Hindawi
Journal of Function Spaces
Volume 2020, Article ID 3805803, 7 pages
https://doi.org/10.1155/2020/3805803

*Research Article*

# Multiplicity of Nodal Solutions for a Class of Double-Phase Problems

**Jie Yang**[1,2] **Haibo Chen**[1] **and Senli Liu**[1]

[1] *School of Mathematics and Statistics, Central South University, Changsha, Hunan 410083, China*
[2] *Department of Mathematics, Huaihua University, Huaihua, Hunan 418008, China*

Correspondence should be addressed to Haibo Chen; math_chb@163.com

Received 6 December 2019; Accepted 18 May 2020; Published 4 June 2020

Academic Editor: Calogero Vetro



We consider the following parametric double-phase problem: $\begin{cases} -\text{div}\,(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u) = \lambda f(x,u) \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega. \end{cases}$. We do not impose any global growth conditions to the nonlinearity $f(x,u)$, which refer solely to its behavior in a neighborhood of $u = 0$. And we will show that they suffice for the multiplicity of signed and nodal solutions of the double-phase problem above when $\lambda$ is large enough.

## 1. Introduction and Statement of Results

In this paper, we deal with the existence and multiplicity of solutions for the following double-phase problem:

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \lambda f(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \quad (P_\lambda), \tag{1}$$

where $\lambda > 0$ is a parameter, $\Omega \subset \mathbb{R}^N (N \geq 2)$ is a bounded domain with smooth boundary, $1 < p < q < \min\{N, p^*\}$, $p^* = Np/(N-p)$,

$$\frac{q}{p} < 1 + \frac{1}{N}, a : \bar{\Omega} \longrightarrow [0, +\infty) \text{ is Lipschitz continuous,} \tag{2}$$

and we also assume that the nonlinearity $f$ satisfies the following conditions:

($f_1$) $f : \Omega \times \mathbb{R} \longrightarrow \mathbb{R}$ is a $C^1$ function, $f(x,0) = 0$ for a.e. $x \in \Omega$

($f_2$) There exists $\gamma \in (q, p^*)$ such that

$$\limsup_{t\to 0^+} \frac{f(x,t)}{|t|^{\gamma-2}t} < +\infty \tag{3}$$

uniformly for a.e. $x \in \Omega$

($f_3$) There exists $\beta \in (p, p + (p(p^* - p)/q))$ such that

$$\liminf_{t\to 0} \frac{F(x,t)}{|t|^\beta} > 0 \tag{4}$$

uniformly for a.e. $x \in \Omega$, where $F(x,t) = \int_0^t f(x,s)ds$

($f_4$) There exists a constant $\theta \in (q, p^*)$, $\delta > 0$ such that

$$0 < \theta F(x,t) \leq f(x,t)t \tag{5}$$

for a.e. $x \in \Omega$ and all $0 < |t| \leq \delta$

($f_5$) For the $\delta$ in ($f_4$), $f(x,-u) = -f(x,u), \forall x \in \Omega, |u| \leq \delta$

Similar problems have been investigated, and it is well known they have a strong physical meaning because they appear in the models of strongly anisotropic materials (see [1, 2]). The energy functionals of the form

$$u \mapsto \int_\Omega \mathscr{H}(x, |\nabla u(x)|)dx, \quad \mathscr{H}(x) = t^p + a(x)t^q, q > p > 1, a(\cdot) > 0, \tag{6}$$

where the integrand $\mathscr{H}$ which switches between two different elliptic behaviors has been intensively studied since the late eighties (see [3–9]). Recently, Colombo and Mingione in [7] have obtained the regularity theory for minimizers of (6).

The double-phase problem

$$\begin{cases} -\text{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = f(x, u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{7}$$

has been studied extensively recently. The existence of a sign-changing ground state solution to problem (7) has been proven by Liu and Dai in [3], when $f$ were assumed to satisfy the p-superlinear growth condition and Ambrosetti-Rabinowitz condition. In [4], by using Morse theory, Perera and Squassina obtained a nontrivial weak solution of problem (7), when $f(x, u) = \lambda|u|^{p-2}u + |u|^{r-2}u + h(x, u)$. In [5], by utilizing the Nehari method, Liu and Dai obtained three ground state solutions.

Motivated by the above works, we intend to establish the multiplicity of both signed and nodal solutions of problem $(P_\lambda)$, when $\lambda > 0$ is large enough. Here, we note that the assumptions $(f_1)$–$(f_5)$ that we make on the nonlinearity $f(x, u)$ refer only to its behavior in a neighborhood of $u = 0$.

We present the main result of this paper as follows:

**Theorem 1.** *Suppose $(f_1) - (f_4)$ are satisfied. Then, there exists $\Lambda > 0$, and if $\lambda > \Lambda$, problem $(P_\lambda)$ has at least one positive solution, one negative solution, and a sign-changing solution.*

**Theorem 2.** *Suppose $(f_1) - (f_5)$ are satisfied. Then, there exists $\Lambda^* > 0$, and if $\lambda > \Lambda^*$, for any given $k \geq 1$, problem $(P_\lambda)$ has k pairs of solutions $\pm u_i, i = 1, \cdots, k$, with $\|u_i\|_\infty \leq \delta$. Moreover, $\pm u_i, i = 2, \cdots, k$, are $k - 1$ pairs of sign-changing solutions.*

*Remark 3.* From $(f_1)$ and $(f_2)$, it can be seen that $\gamma \leq \beta$. We may think of $f(x, t) = |t|^{\beta-2}t$ with $\theta < \beta$, which clearly satisfy $(f_1) - (f_4)$.

We remark that [3, 5] obtained only one sign-changing solution. However, in Theorem 2, since $k$ is arbitrary, we get infinitely many sign-changing solutions. To the best of our knowledge, little has been done in the literature on the existence of multiple nodal solutions for the parametric Dirichlet problem with the minimal conditions on the nonlinearity $f(x, u)$.

The proof will be done by variational techniques. Since we have no information on the behavior of the nonlinearity $f(x, u)$ at the infinity, we adapt the argument introduced by Costa and Wang [10], which consists in making a suitable modification on $f$, solving a modified problem, and then checking that, for a large enough $\lambda$, the solutions of the modified problem are indeed solutions of the original one.

The paper is organized as follows. In Section 2, we modify the original problem and prove the Palais-Smale condition for the modified functional. We present some tools which are useful to establish a multiplicity result. In Section 3, we prove Theorem 1. Theorem 2 is proven in Section 4.

## 2. The Modified Problem

To prove our main results, we need to present the variational setting of our problem. Firstly, we introduce some notations and some necessary definitions. The Musielak-Orlicz space $L^{\mathscr{H}}(\Omega)$ associated with the function

$$\mathscr{H} : \Omega \times [0, \infty] \longrightarrow [0, \infty], (x, t) \mapsto t^p + a(x)t^q, \tag{8}$$

consists of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ with the $\mathscr{H}$-modular

$$\rho_{\mathscr{H}}(u) \coloneqq \int_\Omega \mathscr{H}(x, |u|)dx < \infty. \tag{9}$$

The Musielak-Orlicz space $L^{\mathscr{H}}(\Omega)$ is defined by

$$L^{\mathscr{H}}(\Omega) = \{u : \Omega \longrightarrow \mathbb{R} \text{ measurable} : \rho_{\mathscr{H}}(u) < +\infty\}, \tag{10}$$

endowed with the norm

$$\|u\|_{\mathscr{H}} = \inf\left\{\lambda > 0 : \rho_{\mathscr{H}}\left(\frac{u}{\lambda}\right) \leq 1\right\}. \tag{11}$$

The space $L^{\mathscr{H}}(\Omega)$ is a separable, uniformly convex, and reflexive Banach space. We denote by $\|\cdot\|_p$ the norm in $L^p(\Omega)$ and by $L^q_a(\Omega)$ the space of all measurable functions $u : \Omega \longrightarrow \mathbb{R}$ with the seminorm

$$\|u\|_{q,a} \coloneqq \left(\int_\Omega a(x)|u|^q dx\right)^{1/q} < \infty. \tag{12}$$

It is easy to check that the embeddings

$$L^q(\Omega) \hookrightarrow L^{\mathscr{H}}(\Omega) \hookrightarrow L^p(\Omega) \cap L^p_a(\Omega), \tag{13}$$

are continuous. Since $\rho_{\mathscr{H}}(u/\|u\|_{\mathscr{H}}) = 1$ whenever $u \neq 0$, we have

$$\min\left\{\|u\|^p_{\mathscr{H}}, \|u\|^q_{\mathscr{H}}\right\} \leq \|u\|^p_p + \|u\|^q_{q,a} \leq \max\left\{\|u\|^p_{\mathscr{H}}, \|u\|^q_{\mathscr{H}}\right\}, \quad \forall u \in L^{\mathscr{H}}(\Omega). \tag{14}$$

The related Sobolev space $W^{1,\mathscr{H}}(\Omega)$ is defined by

$$W^{1,\mathscr{H}}(\Omega) \coloneqq \left\{u \in L^{\mathscr{H}}(\Omega) \colon |\nabla u| \in L^{\mathscr{H}}(\Omega)\right\}, \tag{15}$$

equipped with the norm

$$\|u\| = \|u\|_{\mathscr{H}} + \|\nabla u\|_{\mathscr{H}}, \tag{16}$$

where $\|\nabla u\|_{\mathscr{H}} = \||\nabla u|\|_{\mathscr{H}}$. The completion of $C_0^\infty(\Omega)$ in $W^{1,\mathscr{H}}(\Omega)$ is denoted by $W_0^{1,\mathscr{H}}(\Omega)$, and it can be equivalently renormed by

$$\|u\| \coloneqq \|\nabla u\|_{\mathscr{H}}, \tag{17}$$

via a Poincare-type inequality (cf [6], Proposition 2.18(iv)), under assumption (2). The spaces $W^{1,\mathscr{H}}(\Omega)$ and $W_0^{1,\mathscr{H}}(\Omega)$ are uniformly convex and hence are reflexive, Banach spaces. By Proposition 2.15 in [6], we know that the Sobolev embedding $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^r(\Omega)$ is compact since $r < p^*$. By (14), We have

$$\begin{aligned} \min\left\{\|u\|^p, \|u\|^q\right\} &\le \|\nabla u\|_p^p + \|\nabla u\|_{q,a}^q \\ &\le \max\left\{\|u\|^p, \|u\|^q\right\}, \quad \forall u \in W_0^{1,\mathscr{H}}(\Omega). \end{aligned} \tag{18}$$

From now on, we denote by $X \coloneqq W_0^{1,\mathscr{H}}(\Omega)$ for convenience in writing.

We first mention that $(f_2)-(f_3)$ imply that there exist $c_1, c_2 > 0$, such that

$$|f(x,t)| \le c_1 |t|^{\gamma-1}, \tag{19}$$

$$F(x,t) \ge c_2 |t|^\beta, \tag{20}$$

for a.e.$x \in \Omega$ and all $|t| \le \delta$.

Now, define the even cutoff function $\zeta(t) \in C^2(\mathbb{R}, [0,1])$ as

$$\zeta(t) = \begin{cases} 1 & \text{if } |t| \le \delta, \\ 0 & \text{if } |t| > \delta. \end{cases} \tag{21}$$

Moreover, for all $t \in \mathbb{R}$, $\zeta(t)$ satisfies that

$$t\zeta'(t) \le 0, \quad |t\zeta'(t)| \le \frac{2}{\delta}. \tag{22}$$

Let

$$\tilde{F}(x,t) = \zeta(t)F(x,t) + (1-\zeta(t))\frac{c_1|t|^\gamma}{\gamma}. \tag{23}$$

Also, we set

$$\tilde{f}(x,t) = \tilde{F}_t'(x,t) = \frac{\partial \tilde{F}}{\partial t}(x,t). \tag{24}$$

From hypotheses $(f_1)-(f_4)$, it is easy to check that $\tilde{f}$ is a Carathéodory function and satisfies the following properties.

**Lemma 4.** *If hypotheses $(f_1)$, $(f_2)$, and $(f_4)$ hold, then the functions $\tilde{F}$ and $\tilde{f}$ satisfy the following properties:*

*(i) There exist $c_3 > 0$ such that $\tilde{f}(x,t) \le c_3|t|^{\gamma-1}$ for a.e.$x \in \Omega$*

*(ii) There exists $\mu = \min\{\theta, \gamma\}$ such that for a.e.$x \in \Omega$ and $t \ne 0$*

$$0 < \mu \tilde{F}(x,t) \le \tilde{f}(x,t)t \tag{25}$$

*Proof.* The proof is similar to that of Lemma 1.1 in [10].

Now let us consider the modified problem of $(P_\lambda)$:

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \lambda \tilde{f}(x,u), \text{in } \Omega, \\ u = 0, \text{on } \partial\Omega, \end{cases} \quad (P_\lambda^*). \tag{26}$$

The corresponding energy functional of $(P_\lambda^*)$ is

$$\tilde{I}_\lambda(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \lambda\tilde{F}(x,u)\right]dx, \quad \forall u \in X. \tag{27}$$

We easily get that the functional $\tilde{I}_\lambda(u) \in C^2(X, \mathbb{R})$, and its critical points are the solutions of $(P_\lambda^*)$. We note that solutions of $(P_\lambda^*)$ with $\|u\|_\infty \le \delta$ are also solutions of $(P_\lambda)$. We shall search solutions of $(P_\lambda^*)$ as critical points of $\tilde{I}_\lambda(u)$.

Let us now define $J(\cdot)\colon X \longrightarrow \mathbb{R}$ as

$$J(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right]dx, \tag{28}$$

and we denote the derivative operator by $A$, that is, $A = J' : X \longrightarrow X^*$ with

$$\langle A(u), v\rangle = \int_\Omega \left[\left(|\nabla u|^{p-2} + a(x)|\nabla u|^{q-2}\right)\nabla u \cdot \nabla v\right]dx, \quad u, v \in X. \tag{29}$$

In the following lemma, we summarize some properties of $A$, useful to study our problem.

**Lemma 5.** *Under the condition (2), $A$ is a mapping of type $(S_+)$; that is, if $u_n \rightharpoonup u$ in $X$ and $\lim_{n\to+\infty} \sup\langle A(u_n) - A(u), u_n - u\rangle \le 0$, then $u_n \longrightarrow u$ in $X$.*

*Proof.* The proof is similar to that of Proposition 3.1(ii) in [3].

Firstly, we show the functional $\tilde{I}_\lambda$ satisfies the (PS) condition.

**Lemma 6.** *If hypotheses $(f_1)$–$(f_2)$ and $(f_4)$ hold, then $\tilde{I}_\lambda$ satisfies the (PS) condition.*

*Proof.* For every $c \in \mathbb{R}$, let $\{u_n\} \subset X$ be a $(\mathrm{PS})_c$–sequence, that is,

$$\tilde{I}_\lambda(u_n) \longrightarrow c, \text{and } \tilde{I}_\lambda'(u_n) \longrightarrow 0, \text{as } n \longrightarrow \infty. \tag{30}$$

We claim that $\{u_n\}$ is bounded in $X$. Indeed, if $\|u_n\| \le 1$, we have done. If $\|u_n\| > 1$, by Lemma 4(ii), then we have that

$$\begin{aligned}
&\tilde{I}_\lambda(u_n) - \frac{1}{\mu}\left\langle \tilde{I}'_\lambda(u_n), u_n \right\rangle \\
&\quad = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q\right]dx - \lambda\int_\Omega \tilde{F}(x, u_n) \\
&\qquad - \frac{1}{\mu}\int_\Omega \left[|\nabla u|^p + a(x)|\nabla u|^q - \lambda \tilde{f}(x, u_n)u_n\right]dx \\
&\quad \ge \left(\frac{1}{q} - \frac{1}{\mu}\right)\|u_n\|^p,
\end{aligned} \tag{31}$$

where $\mu = \min\{\theta, \gamma\} > q$. Hence, $\{u_n\}$ is bounded. Therefore, there is a subsequence (which we still denote by $\{u_n\}$) that converges weakly to some $u \in X$ and strongly in $L^\gamma(\Omega)$. It is easy to check from Lemma 4(i) and Hölder's inequality that

$$\left|\int_\Omega \left(\tilde{f}(x, u_n) - \tilde{f}(x, u)\right)(u_n - u)dx\right| \le c_4\|u_n - u\|_\gamma \longrightarrow 0. \tag{32}$$

Then,

$$\begin{aligned}
&\langle A(u_n) - A(u), u_n - u\rangle \\
&\quad = \left\langle \tilde{I}'(u_n) - \tilde{I}'(u), u_n - u \right\rangle \\
&\qquad + \lambda\int_\Omega \left(\tilde{f}(x, u_n) - \tilde{f}(x, u)\right)(u_n - u)dx \longrightarrow 0.
\end{aligned} \tag{33}$$

So $u_n \longrightarrow u$ follows from Lemma 5.

Next, we will show the functional $\tilde{I}_\lambda$ satisfies the Mountain Pass Geometry [11].

**Lemma 7.** *Assume that hypotheses $(f_1)$–$(f_4)$ are satisfied. Then, the $\tilde{I}_\lambda$ satisfies the following conditions:*

*(i) For every $\lambda > 0$, there exist $\alpha_\lambda, \rho_\lambda > 0$, such that $\tilde{I}_\lambda(u) \ge \alpha_\lambda > 0$ with $\|u\| = \rho_\lambda > 0$*

*(ii) There exists $\lambda_0 > 0$ such that $|e|_\infty \le \delta/2$, $\|e\| \ge \rho_\lambda$ and $\tilde{I}_\lambda(e) \le 0 < \alpha_\lambda$, for all $\lambda > \lambda_0$*

*Proof.* (i) It follows from Lemma 4(i) that there exists $c_5 > 0$ such that

$$\tilde{F}(x, u) \le c_5|u|^\gamma, \text{ for a.e. } x \in \Omega. \tag{34}$$

Then, for all $u \in X$, $\|u\| \le 1$, we have

$$\begin{aligned}
\tilde{I}_\lambda(u) &\ge \frac{1}{q}\|u\|^q - \lambda c_5\|u\|_\gamma^\gamma \ge \frac{1}{q}\|u\|^q - \lambda c_6\|u\|^\gamma \\
&= \left(\frac{1}{q} - \lambda c_6\|u\|^{\gamma - q}\right)\|u\|^q.
\end{aligned} \tag{35}$$

For $\gamma > q$, we can choose $\rho_\lambda = \min\{(1/2q\lambda c_6)^{1/(\gamma - q)}, 1\}$, then

$$\tilde{I}_\lambda(u) \ge \frac{1}{2q}\rho_\lambda^q > 0, \quad \text{for all } u \in X \text{ with } \|u\| = \rho_\lambda. \tag{36}$$

(ii) Let $e \in C_0^1(\Omega) \setminus \{0\}$, such that

$$|e(x)| \le \frac{\delta}{2}, \quad \text{for all } x \in \Omega. \tag{37}$$

Firstly, we note that $\rho_\lambda \longrightarrow 0$, as $\lambda \longrightarrow +\infty$. So there exists $\lambda_0 > 0$ such that for all $\lambda > \lambda_0$,

$$\|e\| > \rho_\lambda, \ \frac{1}{p}\int_\Omega \left(|\nabla e|^p + a(x)|\nabla e|^q\right)dx \le \lambda c_2\|e\|_\beta^\beta. \tag{38}$$

Then, by (20), (37), and (38), we get

$$\begin{aligned}
\tilde{I}_\lambda(e) &= \int_\Omega \left[\frac{1}{p}|\nabla e|^p + \frac{a(x)}{q}|\nabla e|^q - \lambda \tilde{F}(x, e)\right]dx \\
&\le \frac{1}{p}\int_\Omega \left(|\nabla e|^p + a(x)|\nabla e|^q\right)dx - \lambda c_2\|e\|_\beta^\beta \\
&\le 0 < \alpha_\lambda.
\end{aligned} \tag{39}$$

This completes the proof of Lemma 7.

**Lemma 8.** *Assume that hypotheses $(f_1)$–$(f_2)$ and $(f_4)$ are satisfied, $\lambda > 0$ and $u \in K_{\tilde{I}_\lambda} \coloneqq \{u \in X | \tilde{I}'_\lambda(u) = 0, \tilde{I}_\lambda(u) = c\}$, then there exists $d > 0$ such that* $\min\{\|u\|^q, \|u\|^p\} \le d\tilde{I}_\lambda(u)$.

*Proof.* Let $u \in K_{\tilde{I}_\lambda}$. We observe from Lemma 4(ii) that

$$\begin{aligned}
\mu\tilde{I}_\lambda(u) &= \mu\tilde{I}_\lambda(u) - \left\langle \tilde{I}'_\lambda(u), u \right\rangle \\
&= \mu\int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \lambda\tilde{F}(x, u)\right]dx \\
&\quad - \int_\Omega \left[|\nabla u|^p + a(x)|\nabla u|^q - \lambda\tilde{f}(x, u)u\right] \\
&\ge \left(\frac{\mu}{q} - 1\right)\min\{\|u\|^q, \|u\|^p\}.
\end{aligned} \tag{40}$$

Since $\mu = \min\{\theta, \gamma\} > q$, there exists $d = \mu/((\mu/q) - 1) > 0$ such that

$$\min\{\|u\|^q, \|u\|^p\} \le d\tilde{I}_\lambda(u). \tag{41}$$

From Theorem 8.4 in [12], we say that the solutions of $(P_\lambda^*)$ enjoy the $L^\infty$ estimates given in the next lemma.

**Lemma 9.** *Let $\tilde{u}_\lambda \in W_0^{1,\mathcal{H}}(\Omega)$ $(\lambda > 0)$ be the weak solution to $(P_\lambda^*)$ under the assumptions $(f_1)$–$(f_4)$. Then, the following estimate holds:*

$$\|\tilde{u}_\lambda\|_\infty \le c_7 \lambda^{(p^*-\gamma)^{-1}} \|\tilde{u}\|^{(p^*-p)/(p^*-\gamma)}, \tag{42}$$

*where* $c_7 = c_7(\gamma, N, \Omega)$.

Now we are ready to show the existence of a solution to $(P_\lambda^*)$ for a large $\lambda$.

**Lemma 10.** *Assume that conditions* $(f_1)$*–*$(f_4)$ *hold. Then, for any* $\lambda > \lambda_0$*, there exists a nontrivial solution* $\tilde{u}_\lambda \in X$ *of* $(P_\lambda^*)$*, such that*

$$\|\tilde{u}_\lambda\|_\infty \le \frac{c_9}{\lambda^\kappa}, \tag{43}$$

*where* $\kappa = (p(p^* - p) - (\beta - p)q)/((p^* - \gamma)(\beta - p)q) > 0$.

*Proof.* By Lemma 6 and Lemma 7, we conclude that $\tilde{I}_\lambda(u)$ satisfies the (PS) condition and the Mountain Pass Geometry. Consequently, by the Mountain Pass Theorem (Theorem 2.2 in [13]), there exists a $u_\lambda \in X$ such that

$$\tilde{I}'_\lambda(u_\lambda) = 0, \alpha_\lambda \le \tilde{I}_\lambda(u_\lambda) = \tilde{c}_\lambda = \inf_{\tau\in\Gamma} \max_{0\le t\le 1} \tilde{I}_\lambda(\tau(t)), \tag{44}$$

where $\Gamma = \{\tau \in C([0,1], X), \tau(0) = 0, \tau(1) = e\}$.

From (38), we easily obtain that the function

$$\begin{aligned} h_\lambda(t) = {} & \frac{t^p}{p}\int_\Omega \left(|\nabla e|^p + a(x)|\nabla e|^q\right)dx \\ & - \lambda c_2 t^\beta \|e\|_\beta^\beta \quad t \in [0,1], \lambda > \lambda_0, \end{aligned} \tag{45}$$

is continuous and $h_\lambda(0) = 0, h_\lambda(1) < 0$. It follows from $1 < p < \beta$, for $t \in (0,1)$ which is small enough, we get

$$h_\lambda(t) > 0. \tag{46}$$

Hence, there exists $t_0 \in (0,1)$ such that

$$h_\lambda(t_0) = \max_{0\le t\le 1} h_\lambda(t), \tag{47}$$

where $t_0 = [(\int_\Omega(|\nabla e|^p + a(x)|\nabla e|^q)dx)/c_2\beta\lambda\|e\|_\beta^\beta]^{1/(\beta-p)}$. Then, we obtain

$$h_\lambda(t_0) = \frac{c_8}{\lambda^{p/(\beta-p)}}, \tag{48}$$

where

$$c_8 = \left(\frac{1}{p} - \frac{1}{\beta}\right)\frac{\left[\int_\Omega\left(|\nabla e|^p + a(x)|\nabla e|^q\right)dx\right]^{\beta/(\beta-p)}}{\left(c_2\beta\|e\|_\beta^\beta\right)^{p/(\beta-p)}} > 0. \tag{49}$$

Let $\tau_0 = te$. Obviously, $\tau_0 \in \Gamma$, and then we get from (44) that

$$\tilde{I}_\lambda(u_\lambda) = \tilde{c}_\lambda \le \max_{0\le t\le 1}\left(\tilde{I}_\lambda(te)\right) \le \max_{0\le t\le 1} h_\lambda(t) = \frac{c_8}{\lambda^{p/(\beta-p)}}. \tag{50}$$

Therefore, there exists $\lambda_1 \ge \lambda_0$ such that $dc_8/\lambda^{p/(\beta-p)} < 1$. Together with Lemma 8, we obtain

$$\|u\|^q \le \frac{dc_8}{\lambda^{p/(\beta-p)}}. \tag{51}$$

It follows from Lemma 9, we get that

$$\|\tilde{u}_\lambda\|_\infty \le \frac{c_9}{\lambda^\kappa}, \tag{52}$$

where $c_9 > 0, \kappa = (p(p^* - p) - (\beta - p)q)/((p^* - \gamma)(\beta - p)q) > 0$, recalling that $\beta \in (p, p + (p(p^* - p)/q))$.

## 3. Proof of Theorem 1

In this section, we prove our main result. We will show $(P_\lambda^*)$ has a positive solution, a negative solution, and a nodal solution. And the solutions obtained satisfy the estimate $\|u\|_\infty \le \delta$. This fact implies that these solutions are indeed solutions of the original problem $(P_\lambda)$.

*Proof of Theorem 1.* Consider the following problem:

$$\begin{cases} -\mathrm{div}\left(|\nabla u|^{p-2}\nabla u + a(x)|\nabla u|^{q-2}\nabla u\right) = \lambda \tilde{f}^+(x,u), \text{ in } \Omega, \\ u = 0, \text{ on } \partial\Omega, \end{cases} \tag{53}$$

where

$$\tilde{f}^+(x,t) = \begin{cases} \tilde{f}(x,t), & t > 0, \\ 0, & t \le 0. \end{cases} \tag{54}$$

Define the corresponding functional

$$\tilde{I}_\lambda^+(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{a(x)}{q}|\nabla u|^q - \tilde{F}^+(x,u)\right]dx, \quad \forall u \in X, \tag{55}$$

where $\tilde{F}^+(x,u) = \int_0^u \tilde{f}^+(x,s)ds$. Obviously, $\tilde{I}_\lambda^+ \in C^2(X,R)$ and $\tilde{f}^+$ satisfy all the conditions of Theorem 1. Let $u_\lambda^+$ be a nontrivial critical point of $\tilde{I}_\lambda^+$, which implies that $u_\lambda^+$ is a weak solution of (53). It is known by Lemma 4.1 of [5] that $u_\lambda^+ \ge 0$ a.e. in $\mathbb{R}^N$. From Theorem 3.3 in [5], we conclude that $u_\lambda^+ > 0$. Therefore, by Lemma 10,

$$\|u_\lambda^+\|_\infty \le \frac{c_9}{\lambda^\kappa}, \tag{56}$$

where $\kappa > 0$, so there exists $\lambda_2 \ge \lambda_1$ such that

$$\frac{c_9}{\lambda^\kappa} \le \delta, \tag{57}$$

for all $\lambda > \lambda_2$. Thus, $u_\lambda^+ > 0$ is also a nontrivial solution of original problem $(P_\lambda)$ for all $\lambda \geq \lambda_2$.

Similarly, we can define

$$\tilde{f}^-(x,t) = \begin{cases} \tilde{f}(x,t), & t < 0, \\ 0, & t \geq 0, \end{cases}$$
$$\tilde{I}_\lambda^-(u) = \int_\Omega \left[\frac{1}{p}|\nabla u|^p + \frac{1}{q}a(x)|\nabla u|^q - \tilde{F}^-(x,u)\right]dx, \quad \forall u \in X, \tag{58}$$

where $\tilde{F}^-(x,u) = \int_0^u \tilde{f}^-(x,s)ds$. We also get a negative solution $u_\lambda^- < 0$ of our original problem $(P_\lambda)$ for all $\lambda \geq \lambda_3 \geq \lambda_2$.

We next show that there is a sign-changing solution for $\lambda$ large enough. We can apply the method introduced by Li and Wang [14] to our case. Since $X$ is a real, reflexive, and separable Banach space, there are $\{e_j\} \subset X$ and $\{e_j^*\} \subset X^*$ such that

$$X = \overline{\operatorname{span}\{e_j, j = 1, 2, \cdots\}}, X^* = \overline{\operatorname{span}\left\{e_j^*, j = 1, 2, \cdots\right\}},$$
$$\left\langle e_j, e_j^* \right\rangle = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases} \tag{59}$$

For $k = 1, 2, \cdots$, we denote

$$Y_k := \operatorname{span}\{e_1, e_2, \cdots, e_k\}, Z_k := \overline{\operatorname{span}\{e_k, e_{k+1}, \cdots\}}. \tag{60}$$

On $X$, we define a closed convex cone

$$P_X = \{u \in X | u(x) \geq 0, \text{a.e.in}\, \Omega\}. \tag{61}$$

And $E = C_0^1(\Omega)$ is a Banach space densely embedded in $X$. Assume that $P = P_X \cap E$ has interior points in $P$. As in Section 3 of [14], we may define a partial order relation in $E$: $u, v \in E, u > v \Leftrightarrow u - v \in P \setminus \{0\}$ ; $u \gg v \Leftrightarrow u - v \in P^\circ$. And we define $W = P \cup (-P)$. In order to apply the method introduced by Li and Wang in [14] (Example 3.2 and Corollary 3.2), we take $Q = \{u = s_1 e_1 + s_2 e_2 : |s_1| \leq R, 0 \leq s_2 \leq R, \|u\| \leq R\}$ on $Y_2$, and for $0 < r < R$, $T = \{u \in Z_2 | \|u\| = r\}$. Hence, we obtain $T$ and $\partial Q$ link.

It follows from Lemma 4(ii) that there exist $c_{10}, c_{11} > 0$ such that

$$\tilde{F}(x,u) \geq c_{10}u^\mu - c_{11}. \tag{62}$$

Then, for each $u \in Y_2$, $\|u\| \geq 1$, we obtain that

$$\begin{aligned} \tilde{I}_\lambda(u) &\leq \frac{1}{p}\|u\|^q - \lambda\int_\Omega \tilde{F}(x,u)dx, \\ &\leq \frac{1}{p}\|u\|^q - \lambda c_{12}\|u\|^\mu + \lambda c_{11}|\Omega|, \end{aligned} \tag{63}$$

for all norms on $Y_2$ are equivalent. Since $\mu > q$, we may get $R \geq 1$ which is large enough such that $\tilde{I}_\lambda \leq 0$ for all $u \in Y_2$, $\|u\| = R$ and for all $\lambda > 0$.

Using (34) and the Sobolev embedding $W_0^{1,\mathscr{H}}(\Omega) \hookrightarrow L^r(\Omega)$ for $\gamma \in (q, p^*)$, we can choose some $\alpha > 0$, $0 < \rho < 1$ such that

$$\tilde{I}_\lambda(u) \geq \frac{1}{q}\|u\|^q - \lambda c_{13}\|u\|^\gamma > \alpha, \tag{64}$$

for $\|u\| = \rho$. Let us define

$$c_\lambda = \inf_{h \in \Gamma} \sup_{h(Q)\setminus W} \tilde{I}_\lambda(u), \tag{65}$$

where $\Gamma = \{h \in C(Q,E) | h(Q_1) \in W, h(u) = u, \text{for}\ u \in Q_2\}$. Therefore, by [14], we see that $c_\lambda > 0$ is a critical value of $\tilde{I}_\lambda$ and $\tilde{I}_\lambda$ has a sign-changing critical point $\tilde{u}_\lambda$ at this critical value. Therefore, by Lemma 10,

$$\|\tilde{u}_\lambda\|_\infty \leq \frac{c_9}{\lambda^\kappa}, \tag{66}$$

where $\kappa > 0$, so there exists $\Lambda \geq \lambda_3$ such that

$$\frac{c_9}{\lambda^\kappa} \leq \delta, \tag{67}$$

for all $\lambda > \Lambda$. Thus, $\tilde{u}_\lambda$ is also a nontrivial sign-changing solution of original problem $(P_\lambda)$ for all $\lambda \geq \Lambda$.

## 4. Proof of Theorem 2

In this section, we present a multiplicity result for the modified problem. We shall use arguments in [13] to get solutions for $(P_\lambda^*)$ first. Next, we use proofs in [14] to get the nodal property of the solutions.

*Proof of Theorem 2.* By (63), we can choose $R \geq 0$ such that $\tilde{I}_\lambda \leq 0$ for all $u \in Y_2$, $\|u\| = R$ and for all $\lambda > 0$. Set $D = B_R \cap Y_k$. Let $G = \{h \in C(D,X) | h \text{ is odd and } h = id \text{ on } \partial B_R \cap Y_k\}$. Let $\Sigma$ denote the family of sets $A \subset X \setminus \{0\}$ such that $A$ is closed in $X$ and symmetric with respect to 0. For $A \in \Sigma$, $i(A)$ defines the genus of $A$. Then, for $j = 1, \cdots k$, we can define

$$c_{j,\lambda} = \inf_{A \in \Gamma_j} \sup_{u \in A} \tilde{I}_\lambda(u), \tag{68}$$

where $\Gamma_j = \{h(\overline{D \setminus Y}) | h \in G, k \geq j, Y \in \Sigma, i(Y) \leq k - j\}$. Now we can apply Proposition 9.30 in [13] to functional $\tilde{I}_\lambda$, we get that $0 < c_{1,\lambda} \leq c_{2,\lambda} \leq \cdots \leq c_{k,\lambda}$ are all critical values of $\tilde{I}_\lambda$. And $(P_\lambda^*)$ possesses at least $k$ pairs of distinct nontrivial solutions. Then, by Lemma 10, there exists $\Lambda^* > 0$, such that $\lambda > \Lambda^*$; these $k$ pairs solutions of $(P_\lambda^*)$ are also solutions of the original problem $(P_\lambda)$.

Secondly, we will obtain the nodal property of the solutions by Theorem 2.3 in [14]. We need a procedure similar to the one we used earlier to rule this out. Let

$$\Sigma_E = \{A \subset E | A \text{ is compact}, A = -A\}. \quad (69)$$

Let $G = \{h \in C(D, E) | h \text{ is odd and } h = id \text{ on } \partial B_R \cap Y_k\}$. For $A \in \Sigma_E$, $i(A)$ also denotes the genus of $A$. By (63), we can also choose $R \geq 0$ such that $\tilde{I}_\lambda \leq 0$ for all $u \in Y_2$, $\|u\| = R$ and for all $\lambda > 0$. By (64), we can also choose $\alpha > 0$, $0 < r < R$ such that $\tilde{I}_\lambda \geq \alpha$ for all $u \in X$, $\|u\| = r$ and for all $\lambda > 0$. By Theorem 2.3 in [14], $\tilde{I}_\lambda$ has at least $k-1$ pairs of critical points $\pm u_j$ in $E \setminus (P \cup -P)$, with critical values

$$\tilde{I}_\lambda(u_j) = c_{j,\lambda} = \inf_{A \in \Gamma_j} \sup_{A \setminus W} \tilde{I}_\lambda(u), \quad (70)$$

where $\Gamma_j = \{h(\overline{D \setminus Y}) | h \in G, k \geq j, Y \in \Sigma, i(Y) \leq k - j\}$, $j = 2, \cdots k$. Moreover, $0 < c_{2,\lambda} \leq c_{3,\lambda} \leq \cdots \leq c_{k,\lambda}$. Finally, by Lemma 10,

$$\|u_j\|_\infty \leq \frac{c_9}{\lambda^\kappa}, \quad (71)$$

where $\kappa > 0$, so there exists $\Lambda^* > 0$ such that

$$\frac{c_9}{\lambda^\kappa} \leq \delta, \quad (72)$$

for all $\lambda > \Lambda^*$. Thus, $\pm u_j$, $j = 2, \cdots, k$, are also nontrivial sing-changing solutions of the original problem $(P_\lambda)$ for all $\lambda \geq \Lambda^*$.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

## Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

## Acknowledgments

H. B. Chen is supported by the National Natural Science Foundation of China (No. 11671403); J. Yang is supported by the Scientific Research Fund of Hunan Provincial Education Department (No. 17C1263), the Natural Science Foundation of Hunan Province of China (No. 2019JJ50473), and the Research Foundation of Education Bureau of Hunan Province, China (Grant No. 19B450).

## References

[1] V. V. Zhikov, "On Lavrentiev's phenomenon," *Russian Journal of Mathematical Physics*, vol. 3, pp. 249–269, 1995.

[2] V. V. Zhikov, "On some variational problems," *Russian Journal of Mathematical Physics*, vol. 5, pp. 105–116, 1997.

[3] W. Liu and G. Dai, "Existence and multiplicity results for double phase problem," *Journal of Differential Equations*, vol. 265, no. 9, pp. 4311–4334, 2018.

[4] K. Perera and M. Squassina, "Existence results for double-phase problems via Morse theory," *Communications in Contemporary Mathematics*, vol. 20, no. 2, article 1750023, 2017.

[5] W. Liu and G. Dai, "Three ground state solutions for double phase problem," *Journal of Mathematical Physics*, vol. 59, no. 12, article 121503, 2018.

[6] F. Colasuonno and M. Squassina, "Eigenvalues for double phase variational integrals," *Annali di Matematica Pura ed Applicata*, vol. 195, no. 6, pp. 1917–1959, 2016.

[7] M. Colombo and G. Mingione, "Regularity for double phase variational problems," *Archive for Rational Mechanics and Analysis*, vol. 215, no. 2, pp. 443–496, 2015.

[8] W. Xie and H. Chen, "Existence and multiplicity of solutions for $p(x)$-Laplacian equations in $\mathrm{R}^N$," *Mathematische Nachrichten*, vol. 16, no. 291, pp. 2476–2488, 2018.

[9] L. Xu and H. Chen, "Nontrivial solutions for Kirchhoff-type problems with a parameter," *Journal of Mathematical Analysis and Applications*, vol. 433, no. 1, pp. 455–472, 2016.

[10] D. Costa and Z.-Q. Wang, "Multiplicity results for a class of superlinear elliptic problems," *Proceedings of American Mathematical Society*, vol. 133, no. 3, pp. 787–795, 2005.

[11] M. Williem, "Minimax theorems," in *Progress in Nonlinear Differential Equations and their Applications*, vol. 24, Birkhäuser, 1996.

[12] D. Motreanu, V. Motreanu, and N. Papageorgiou, *Topological and Variational Methods with Applications to Nonlinear Boundary Value Problems*, Springer, New York, NY, USA, 2014.

[13] P. Rabinowitz, "Minimax methods in critical point theory with applications to differential equations," in *CBMS Regional Conference Series in Mathematics*, vol. 65, American Mathematical Society, 1986.

[14] S. Li and Z. Q. Wang, "Ljusternik-Schnirelman theory in partially ordered Hilbert spaces," *Transactions of the American Mathematical Society*, vol. 354, no. 8, pp. 3207–3228, 2002.