Research Article

Stability and Stabilization for a Class of Semilinear Fractional Differential Systems

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1. Introduction

Over the past several decades, fractional calculus has attracted much attention from scientists and engineers. This is because fractional differential equations have proven to be effective in modelling many physical phenomena and have been applied in different science and engineering fields. Significant contributions have been proposed in fractional differential equations both in theory and applications. For example, see [1–4] and references therein.

In recent years, the stability of fractional-order systems has gained increasing interest due to its importance in control theory. Also, stability theory is an important topic in the study of differential equations. In 1996, Matignon studied the stability of linear fractional differential equations in [5], which is regarded as the first work in this area. Li et al. investigated the Mittag-Leffler stability of nonlinear fractional dynamic systems in [6] and suggested the Lyapunov direct method for nonlinear fractional-order stability systems [7]. There has been more literature on stability of dynamic fractional-order systems, in which important and sufficient conditions were discussed for the stability of linear and linear time-delay fractional differential equations as stated in [8–10]. The modelling and stability of the water jet mixed-flow pump fractional-order shafting system has also been studied [11]. The stability of fractional-order nonlinear systems with $0 < \alpha < 1$ was derived in [6, 12, 13], according to the Lyapunov approach. Based on the uncertain Takagi–Sugeno fuzzy model, the stability problems of nonlinear fractional-order systems were studied, whereas the sliding-mode control approach was used to investigate the stabilization and synchronization problems of the nonlinear fractional-order system (e.g., [14–18]). As noted, a growing number of scientists are dedicated to the stability of fractional systems, with most of the above findings concentrating only on nonlinear fractional systems of $0 < \alpha < 1$.

In [19–24], the authors studied the stability of fractional nonlinear systems with order $0 < \alpha \leq 2$. Various sufficient conditions for asymptotic stability (local or global) are obtained by using Mittag-Leffler function, Laplace transform, and the generalized Gronwall inequality. In summary, the authors of [19, 20, 23, 25] conducted studies on the stability relying on a class of commensurate and incommensurate fractional-order systems together with fractionally controlled systems with linear feedback inputs. By using Mittag-Leffler, Laplace transforms, and Gronwall–Bellman lemma, Zhang et al. [21] discussed the stability of n-dimensional nonlinear fractional order.
In this paper, we discuss the stability of a class of semilinear fractional differential systems with the fractional order between 0 and 2. Theoretically, a stability theorem is developed with the property of convolution and the asymptotic properties of Mittag-Leffler functions. Therefore, based on this theory of stability, a basic criterion for stabilizing a class of nonlinear fractional-order systems is derived, in which control parameters can be selected through the linear control theory pole placement technique. Our results give us a simple method for determining the stability of nonlinear fractional systems with Caputo derivative with order 0 < \alpha < 2. Compared with the abovementioned stability method, the conditions we have proposed for the nonlinear component f(t, x(t)) are new and much simpler to test. Thus, there is no need to arrive at an exact solution if only the nonlinear term satisfies certain conditions. What is needed is to calculate the eigenvalues of the linear coefficient matrix A and test that (see [3]). Therefore, based on the theory of stability, a basic criterion for stabilizing a class of nonlinear fractional-order systems is derived, in which control parameters can be selected through the linear control theory pole placement technique. Our results give us a simple method for determining the stability of nonlinear fractional systems with Caputo derivative with order 0 < \alpha < 2. Compared with the abovementioned stability method, the conditions we have proposed for the nonlinear component f(t, x(t)) are new and much simpler to test. Thus, there is no need to arrive at an exact solution if only the nonlinear term satisfies certain conditions. What is needed is to calculate the eigenvalues of the linear coefficient matrix A and test that (see [3]).

2. Preliminaries

In this section, we state some definitions and results that are going to be used in our investigations.

Definition 1 (see [3]). The Caputo fractional derivative of order \alpha of function x(t) is defined as

\[ C^\alpha_{t_0}D^\alpha_x(t) = \frac{1}{\Gamma(n-\alpha)} \int_{t_0}^{t} (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \]

where \Gamma(\cdot) is the gamma function, i.e., \Gamma(z) = \int_{0}^{\infty} e^{-t} t^z-1 dt, and n is an integer satisfying n - 1 < \alpha < n.

The Laplace transform of the Caputo fractional derivative \[ C^\alpha_{t_0}D^\alpha_x(t) \] is

\[ \int_{0}^{\infty} e^{-st} C^\alpha_{t_0}D^\alpha_x(t) dt = s^\alpha X(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} x^{(k)}(t_0), \]

where n is an integer such that n - 1 < \alpha < n.

Similar to the exponential function, the function frequently used in the fractional differential equations is the Mittag-Leffler function. The definitions and properties are therefore given as follows.

Definition 2 (see [3]). The Mittag-Leffler function is defined as

\[ E_{\alpha}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+1)}, \quad \text{Re}(\alpha) > 0, z \in \mathbb{C}. \]

The Mittag-Leffler function with two parameters is defined as

\[ E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(ak+\beta)}, \quad \text{Re}(\alpha) > 0, \beta \in \mathbb{R}, z \in \mathbb{C}. \]

It is easy to see that \( E_{\alpha}(z) = E_{\alpha,1}(z) \) and \( E_{1}(z) = E_{1,1}(z) = e^z \). The Laplace transform of Mittag-Leffler function is formulated as

\[ \int_{0}^{\infty} e^{-st} C^\alpha_{t_0}D^\alpha_x(t) dt = \frac{k! s^{\alpha-\beta}}{\Gamma(\alpha+\beta) s^{\alpha+\beta}}, \quad \text{Re}(s) > |\alpha|^{1/\alpha}. \]

Definition 3 (see [2]). For \( A \in C^{n \times n} \), the matrix Mittag-Leffler function is defined by

\[ E_{\alpha,\beta}(A) = \sum_{k=0}^{\infty} \frac{A^k}{\Gamma(ak+\beta)}, \quad \beta \in \mathbb{C}, \text{Re}(\alpha) > 0. \]

Lemma 1 (see [26]). The following properties hold:

(i) There exist finite real constants \( M_1 \geq 1, M_2 \geq 1, M_3 \geq 1 \) such that for any 0 < \alpha < 1,

\[ E_{\alpha,1}(A) \leq M_1 \left\| e^{At} \right\|, \]

\[ E_{\alpha,\alpha}(A) \leq M_2 \left\| e^{At} \right\|. \]

where \( A \) denotes matrix and \( \| \cdot \| \) denotes any vector or induced matrix norm.

(ii) If \( \alpha > 1 \), then for \( \beta = 1, 2, \alpha \),

\[ E_{\alpha,\beta}(A) \leq M_3 \left\| e^{At} \right\|. \]

Definition 4 (see [7]). The constant \( x_0 \in \mathbb{R}^n \) is an equilibrium point of the Caputo dynamic system \( C^\alpha_{t_0}D^\alpha_x(t) = f(t, x(t)) \) if and only if \( f(t, x_0) = 0 \).

Without loss of generality, we may assume that the equilibrium point is \( x_0 = 0 \), representing the origin of \( \mathbb{R}^n \) (See [7]). Hence, in the rest of this paper, we always assume that the nonlinear function \( f \) satisfies \( f(t, 0) = 0 \).

Definition 5. The zero solution of \( C^\alpha_{t_0}D^\alpha_x(t) = f(t, x(t)) \), with order 0 < \alpha < 1 (1 < \alpha < 2) is said to be stable if for any initial values \( x_0 \) \( (k=0)(x_0(k=0, 1)) \) and any \( \epsilon > 0 \) there exists \( t_0 > 0 \) such that \( \|x(t)\| < \epsilon \) for all \( t > t_0 \). The zero solution is said to be asymptotically stable if \( \lim_{t \to \infty} \|x(t)\| = 0 \).

The following property of convolution plays a key role in the proof of the main results.

Lemma 2 (see [27]). Let \( I < p, q < \infty \) satisfy \( (1/p) + (1/q) = 1 \). If \( f \in L^p(\mathbb{R}, X) \) and \( g \in L^q(\mathbb{R}, X) \), then \( f \ast g \in C_0(\mathbb{R}, X) \). Here, \( X \) is a Banach space, \( f \ast g \) denotes the convolution of the functions \( f \) and \( g \).
i.e., $f \ast g(t) = \int_0^t f(t - \tau)g(\tau)d\tau$ for $t \geq 0$, and $C_0(\mathbb{R}, X) = \{f \in C(\mathbb{R}, X): f$ vanishes at infinity).}

3. Existence Results

In this section, we consider the following system of fractional differential equation:

$$\frac{D^\alpha t}{\alpha} x(t) = Ax(t) + f(t, x(t)),$$

(9)

where $x(t) = (x_1(t), x_2(t), \ldots, x_n(t))^T \in \mathbb{R}^n$ denotes the state vector of the system, $\alpha \in (0, 2)$ is the order of the fractional-order derivative, $f : [0, +\infty) \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ defines a nonlinear vector field in the $n$-dimensional vector space, and $A \in \mathbb{R}^{n \times n}$ is a constant matrix. Hereafter, we assume that $\lambda_i (i = 1, 2, \ldots, n)$ are the eigenvalues of the matrix $A$ and that $f(t, 0) = 0$, i.e., $0$ is an equilibrium point of system (9).

3.1. Stability for the Case $0 < \alpha \leq 1$. We first present a stability result for the case $0 < \alpha \leq 1$. In this case, the solution to equation (9), with initial condition $x(0) = x_0$, can be expressed as

$$x(t) = E_{\alpha,1}(At)x_0 + \int_0^t (t - \tau)^{\alpha - 1}E_{\alpha,\alpha}(A(t - \tau)\tau) \cdot f(\tau, x(\tau))d\tau.$$

(10)

The existence and uniqueness of solutions to this problem are widely studied [1–3].

Theorem 1. The zero solution of system (9) is locally asymptotically stable if the following conditions are satisfied:

1. The matrix $A$ is stable.
2. There is a function $g \in L^q(0, \infty)$ such that

$$\|f(t, x(t))\| \leq g(t),$$

(11)

where $g$ is a positive constant satisfying $1/p + 1/q = 1$ and $p > (1/\alpha)$.

Proof. Let the initial condition be $x(0) = x_0$. Then, the solution of (9) is given by (10). This is obtained from condition (1) that $|\arg(\lambda_i(A))| > (\alpha \pi/2)$. It then follows from Lemma 1 that there exist constants $M_1 > 0$ and $M_2 > 0$ such that

$$\|x(t)\| \leq M_1 e^{\omega t}\|x_0\| + M_2 \int_0^t (t - \tau)^{\alpha - 1}e^{A(t - \tau)\tau} \cdot \|f(\tau, x(\tau))\|d\tau.$$

(12)

Because the matrix $A$ is stable, there exist constants $M > 0$ and $\omega > 0$ such that

$$\|e^{At}\| \leq Me^{-\alpha t}.$$

(13)

Substituting it into (12) and using condition (2), one has

$$\text{lim}_{t \rightarrow +\infty} \|x(t)\| = 0.$$
for \( t \in [0, T] \). Then, equation (9) has a unique solution if and only if \( Q \) has a unique fixed point. Taking \( x, y \in C([0, T], \mathbb{R}^n) \) arbitrarily and \( t \in [0, T] \). We obtain

\[
\|Qx(t) - Qy(t)\| \leq \int_0^t (t - \tau)^{\alpha-1} E_{\alpha,\alpha} (A(t - \tau)^\alpha) f(\tau, x(\tau)) d\tau,
\]

(21)

for \( t > 0 \). Because \( 1 < \alpha \leq 2 \), it follows from (25) and condition (2) that

\[
\|x(t)\| \leq M_3 \|e^{At}\| \|x_0\| + M_3 \|e^{At}\| \|x\| t
\]

\[
+ M_3 \int_0^t (t - \tau)^{\alpha-1} \|e^{A(t-\tau)^\alpha}\| f(\tau, x(\tau)) d\tau
\]

\[
\leq M_3 \|e^{At}\| \|x_0\| + M_3 \|e^{At}\| \|x\| t
\]

\[
+ M_3 \int_0^t (t - \tau)^{\alpha-1} \|e^{A(t-\tau)^\alpha}\| g(\tau) d\tau,
\]

(26)

for all \( t > 0 \). From condition (1) we know that the matrix \( A \) is stable. Hence, the inequality (13) holds. Therefore, we obtain

\[
\|x(t)\| \leq M M_3 e^{-\omega t \alpha} \|x_0\| + M M_3 e^{-\omega t \alpha} \|x\| t
\]

\[
+ M M_3 \int_0^t (t - \tau)^{\alpha-1} e^{-\omega (t-\tau)^\alpha} g(\tau) d\tau,
\]

(27)

for all \( t > 0 \). Similar to the proof of Theorem 1, we define the function \( \varphi: [0, +\infty) \rightarrow [0, +\infty) \) as

\[
\varphi(u) = u^{\alpha-1} e^{-\omega u^\alpha}.
\]

(28)

Because \( 1 < \alpha \leq 2 \), it is easy to see that \( \varphi \in L^p([0, +\infty)) \) for any \( p > 1 \). Because of Lemma 2 and condition (2), we obtain \( \varphi^* g \in C_0([0, +\infty)) \). Therefore, \( \lim_{t \rightarrow +\infty} \|x(t)\| = 0 \). The proof is completed.\( \square \)

Remark 1. Theorems 1 and 3 give us a simple procedure for determining stability of the fractional-order nonlinear system with Caputo derivative of order \( 0 < \alpha < 2 \). If the nonlinear term \( f(t, x(t)) \) fulfills condition (2), then the exact solution need not be reached. Importantly, it is required to calculate matrix \( A \)'s eigenvalues and test their arguments. If \( \|a(\lambda_i(A))\| > (\alpha/2), i = 1, 2, \ldots, n \), we conclude that the origin is stable asymptotically.

3.3 Stabilization of a Class of Fractional-Order Semilinear System. In this subsection, we propose the stabilization theory of a class of fractional-order semilinear controlled systems. We consider the controlled systems of the following form:

\[
C_0^\alpha D_t^\alpha x(t) = Ax(t) + f(t, x(t)) + Bu(t),
\]

(29)

where \( x, A \) and \( f \) are as in system (9), \( B \in \mathbb{R}^{n \times n} \) is the input matrix, and \( u \) is the control input. If \( u \) is chosen to be a linear state feedback control, i.e., \( u = Kx \) for some feedback gains \( K \), then system (29) becomes a closed-loop system:

\[
C_0^\alpha D_t^\alpha x(t) = Ax(t) + f(t, x(t)) + Bu(t)
\]

\[
= Ax(t) + f(t, x(t)) + BKx(t)
\]

\[
= (A + BK)x(t) + f(t, x(t))
\]

\[
= Ax(t) + f(t, x(t)).
\]

(30)

Suppose that \( (A, B) \) is controllable, then the feedback gain \( K \) can be chosen such that system (30) is asymptotically stable.
Theorem 4. If $0 < a < 2$, feedback gain $K$ is chosen such that the following conditions hold:

(i) $\|\arg(\lambda_i(\tilde{A}))\| > (ax/2), (i = 1, 2, \ldots, n)$, where $\lambda_i(\tilde{A})$ is the eigenvalue of matrix $\tilde{A}$.

(ii) $f(t, x(t))$ satisfies $\|f(t, x(t))\| \leq g(t)$, where $g \in L^q(0, \infty)$.

(iii) The matrix $\tilde{A}$ is stable. Then, the controlled system (29) is locally asymptotically stable.

Proof. The proof of Theorem 4 is similar to that of Theorems 1 and 3.

Remark 2. The nonlinear term of Chaotic fractional-order systems satisfies $\|f(t, x(t))\| \leq g(t)$, where $g \in L^q(0, \infty)$, i.e., the hyperchaotic fractional-order novel system. Therefore, in a large class of generalized fractional-order chaotic or hyperchaotic systems, Theorem 4 can be applied to control chaotic. Of all control methods, linear feedback control is particularly attractive and has been widely extended to practical implementation due to its ease among configuration and implementation.

4. Applications

In this section, we apply the obtained results to some semilinear systems to illustrate the effectiveness of the theory.

Example 1. Consider the fractional nonlinear system with Caputo derivative:

\[
\begin{align*}
\mathcal{C}D^\alpha_0 x_1(t) &= 3x_1 + 7x_2 + 2x_3 + e^{-w_1t}\sin x_1x_3, \\
\mathcal{C}D^\alpha_0 x_2(t) &= -2x_1 - 4x_2 + 4x_3 + e^{-w_2t}\sin x_2^2, \\
\mathcal{C}D^\alpha_0 x_3(t) &= e^{-w_3t}\sin x_1x_2 - x_3,
\end{align*}
\]

(31)

where $w_i > 0, i = 1, 2, 3$. This system can be rewritten as (9), where

\[
A = \begin{bmatrix}
3 & 7 & 2 \\
-2 & -4 & 4 \\
0 & 0 & -1
\end{bmatrix},
\]

\[
f(t, x) = \begin{bmatrix}
e^{-w_1t}\sin x_1x_3 \\
e^{-w_2t}\sin x_2^2 \\
e^{-w_3t}\sin x_1x_2
\end{bmatrix}.
\]

It is easy to verify that

\[
\|f(t, x)\| \leq \sqrt{3}e^{-w_0t} \in L^q(0, +\infty),
\]

where $w_0 = \max\{w_1, w_2, w_3\}$. So, condition (2) in Theorems 1 and 3 is satisfied. The eigenvalues of $A$ are $\lambda_{1,2} = -0.500 \pm 1.3229i$ and $\lambda_3 = -1$. According to Theorems 1 and 3, if $\alpha < 1.2300$, then the zero solution of (31) is asymptotically stable. For the initial values $(x_1(0), x_2(0), x_3(0)) = (0.1, -0.2, 0.3)$, simulation results are displayed in Figures 1–3. Figures 1 and 2 show that the zero solution of system (31) is asymptotically stable with $\alpha = 1.21$ and $\alpha = 1.22$, respectively. Figure 3 also shows that the zero solution of system (31) is unstable with $\alpha = 1.23$.

Example 2. Consider the following fractional nonlinear system:

\[
\begin{align*}
\mathcal{C}D^\alpha_0 x_1(t) &= -x_1 + e^{-w_1t}\sin x_2x_3, \\
\mathcal{C}D^\alpha_0 x_2(t) &= x_3, \\
\mathcal{C}D^\alpha_0 x_3(t) &= x_1 - x_2 - x_3 - e^{-w_2t}\sin x_1x_2.
\end{align*}
\]

(34)

The system can be rewritten as (9), where

\[
A = \begin{bmatrix}
-1 & 0 & 0 \\
0 & 0 & 1 \\
1 & -1 & -1
\end{bmatrix},
\]

\[
f(t, x) = \begin{bmatrix}
e^{-w_1t}\sin x_2x_3 \\
0 \\
e^{-w_2t}\sin x_1x_2
\end{bmatrix}.
\]

Obviously,

\[
\|f(t, x(t))\| \leq e^{-w_0} := g(t) \in L^q.
\]

The eigenvalues of $A$ are $\lambda_{1,2} = (-1/2) \pm (\sqrt{3}i/2)$ and $\lambda_3 = -1$. According to Theorems 1 and 3, if $\alpha < (4/3)$, then the zero solution of (32) is asymptotically stable.

Simulation results with initial values $(x_1(0), x_2(0), x_3(0)) = (-0.01, -0.02, 0.03)$ are displayed in Figures 4–6. Figures 4 and 5 show that the zero solution of system (34) is asymptotically stable with $\alpha = 1.1$ and $\alpha = 1.3$, respectively. Figure 6 further shows that the zero solution of system (34) is unstable when $\alpha = 1.34$.

Example 3. The fractional-order novel hyperchaotic system can be written as

\[
\begin{align*}
\mathcal{C}D^\alpha_0 x_1(t) &= a(x_2 - x_1), \\
\mathcal{C}D^\alpha_0 x_2(t) &= -bx_1 + e^{-w_1t}\sin x_1x_3 + x_4, \\
\mathcal{C}D^\alpha_0 x_3(t) &= -2e^{-w_2t}\sin x_1^2 - 2e^{-w_3t}\sin x_2^2 - cx_3 - x_4, \\
\mathcal{C}D^\alpha_0 x_4(t) &= -dx_1,
\end{align*}
\]

(37)

where $a, b, c,$ and $d$ are some parameters. System (37) can be rewritten as (9) if we write
Figure 1: System (31) is asymptotically stable with $\alpha = 1.21$.

Figure 2: System (31) is asymptotically stable with $\alpha = 1.22$.

Figure 3: System (31) is unstable with $\alpha = 1.23$. 
Figure 4: System (34) is asymptotically stable with $\alpha = 1.1$.

Figure 5: System (34) is asymptotically stable with $\alpha = 1.3$.

Figure 6: System (34) is unstable with $\alpha = 1.34$. 
Figure 7: Asymptotical stabilization of the fractional-order novel hyperchaotic system (37) with $\alpha = 1.14$ and feedback gain $K = (2, -8, -3.5, -1)$.

Figure 8: Attractor of the fractional-order novel hyperchaotic system with order $\alpha = 1.14$ ($a = 10$, $b = 40$, $c = 2.5$, and $d = 10$).
Figure 9: Asymptotical stabilization of the fractional-order novel hyperchaotic system (37) with $\alpha = 1.15$ and feedback gain $K = (2, -8, -3.5, -1)$.

Figure 10: Attractor of the fractional-order novel hyperchaotic system with order $\alpha = 1.15$ ($a = 10$, $b = 40$, $c = 2.5$, and $d = 10$).
According to Theorem 4, feedback gain can be obtained by the authors. They declare that they have no conflicts of interest.

Conflicts of Interest

The data used in the examples were originally taken from MATLAB, and they are available from the corresponding author upon request.

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