m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{f(a)^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right)+\frac{(b-x)^{\mu-1}}{\Gamma(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f(x/m)}{e^{\alpha(x/m)}}\cdot g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(x)-m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right). \tag{25}$$

Corollary 15. *If we take $\phi(t)=\Gamma(\mu)t^{\mu/k}/k\Gamma_k(\mu)$, $p=\omega=0$, and $g(x)=I(x)=x$, then (12) gives the following bound for fractional integral operators defined in [7] for $\mu \geq k$:*

$$\left({}^{\mu}I_{a^+}^{k}f\right)(x)+\left({}^{\mu}I_{b^-}^{k}f\right)(x)\leq \frac{(x-a))^{(\mu/k)-1}}{k\Gamma_k(\mu)}\cdot\left(m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{f(a)^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right)+\frac{(b-x))^{(s/\mu)-1}}{k\Gamma_k(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{\Gamma(s+1)}{(b-x)^s}\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(x)-m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right). \tag{26}$$

Corollary 16. *If we take $\phi(t)=t^\mu$, $p=\omega=0$, and $g(x)=x^\rho/\rho$, $\rho>0$, then (12) gives the following bound for fractional integral operators defined in [1]:*

$$\left({}^{\rho}I_{a^+}^{\mu}f\right)(x)+\left({}^{\rho}I_{b^-}^{\mu}f\right)(x)\leq \frac{(x^\rho-a^\rho)^{\mu-1}}{\Gamma(\mu)\rho^{\mu-1}}\cdot\left(m\frac{f(x/m)}{e^{\alpha(x/m)}}g(x)-\frac{f(a)}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{f(a)^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right)+\frac{(b^\rho-x^\rho)^{\mu-1}}{\Gamma(\mu)\rho^{\mu-1}}\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f(x/m)}{e^{\alpha(x/m)}}\cdot g(x)-\frac{\Gamma(s+1)}{(x-a)^s}\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(x)-m\frac{f(x/m)^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right). \tag{27}$$

The following lemma is essential to prove the next result.

Lemma 17. *Let $f:[0,\infty]\longrightarrow\mathbb{R}$ be an exponentially (s,m)-convex function with $m\in(0,1]$. If $0\leq a<b$ and $f(x)/e^{\alpha x}=f((a+b-x)/m)/e^{\alpha((a+b-x)/m)}$, then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2^s}(1+m)\frac{f(x)}{e^{\alpha x}},\quad x\in[a,b]. \tag{28}$$

Proof. Since f is an exponentially (s,m)-convex, therefore, the following inequality is valid:

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{2^s}\frac{f(x)}{e^{\alpha x}}+\frac{m}{2^s}\frac{f((a+b-x)/m)}{e^{\alpha((a+b-x)/m)}}. \tag{29}$$

By using $f(x)/e^{\alpha x}=f((a+b-x)/m)/e^{\alpha((a+b-x)/m)}$ in the above inequality, we get (28).

The following result provides the Hadamard inequality for unified integral operators defined in (9) and (10).

Theorem 18. *Under the assumptions of Theorem 9, in addition if $f(x)/e^{\alpha x}=f((a+b-x)/m)/e^{\alpha((a+b-x)/m)}$, then we have*

$$\frac{h(\alpha)2^sf((a+b)/2)}{(1+m)}\left(\left({}_{g}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}1\right)(a,\omega;p)+\left({}_{g}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}1\right)(b,\omega;p)\right)\leq\left({}_{g}F_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(a,\omega;p)+\left({}_{g}F_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(b,\omega;p)\leq 2K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\cdot\left(\frac{f(b)}{e^{\alpha b}}g(b)-m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a)-\frac{\Gamma(s+1)}{(b-a)^s}\cdot\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(a)-m\frac{f(a/m)^s}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \tag{30}$$

Proof. Under the assumptions of ϕ and g, we can obtain:

$$\frac{\phi(g(x)-g(a))}{g(x)-g(a)} \leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)}. \tag{31}$$

Multiplying with $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)g'(x)$, we can obtain:

$$K^{a}_{x}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x) \leq \frac{\phi(g(b)-g(a))}{g(b)-g(a)} \cdot E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p)g'(x). \tag{32}$$

By using $E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(x)-g(a))^{\mu};p) \leq E^{\gamma,\delta,k,c}_{\mu,\alpha,l}(\omega(g(b)-g(a))^{\mu};p)$, the following inequality is obtained:

$$K^{a}_{x}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x) \leq K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x). \tag{33}$$

Using exponentially (s,m)-convexity of f, we have

$$f(x) \leq \left(\frac{x-a}{b-a}\right)^{s}\frac{f(b)}{e^{\alpha b}} + m\left(\frac{b-x}{b-a}\right)^{s}\frac{f(a/m)}{e^{\alpha(a/m)}}. \tag{34}$$

Multiplying (33) and (34) and integrating the resulting inequality over $[a,b]$, we can obtain

$$\int_a^b K^{a}_{x}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)d(g(x)) \leq m\frac{f(a/m)}{e^{\alpha(a/m)}}K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right) \cdot \int_a^b\left(\frac{b-x}{b-a}\right)^{s}d(g(x)) + \frac{f(b)}{e^{\alpha b}}K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right) \cdot \int_a^b\left(\frac{x-a}{b-a}\right)^{s}d(g(x)). \tag{35}$$

By using Definition 4 and integrating by parts, the following inequality is obtained:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p) \leq K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right) \cdot \left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a) - \frac{\Gamma(s+1)}{(b-a)^{s}} \cdot \left(\frac{f(b)^{s}}{e^{\alpha b}}I_{b^-}g(a) - m\frac{f(a/m)^{s}}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \tag{36}$$

On the other hand, for $x \in (a,b)$, the following inequality holds true:

$$K^{x}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x) \leq K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)g'(x). \tag{37}$$

Adopting the same pattern of simplification as we did for (33) and (34), the following inequality can be observed for (34) and (37):

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p) \leq K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right) \cdot \left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a) - \frac{\Gamma(s+1)}{(b-a)^{s}} \cdot \left(\frac{f(b)^{s}}{e^{\alpha b}}I_{b^-}g(a) - m\frac{f(a/m)^{s}}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \tag{38}$$

By adding (36) and (38), following inequality can be obtained:

$$\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p) \leq K^{a}_{b}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}} \cdot g(a) - \frac{\Gamma(\alpha+1)}{(b-a)^{s}}\left(\frac{f(b)^{s}}{e^{\alpha b}}I_{b^-}g(b) - m\frac{f(a/m)^{s}}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \tag{39}$$

Multiplying both sides of (28) by $K^{a}_{x}(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x)$, and integrating over $[a,b]$ we have

$$f\left(\frac{a+b}{2}\right)\int_a^b K^{a}_{x}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)d(g(x)) \leq \left(\frac{1}{2^{s}}\right)(1+m) \cdot \int_a^b K^{a}_{x}\left(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi\right)\frac{f(x)}{e^{\alpha x}}d(g(x)). \tag{40}$$

From Definition 4, the following inequality is obtained:

$$h(\alpha)f\left(\frac{a+b}{2}\right)\frac{2^{s}}{(1+m)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right) \cdot (a,\omega;p) \leq \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right)(a,\omega;p). \tag{41}$$

Similarly multiplying both sides of (28) by $K^{x}_{b}(E^{\gamma,\delta,k,c}_{\mu,\alpha,l},g;\phi)g'(x)$ and integrating over $[a,b]$, we have

$$h(\alpha)f\left(\frac{a+b}{2}\right)\frac{2^{s}}{(1+m)}\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right) \cdot (b,\omega;p) \leq \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p). \tag{42}$$

By adding (41) and (42) following inequality is obtained:

$$h(\alpha)f\left(\frac{a+b}{2}\right)\frac{2^{s}}{(1+m)}\left(\left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}1\right)(a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}1\right)(b,\omega;p)\right) \leq \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,b^-}f\right) \cdot (a,\omega;p) + \left({}_gF^{\phi,\gamma,\delta,k,c}_{\mu,\alpha,l,a^+}f\right)(b,\omega;p). \tag{43}$$

Using (39) and (43), inequality (30) can be achieved.

Remark 19.

(1) If we put $p = \omega = 0$ in (30), then result for exponentially (s, m)-convex functions can be obtained for the integral operators defined in [23]

(2) If we put $(s, m) = (1, 1)$ and $\alpha = 0$ in (30), then Theorem 22 in [20] can be obtained

(3) If we put $(s, m) = (1, 1)$ and $p = \omega = 0$ in (30), then the result for exponentially convex functions can be obtained for the integral operators defined in [23]

(4) If we put $p = \omega = \alpha = 0$ in (30), then Theorem 3 in [23] can be obtained

(5) If we put $(s, m) = (1, 1)$ and $p = \omega = \alpha = 0$ in (30), then the result for convex functions can be obtained for the integral operators defined in [23]

Corollary 20. *If we put $\phi(t) = \Gamma(\mu)t^{\mu/k}/k\Gamma_k(\mu)$ and $p = \omega = 0$, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [17]:*

$$\begin{aligned} &\frac{h(\alpha)2^{s+1}f((a+b)/2)}{\mu\Gamma_k(\mu)(m+1)}(g(b)-g(a))^{\mu/k} \leq \left({}_g^{\mu}I_{b\_}^{k}f(a) + {}_g^{\mu}I_{a^+}^{k}f(b)\right) \\ &\leq \frac{2(g(b)-g(a))^{(\mu/k)-1}}{k\Gamma_k(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a) - \frac{\Gamma(s+1)}{(b-a)^s}\right. \\ &\left.\cdot\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(a) - m\frac{f(a/m)^s}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \end{aligned} \tag{44}$$

Corollary 21. *If we put $\phi(t) = t^{\mu}$ and $p = \omega = 0$, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [6]:*

$$\begin{aligned} &\frac{h(\alpha)2^{s+1}f((a+b)/2)}{\mu\Gamma(\mu)(m+1)}(g(b)-g(b))^{\mu} \leq \left({}_g^{\mu}I_{b\_}f(a) + {}_g^{\mu}I_{a^+}f(b)\right) \\ &\leq \frac{2(g(b)-g(a))^{\mu-1}}{\Gamma(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a) - \frac{\Gamma(s+1)}{(b-a)^s}\right. \\ &\left.\cdot\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(a) - m\frac{f(a/m)^s}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \end{aligned} \tag{45}$$

Corollary 22. *If we put $\phi(t) = \Gamma(\mu)t^{\mu/k}/k\Gamma_k(\mu)$, $p = \omega = 0$ and take g as identity function, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [7]:*

$$\begin{aligned} &\frac{h(\alpha)2^{s+1}f((a+b)/2)}{\mu\Gamma_k(\mu)(m+1)}(b-a)^{\mu/k} \leq \left({}^{\mu}I_{b\_}^{k}f(a) + {}^{\mu}I_{a^+}^{k}f(b)\right) \\ &\leq \frac{2(b-a)^{(\mu/k)-1}}{k\Gamma_k(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a) - \frac{\Gamma(s+1)}{(b-a)^s}\right. \\ &\left.\cdot\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(a) - m\frac{f(a/m)^s}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \end{aligned} \tag{46}$$

Corollary 23. *If we put $\phi(t) = t^{\mu}$, $p = \omega = 0$ and take g as the identity function, then the inequality (30) produces the following Hadamard inequality for fractional integral operators defined in [6]:*

$$\begin{aligned} &\frac{h(\alpha)2^{s+1}f((a+b)/2)}{\mu\Gamma(\mu)(m+1)}(g(b)-g(a))^{\mu} \leq \left({}^{\mu}I_{b\_}f(a) + {}^{\mu}I_{a^+}f(b)\right) \\ &\leq \frac{2(b-a)^{\mu-1}}{\Gamma(\mu)}\left(\frac{f(b)}{e^{\alpha b}}g(b) - m\frac{f(a/m)}{e^{\alpha(a/m)}}g(a) - \frac{\Gamma(s+1)}{(b-a)^s}\right. \\ &\left.\cdot\left(\frac{f(b)^s}{e^{\alpha b}}I_{b^-}g(a) - m\frac{f(a/m)^s}{e^{\alpha(a/m)}}I_{a^+}g(b)\right)\right). \end{aligned} \tag{47}$$

Theorem 24. *Let $f : I \longrightarrow \mathbb{R}$ be a differentiable function. If $|f'|$ is exponentially (s, m)-convex with $m \in (0, 1]$ and $g : I \longrightarrow \mathbb{R}$ be a differentiable and strictly increasing function. Also let ϕ/x be an increasing function on I then for $a, b \in I$, $a < b$. If $\alpha, l, \gamma, c \in \mathbb{C}$, $p, \mu \geq 0$, $\delta \geq 0$ and $0 < k \leq \delta + \mu$, then for $x \in (a, b)$, we have*

$$\begin{aligned} &\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f * g\right)(x,\omega;p) + \left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f * g\right)(x,\omega;p)\right| \\ &\leq K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right)\left(m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}g(x) - \frac{|f'(a)|}{e^{\alpha a}}g(a)\right. \\ &\left.- \frac{\Gamma(s+1)}{(x-a)^s}\left(m\frac{|f'(x/m)|^s}{e^{\alpha(x/m)}}I_{x^-}g(a) - \frac{|f'(a)|^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right) \\ &+ K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right)\left(\frac{|f'(b)|}{e^{\alpha b}}g(b) - m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}\right. \\ &\left.\cdot g(x) - \frac{\Gamma(s+1)}{(x-a)^s}\left(\frac{|f'(b)|^s}{e^{\alpha b}}I_{b^-}g(x) - m\frac{|f'(x/m)|^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right), \end{aligned} \tag{48}$$

where

$$\begin{aligned} &\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f * g\right)(x,\omega;p) := \int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right)f'(t)d(g(t)), \\ &\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f * g\right)(x,\omega;p) := \int_x^b K_t^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c}, g; \phi\right)f'(t)d(g(t)). \end{aligned} \tag{49}$$

Proof. Using exponentially (s, m)-convexity of $|f'|$, we have

$$\begin{aligned} |f'(t)| &\leq \left(\frac{x-t}{x-a}\right)^s\frac{|f'(a)|}{e^{\alpha a}} \\ &+ m\left(\frac{t-a}{x-a}\right)^s\frac{|f'(x/m)|}{e^{\alpha(x/m)}}. \end{aligned} \tag{50}$$

The inequality (50) can be written as follows:

$$-\left(\left(\frac{x-t}{x-a}\right)^s\frac{|f'(a)|}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^s\frac{|f'(x/m)|}{e^{\alpha(x/m)}}\right)\le f'(t)\le\left(\frac{x-t}{x-a}\right)^s\frac{|f'(a)|}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^s\frac{|f'(x/m)|}{e^{\alpha(x/m)}}. \tag{51}$$

Let us consider the second inequality of (51)

$$f'(t)\le\left(\frac{x-t}{x-a}\right)^s\frac{|f'(a)|}{e^{\alpha a}}+m\left(\frac{t-a}{x-a}\right)^s\frac{|f'(x/m)|}{e^{\alpha(x/m)}}. \tag{52}$$

Multiplying (15) and (52) and integrating over $[a,x]$, we can obtain:

$$\int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)f(t)d(g(t))\le\frac{|f'(a)|}{e^{\alpha a}}K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\cdot\int_a^x\left(\frac{x-t}{x-a}\right)^s d(g(t))+m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}\times K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\cdot\int_a^x\left(\frac{t-a}{x-a}\right)^s d(g(t)). \tag{53}$$

By using (9) of Definition 4 and integrating by parts, the following inequality is obtained:

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\times\left(m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}g(x)-\frac{|f'(a)|}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{|f'(x/m)|^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{|f'(a)|^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right). \tag{54}$$

Now we consider the left-hand side from the inequality (51), and adopting the same pattern as we did for the right-hand side inequality, we have

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\ge -K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\times\left(m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}g(x)-\frac{|f'(a)|}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{|f'(x/m)|^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{|f'(a)|^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right). \tag{55}$$

From (54) and (55), the following inequality is observed:

$$\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}(f*g)\right)(x,\omega;p)\right|\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\times\left(m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}g(x)-\frac{|f'(a)|}{e^{\alpha a}}g(a)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(m\frac{|f'(x/m)|^s}{e^{\alpha(x/m)}}I_{x^-}g(a)-\frac{|f'(a)|^s}{e^{\alpha a}}I_{a^+}g(x)\right)\right). \tag{56}$$

Now using exponentially (s,m)-convexity of $|f'|$, we have

$$|f'(t)|\le\left(\frac{t-x}{b-x}\right)^s\frac{|f'(b)|}{e^{\alpha b}}+m\left(\frac{b-t}{b-x}\right)^s\frac{|f'(x/m)|}{e^{\alpha(x/m)}}. \tag{57}$$

On the same pattern as we did for (15) and (50), one can obtain the following inequality from (19) and (57):

$$\left|\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}(f*g)\right)(x,\omega;p)\right|\le K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\times\left(\frac{|f'(b)|}{e^{\alpha b}}g(b)-m\frac{|f'(x/m)|}{e^{\alpha(x/m)}}g(x)-\frac{\Gamma(s+1)}{(x-a)^s}\cdot\left(\frac{|f'(b)|^s}{e^{\alpha b}}I_{b^-}g(x)-m\frac{|f'(x/m)|^s}{e^{\alpha(x/m)}}I_{x^+}g(b)\right)\right). \tag{58}$$

By adding (56) and (58), inequality (48) can be achieved.

Remark 25.

(1) If we put $\alpha=0$ in (48), then Theorem 2 in [23] can be obtained

(2) If we put $(s,m)=(1,1)$ in (48), then the result for exponentially convex functions can be obtained for the integral operators defined in [23]

(3) If we put $p=\omega=0$ in (48), then the result for exponentially (s,m)-convex functions can be obtained for the integral operators defined in [23]

(4) If we put $(s,m)=(1,1)$ and $\alpha=0$ in (48), then Theorem 25 in [20] can be obtained

3. Boundedness and Continuity

Theorem 26. *Under the assumptions of Theorem 9, the following inequality holds for m-convex functions:*

$$\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)+\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)(g(x)-g(a))\left(mf\left(\frac{x}{m}\right)+f(a)\right)+K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)(g(b)-g(x))\left(mf\left(\frac{x}{m}\right)+f(b)\right). \tag{59}$$

Proof. If we put $s=1$ and $\alpha=0$ in (17), we have

$$\begin{aligned}\int_a^x K_x^t\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)f(t)d(g(t)) &\le f(a)K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\ &\times\int_a^x\left(\frac{x-t}{x-a}\right)d(g(t))+mf\left(\frac{x}{m}\right)K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\ &\cdot\int_a^x\left(\frac{t-a}{x-a}\right)d(g(t)).\end{aligned}\tag{60}$$

By using Definition 4 and integrating by part, the following inequality is obtained:

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p) &\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\left(mg(x)f\left(\frac{x}{m}\right)\right.\\ &\left.+g(a)f(a)-mg(x)f\left(\frac{x}{m}\right)-g(a)f(a)\right),\end{aligned}\tag{61}$$

which can be written as

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p) &\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)(g(x)-g(a))\\ &\cdot\left(mf\left(\frac{x}{m}\right)+f(a)\right).\end{aligned}\tag{62}$$

Similarly from (21), the following inequality holds:

$$\begin{aligned}\int_x^b K_t^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)f(t)d(g(t)) &\le f(b)K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\ &\times\int_x^b\left(\frac{t-x}{b-x}\right)d(g(t))+mf\left(\frac{x}{m}\right)K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\ &\cdot\int_x^b\left(\frac{b-t}{b-x}\right)d(g(t)),\end{aligned}\tag{63}$$

which further simplifies as follows:

$$\begin{aligned}\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p) &\le K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\ &\cdot(g(b)-g(x))\left(mf\left(\frac{x}{m}\right)+f(b)\right).\end{aligned}\tag{64}$$

From (62) and (64), (59) can be obtained.

Corollary 27. *If we take $m=1$ in Theorem 26, then the following inequality holds for convex functions:*

$$\begin{aligned}&\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)+\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\\ &\le K_x^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)(g(x)-g(a))(f(x)+f(a))\\ &+K_b^x\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)(g(b)-g(x))(f(x)+f(b)).\end{aligned}\tag{65}$$

Theorem 28. *With assumptions of Theorem 26, if $f\in L_\infty[a,b]$, then unified integral operators for m-convex functions are bounded and continuous.*

Proof. From (62), we have

$$\begin{aligned}\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\right| &\le K_b^a\left(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi\right)\\ &\cdot(g(b)-g(a))(m+1)\|f\|_\infty,\end{aligned}\tag{66}$$

which further gives

$$\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\right|\le K\|f\|_\infty,\tag{67}$$

where $K=(g(b)-g(a))(m+1)K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)$.

Similarly, from (64), the following inequality holds:

$$\left|\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\right|\le K\|f\|_\infty.\tag{68}$$

The boundedness is established; also, they are linear therefore the continuity is obtained.

Corollary 29. *If we take $m=1$ in Theorem 28, then unified integral operators for convex functions are bounded and continuous and following inequalities hold:*

$$\begin{aligned}\left|\left({}_gF_{\mu,\alpha,l,a^+}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\right| &\le K\|f\|_\infty,\\ \left|\left({}_gF_{\mu,\alpha,l,b^-}^{\phi,\gamma,\delta,k,c}f\right)(x,\omega;p)\right| &\le K\|f\|_\infty,\end{aligned}\tag{69}$$

where $K=2(g(b)-g(a))K_b^a(E_{\mu,\alpha,l}^{\gamma,\delta,k,c},g;\phi)$.

4. Concluding Remarks

The class of exponentially (s,m)-convex functions contains (s,m)-convex, exponentially convex, s-convex, m-convex, and convex functions. The unified integral operators defined in (9) and (10) produce almost all fractional and conformable integral operators which have been defined independently in recent decades. We have established the bounds of sum of unified integral operators of exponentially (s,m)-convex functions. Also a Hadamard inequality for these operators is established. In conclusion, the presented results reproduce plenty of results for operators composed in Remarks 5 and 6 for functions deduced in Remark 8.

Data Availability

There is no additional data required for the finding of results of this paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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