Research Article

Fixed Point Theorems for Mizoguchi-Takahashi Type Contraction in Bicomplex-Valued Metric Spaces and Applications

Fuli He ¹, Z. Mostefaoui,² and M. Abdalla ^{3,4}

¹School of Mathematics and Statistics, Central South University, Changsha 410083, China

²Department of Mathematics, E.N.S. B.P. 92 Vieux Kouba 16050 Algiers, Algeria

³Mathematics Department, Faculty of Science, King Khalid University, Abha, Saudi Arabia

⁴Mathematics Department, Faculty of Science, South Valley University, Qena 83523, Egypt

Correspondence should be addressed to M. Abdalla; moabdalla@kku.edu.sa

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The main aim of this paper is to study and establish some new fixed point theorems for contractive maps that satisfied Mizoguchi-Takahashi's condition in the setting of bicomplex-valued metric spaces. These new results improve and generalize the Banach contraction principle and some well-known results in the literature. Finally, as applications of our results, we give the existence and uniqueness of the solution of a nonlinear integral equation.

1. Introduction

Fixed point theorems concern maps f of a set X into itself that, under certain conditions, admit a fixed point, that is, a point $x \in X$ such that $f(x) = x$. The knowledge of the existence of fixed points has relevant applications in many branches of analysis and topology. In 2011, Azam et al. [1] introduced the so-called complex-valued metric spaces and proved the existence of fixed points under some contraction conditions. Very recently, Choi et al. [2] introduced the notion of bicomplex-valued metric space which is a generalization of the complex-valued metric space and established sufficient conditions for the existence of common fixed points of a pair of mappings satisfying a contractive condition.

The following new fixed point theorem is one of the main results of this paper. It can be considered a bicomplex-valued metric version of the Banach contraction principle and will generalize and improve some well-known results in the literature. Finally, we strengthen our results by giving an application to find an analytical solution for a nonlinear integral equation. The results thus derived in this study are generally extended and generalized in character. Moreover, the results presented here may be very useful in the study of engineering problems like boundary value problems (see, e. g., [3–7]) in future works.

2. Preliminaries

In this section, we recall some definitions and terminologies, used to prove the main results.

2.1. Bicomplex Numbers. The set of bicomplex numbers denoted by $\mathbb{BC}$ is the first setting in an infinite sequence of multicomplex sets which are generalizations of the set of complex numbers $\mathbb{C}$. Here, we recall the set of bicomplex numbers $\mathbb{BC}$ (see, for example, [8–10]):

$$\mathbb{BC} = \{w = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3; x_k \in \mathbb{R}, (k = 0, 1, 2, 3)\}. \quad (1)$$

Since each element w in $\mathbb{BC}$ can be written as $w = x_0 + i_1 x_1 + i_2(x_2 + i_1 x_3)$ or $w = z_1 + i_2 z_2; (z_1, z_2 \in \mathbb{C})$, we can also express $\mathbb{BC}$ as

$$\mathbb{BC} := \{w = z_1 + i_2 z_2, z_1, z_2 \in \mathbb{C}\}, \quad (2)$$

where $z_1 = x_0 + i_1 x_1$, $z_2 = x_2 + i_1 x_3$, and i_1, i_2 are independent imaginary units such that $i_1^2 = -1 = i_2^2$. The product of i_1 and i_2 defines a hyperbolic unit j such that $j^2 = 1$. The product

of units is commutative and is defined as

$$\begin{aligned} i_1 i_2 &= j, \\ i_1 j &= -i_2, \\ i_2 j &= -i_1. \end{aligned} \quad (3)$$

With the addition and multiplication of two bicomplex numbers defined in the obvious way, the set $\mathbb{BC}$ makes up a commutative ring. Three important subsets of $\mathbb{BC}$ can be specified as

$$\begin{aligned} \mathbb{C}(i_k) &:= \{x + yi_k; x, y \in \mathbb{R}\}, k = 1, 2, \\ \mathbb{D} &:= \{x + yj; x, y \in \mathbb{R}\}. \end{aligned} \quad (4)$$

Each of the set $\mathbb{C}(i_k)$ is isomorphic to the field of complex numbers, while $\mathbb{D}$ is the set of the so-called hyperbolic numbers.

2.2. Conjugation and Moduli. Three kinds of conjugation $w^{\dagger_1}$, $w^{\dagger_2}$, and $w^{\dagger_3}$ can be defined on bicomplex numbers. With the bicomplex number w specified as in (2) and the bar ($\bar{}$) denoting complex conjugation in $\mathbb{C}$, we define

$$\begin{aligned} w^{\dagger_1} &:= \bar{z}_1 + \bar{z}_2 i_2, \\ w^{\dagger_2} &:= z_1 - z_2 i_2, \\ w^{\dagger_3} &:= \bar{z}_1 - \bar{z}_2 i_2. \end{aligned} \quad (5)$$

It is easy to check that each conjugation has the following properties:

$$\begin{aligned} (u + v)^{\dagger_k} &= u^{\dagger_k} + v^{\dagger_k}, \\ (u^{\dagger_k})^{\dagger_k} &= u, \\ (u \cdot v)^{\dagger_k} &= u^{\dagger_k} \cdot v^{\dagger_k}. \end{aligned} \quad (6)$$

Here, $u, v \in \mathbb{BC}$ and $k = 1, 2, 3$.

With each kind of conjugation, one can define a specific bicomplex modulus as

$$\begin{aligned} |w|_{i_1}^2 &:= w \cdot w^{\dagger_2} = z_1^2 + z_2^2 \in \mathbb{C}(i_1) \\ |w|_{i_2}^2 &:= w \cdot w^{\dagger_1} = (|z_1|^2 - |z_2|^2) + 2 \operatorname{Re}(z_1 \bar{z}_2) i_2 \in \mathbb{C}(i_2) \\ |w|_j^2 &:= w \cdot w^{\dagger_3} = (|z_1|^2 + |z_2|^2) - 2 \operatorname{Im}(z_1 \bar{z}_2) j \in D \end{aligned} \quad (7)$$

It can be shown that $|u \cdot v|_s^2 = |u|_s^2 \cdot |v|_s^2$, where $s = i_1, i_2$ or j .

A norm of a bicomplex number $w = z_1 + i_2 z_2$ denoted by $\|w\|$ is defined by

$$\|w\| = \|z_1 + i_2 z_2\| = (|z_1|^2 + |z_2|^2)^{1/2}, \quad (8)$$

which, upon choosing $w = x_0 + i_1 x_1 + i_2 x_2 + i_1 i_2 x_3$, ($x_k \in \mathbb{R}$,

$k = 0, 1, 2, 3$), gives

$$\|w\| = (x_0^2 + x_1^2 + x_2^2 + x_3^2)^{1/2}. \quad (9)$$

For any two bicomplex numbers $u, v \in \mathbb{BC}$, one can easily verify that $0 \prec_{i_2} u \prec_{i_2} v \implies \|u\| \leq \|v\|$, $\|u + v\| \leq \|u\| + \|v\|$, $\|\alpha u\| = \alpha \|u\|$, where α is a nonnegative real number. Further, for any two bicomplex numbers, $u, v \in \mathbb{BC}$, $\|uv\| \leq \sqrt{2} \|u\| \|v\|$ holds.

Next, we recall some necessary definitions and lemmas.

First, define a partial order $\leq$ on $\mathbb{BC}$ as follows:

Let $u = u_1 + i_2 u_2 \in \mathbb{BC}$ and $v = v_1 + i_2 v_2 \in \mathbb{BC}$. Define a partial order relation $\leq_{i_2}$ on $\mathbb{BC}$ as follows:

$u \leq_{i_2} v$ if and only if $u_1 \leq_{i_1} v_1$ and $u_2 \leq_{i_1} v_2$. We find that $u \leq_{i_2} v$ if any one of the following properties holds:

$$\begin{aligned} (a_1) \quad & u_1 = v_1 \\ & u_2 = v_2 \\ (a_2) \quad & u_1 \prec_{i_1} v_1 \\ & u_2 = v_2 \\ (a_3) \quad & u_1 = v_1 \\ & u_2 \prec_{i_1} v_2 \\ (a_4) \quad & u_1 \prec_{i_1} v_1 \\ & u_2 \prec_{i_1} v_2 \end{aligned} \quad (10)$$

where $\leq_{i_1}$ is a partial order relation on $\mathbb{C}$. So that

$$\begin{aligned} (b_1) \quad & \operatorname{Re}(u_1) = \operatorname{Re}(v_1), \operatorname{Im}(u_1) = \operatorname{Im}(v_1), \\ & \operatorname{Re}(u_2) = \operatorname{Re}(v_2), \operatorname{Im}(u_2) = \operatorname{Im}(v_2) \\ (b_2) \quad & \operatorname{Re}(u_1) < \operatorname{Re}(v_1), \operatorname{Im}(u_1) < \operatorname{Im}(v_1), \\ & \operatorname{Re}(u_2) = \operatorname{Re}(v_2), \operatorname{Im}(u_2) = \operatorname{Im}(v_2) \\ (b_3) \quad & \operatorname{Re}(u_1) = \operatorname{Re}(v_1), \operatorname{Im}(u_1) = \operatorname{Im}(v_1), \\ & \operatorname{Re}(u_2) < \operatorname{Re}(v_2), \operatorname{Im}(u_2) < \operatorname{Im}(v_2) \\ (b_4) \quad & \operatorname{Re}(u_1) < \operatorname{Re}(v_1), \operatorname{Im}(u_1) < \operatorname{Im}(v_1), \\ & \operatorname{Re}(u_2) < \operatorname{Re}(v_2), \operatorname{Im}(u_2) < \operatorname{Im}(v_2) \end{aligned} \quad (11)$$

We write $u \prec v$ if $u \leq_{i_2} v$ and $u \neq v$, i.e., one of (b_2) , (b_3) , and (b_4) is satisfied and we write $u \prec_{i_2} v$ if only (b_4) is satisfied.

Given the definition of the complex metric space introduced in [1], we extend the definition of bicomplex analysis as follows.

Definition 1. Let X be a nonempty set. A functional $d_{\mathbb{BC}} : X \times X \rightarrow \mathbb{BC}$ is called a bicomplex-valued metric on X if for $x, y, z \in X$ the following conditions are satisfied:

$$\begin{aligned} (m_1) & 0 \leq_{i_2} d_{\mathbb{BC}}(x, y) \\ (m_2) & d_{\mathbb{BC}}(x, y) = 0 \text{ if and only if } x = y \\ (m_3) & d_{\mathbb{BC}}(x, y) = d_{\mathbb{BC}}(y, x) \\ (m_4) & d_{\mathbb{BC}}(x, y) \leq_{i_2} d_{\mathbb{BC}}(x, z) + d_{\mathbb{BC}}(z, y) \end{aligned} \quad (12)$$

Then, $(X, d_{\mathbb{BC}})$ is called a bicomplex-valued metric space.

Some known examples of bicomplex-valued metric, which show a bicomplex-valued metric space, are the following.

Example 2. The set of real numbers together with the functional

$$\begin{aligned} d_{\mathbb{BC}}^1(x, y) &= (1 + i_1 + i_2 + i_1 i_2)|x - y|, \\ d_{\mathbb{BC}}^2(x, y) &= i_1 i_2 |x - y|, \end{aligned} \quad (13)$$

for all $x, y \in \mathbb{R}$ where $|\cdot|$ is the usual real modulus. One can easily check that $(X, d_{\mathbb{BC}}^1)$, $(X, d_{\mathbb{BC}}^2)$ are bicomplex-valued metric spaces.

Definition 3. Let (x_n) be a sequence in a bicomplex-valued metric space $(X, d_{\mathbb{BC}})$. The sequence (x_n) is said to converge to $x \in X$ if and only if for any $0 <_{i_2} \varepsilon \in \mathbb{BC}$, there exists $N \in \mathbb{N}$ depending on ε such that $d_{\mathbb{BC}}(x_n, x) <_{i_2} \varepsilon$ for all $n > N$. It is denoted by $x_n \rightarrow x$ as $n \rightarrow +\infty$ or $\lim_{n \rightarrow +\infty} x_n = x$.

Definition 4. A sequence (x_n) in a bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ is said to be a Cauchy sequence if and only if for any $0 <_{i_2} \varepsilon \in \mathbb{BC}$, there exists $N \in \mathbb{N}$ depending on ε such that $d_{\mathbb{BC}}(x_n, x_m) <_{i_2} \varepsilon$ for all $n, m > N$.

Definition 5. A bicomplex-valued metric space $(X, d_{\mathbb{BC}})$ is said to be complete if and only if every Cauchy sequence in X converges in X .

Lemma 6 (see [2]). Let $(X, d_{\mathbb{BC}})$ be a bicomplex-valued metric space and (x_n) be a sequence in X . Then, (x_n) converges to $x \in X$ if and only if $\|d_{\mathbb{BC}}(x_n, x)\| \rightarrow 0$ as $n \rightarrow +\infty$.

Lemma 7 (see [2]). Let $(X, d_{\mathbb{BC}})$ be a bicomplex-valued metric space and (x_n) be a sequence in X such that $\lim_{n \rightarrow +\infty} x_n = x$. Then, for any $a \in X$, $\lim_{n \rightarrow +\infty} \|d_{\mathbb{BC}}(x_n, a)\| = \|d_{\mathbb{BC}}(x, a)\|$.

3. Statement of Results

Definition 8 (see [11, 12]). A function $\psi : [0, +\infty) \rightarrow [0, 1]$ is said to be a MT-function if it satisfies Mizoguchi-Takahashi's condition, i.e., $\limsup_{s \rightarrow t^+} \psi(s) < 1$ for all $t \in [0, +\infty)$.

Clearly, if $\psi : [0, +\infty) \rightarrow [0, 1]$ is a nondecreasing function, then ψ is a MT-function.

Notice that $\psi : [0, +\infty) \rightarrow [0, 1]$ is a MT-function if and only if for each $t \in [0, +\infty)$ there exist $r_t \in [0, 1]$ and $\varepsilon_t > 0$ such that $\psi(s) \leq r_t$ for all $s \in [t, t + \varepsilon_t)$, for more detail, (see [4], Remark 2.5 (iii)).

An example which is not an MT-function is given hereunder. Let $\psi : [0, +\infty) \rightarrow [0, 1]$ be defined by

$$\psi(t) = \begin{cases} \frac{\sin t}{t}, & \text{if } t \in \left(0, \frac{\pi}{2}\right], \\ 0, & \text{otherwise.} \end{cases} \quad (14)$$

Since $\limsup_{s \rightarrow 0^+} \psi(s) = 1$, ψ is not a MT-function.

Recently, Du [13] first proved the following characterizations of MT-functions which are quite useful for proving our main results.

Theorem 9. [13] Let $\psi : [0, +\infty) \rightarrow [0, 1]$ be a function. Then, the following statements are equivalent.

- (a) ψ is a MT-function
- (b) For each $t \in [0, +\infty)$, there exist $r_t^1 \in [0, 1]$ and $\varepsilon_t^1 > 0$ such that $\psi(s) \leq r_t^1$ for all $s \in (t, t + \varepsilon_t^1)$
- (c) For each $t \in [0, +\infty)$, there exist $r_t^2 \in [0, 1]$ and $\varepsilon_t^2 > 0$ such that $\psi(s) \leq r_t^2$ for all $s \in [t, t + \varepsilon_t^2]$
- (d) For each $t \in [0, +\infty)$, there exist $r_t^3 \in [0, 1]$ and $\varepsilon_t^3 > 0$ such that $\psi(s) \leq r_t^3$ for all $s \in (t, t + \varepsilon_t^3]$
- (e) For each $t \in [0, +\infty)$, there exist $r_t^4 \in [0, 1]$ and $\varepsilon_t^4 > 0$ such that $\psi(s) \leq r_t^4$ for all $s \in [t, t + \varepsilon_t^4)$
- (f) For any nonincreasing sequence $(x_n)_{n \in \mathbb{N}}$ in $[0, +\infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \psi(x_n) < 1$
- (g) ψ is a function of contractive factor, that is, for any strictly decreasing sequence $(x_n)_{n \in \mathbb{N}}$ in $[0, +\infty)$, we have $0 \leq \sup_{n \in \mathbb{N}} \psi(x_n) < 1$

Theorem 10. Let $(X, d_{\mathbb{BC}})$ be a complete bicomplex-valued metric space and let $f : X \rightarrow X$ be a mapping on X . Suppose that there exists a MT-function $\psi : [0, +\infty) \rightarrow [0, 1]$ such that

$$d_{\mathbb{BC}}(fx, fy) \leq_{i_2} \psi(\|d_{\mathbb{BC}}(x, y)\|) d_{\mathbb{BC}}(x, y), \text{ for all } x, y \in X. \quad (15)$$

Then, f has a unique fixed point on X .

Notice that f need not be continuous.

Proof. Let $x_0 \in X$ be given. Define the sequence $(x_n)_{n \in \mathbb{N}}$ by

$$x_n = f^n x_0 = f x_{n-1}, \text{ for each } n \in \mathbb{N}. \quad (16)$$

For each $n \in \mathbb{N}$, by (15), we have

$$d_{\mathbb{BC}}(x_n, x_{n+1}) = d_{\mathbb{BC}}(fx_{n-1}, fx_n) \leq_{i_2} \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) \cdot d_{\mathbb{BC}}(x_{n-1}, x_n), \quad (17)$$

which implies

$$\|d_{\mathbb{BC}}(x_n, x_{n+1})\| \leq \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) \|d_{\mathbb{BC}}(x_{n-1}, x_n)\|. \quad (18)$$

Let $\delta_n = \|d_{\mathbb{BC}}(x_{n-1}, x_n)\|$ for $n \in \mathbb{N}$. Then, by (18), we have

$$\delta_{n+1} \leq \psi(\delta_n) \delta_n < \delta_n, \forall n \in \mathbb{N}. \quad (19)$$

So, we know that $(\delta_n)_{n \in \mathbb{N}}$ is a strictly decreasing sequence in $[0, +\infty)$. Applying (g) of Theorem 9, we obtain

$$0 \leq \sup \psi(\delta_n) < 1. \quad (20)$$

That is,

$$0 \leq \sup \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) < 1. \quad (21)$$

Let

$$k = \sup \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|). \quad (22)$$

Then, $k \in [0, 1)$. For each $n \in \mathbb{N}$, by (18) again, we have

$$\begin{aligned} \|d_{\mathbb{BC}}(x_n, x_{n+1})\| &\leq \psi(\|d_{\mathbb{BC}}(x_{n-1}, x_n)\|) \|d_{\mathbb{BC}}(x_{n-1}, x_n)\| \\ &\leq k \|d_{\mathbb{BC}}(x_{n-1}, x_n)\| \leq k^2 \|d_{\mathbb{BC}}(x_{n-2}, x_{n-1})\| \\ &\leq \dots \leq k^n \|d_{\mathbb{BC}}(x_0, x_1)\|. \end{aligned} \quad (23)$$

For any $m, n \in \mathbb{N}$ with $m > n$, we get

$$\begin{aligned} \|d_{\mathbb{BC}}(x_n, x_m)\| &\leq \|d_{\mathbb{BC}}(x_n, x_{n+1})\| \\ &\quad + \|d_{\mathbb{BC}}(x_{n+1}, x_{n+2})\| + \dots + \|d_{\mathbb{BC}}(x_{m-1}, x_m)\| \\ &\leq (k^n + k^{n+1} + \dots + k^{m-1}) \|d_{\mathbb{BC}}(x_0, x_1)\| < \frac{k^n}{1-k} \|d_{\mathbb{BC}}(x_0, x_1)\|. \end{aligned} \quad (24)$$

Since $k \in [0, 1)$, $\lim_{n \rightarrow +\infty} (k^n/1 - k) \|d_{\mathbb{BC}}(x_0, x_1)\| = 0$. Hence, by the last inequality, we obtain

$$\|d_{\mathbb{BC}}(x_n, x_m)\| \longrightarrow 0, \text{ as } m, n \longrightarrow +\infty, \quad (25)$$

that is, $d_{\mathbb{BC}}(x_n, x_m) \longrightarrow 0$ as $n \longrightarrow +\infty$. Therefore, (x_n) is a Cauchy sequence in the complete bicomplex-valued metric space. Then, there exists $u \in X$ such that

$$\lim_{n \rightarrow +\infty} d_{\mathbb{BC}}(x_n, u) = 0. \quad (26)$$

Next, we prove that $fu = u$. Assume that $fu \neq u$. For each $n \in \mathbb{N}$, by (15), we have

$$d_{\mathbb{BC}}(x_{n+1}, fu) = d_{\mathbb{BC}}(fx_n, fu) \leq_{i_2} \psi(\|d_{\mathbb{BC}}(x_n, u)\|) d_{\mathbb{BC}}(x_n, u), \quad (27)$$

which deduces

$$\|d_{\mathbb{BC}}(x_{n+1}, fu)\| \leq \psi(\|d_{\mathbb{BC}}(x_n, u)\|) \|d_{\mathbb{BC}}(x_n, u)\| < \|d_{\mathbb{BC}}(x_n, u)\|. \quad (28)$$

Since $x_n \longrightarrow u$ as $n \longrightarrow +\infty$, from Lemma 6 and Lemma 7 and by taking the limit from both sides of (29), we get

$$\|d_{\mathbb{BC}}(u, fu)\| \leq \|d_{\mathbb{BC}}(u, u)\| = 0. \quad (29)$$

Since $0 < d_{\mathbb{BC}}(u, fu)$, then

$$0 < \|d_{\mathbb{BC}}(u, fu)\| \leq 0, \quad (30)$$

which is a contradiction. Therefore, $fu = u$, that is, u is a fixed point of f .

Finally, we want to show the uniqueness of the fixed point of f . Let v be a fixed point of f and $v \neq u$. By (15), we obtain

$$d_{\mathbb{BC}}(fv, fu) = d_{\mathbb{BC}}(v, u) \leq_{i_2} \psi(\|d_{\mathbb{BC}}(v, u)\|) d_{\mathbb{BC}}(v, u), \quad (31)$$

which implies

$$\|d_{\mathbb{BC}}(v, u)\| \leq \psi(\|d_{\mathbb{BC}}(v, u)\|) \|d_{\mathbb{BC}}(v, u)\|, \quad (32)$$

then

$$(1 - \psi(\|d_{\mathbb{BC}}(v, u)\|)) \|d_{\mathbb{BC}}(v, u)\| \leq 0. \quad (33)$$

Since $\psi(\|d_{\mathbb{BC}}(v, u)\|) \in [0, 1)$, we have

$$\|d_{\mathbb{BC}}(v, u)\| \leq 0, \quad (34)$$

which deduces

$$\|d_{\mathbb{BC}}(v, u)\| = 0, \quad (35)$$

and hence $d_{\mathbb{BC}}(v, u) = 0$ which is a contradiction. Therefore $v = u$, that is, f has a unique fixed point.

Example 11. Let $X = \mathbb{R}$ and $d_{\mathbb{BC}} : X \times X \longrightarrow \mathbb{BC}$ be defined by $d_{\mathbb{BC}}(x, y) = i_1 i_2 |x - y|$, $(x, y \in X)$, where $|\cdot|$ is the usual real modulus. One can easily check that $(X, d_{\mathbb{BC}})$ is a complete bicomplex-valued metric on X and $\|d_{\mathbb{BC}}(x, y)\| = |x - y|$ for all $x, y \in X$.

Define $f : X \longrightarrow X$ and $\psi : [0, +\infty) \longrightarrow [0, 1]$ by $f(x) = 1/12x$ for $x \in X$,

$$\psi(t) = \begin{cases} \frac{5}{6} & \text{if } t = 0 \\ \frac{1}{4} & \text{if } t > 0 \end{cases} \quad (36)$$

Then, ψ is a MT-function.

For all $x, y \in X$ we have

$$\begin{aligned} d_{\mathbb{B}\mathbb{C}}(fx, fy) &= i_1 i_2 |fx - fy| = \begin{cases} \frac{1}{2} i_1 i_2 |x - y| & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \\ \psi(\|d_{\mathbb{B}\mathbb{C}}(x, y)\|) d_{\mathbb{B}\mathbb{C}}(x, y) &= \psi(|x - y|) d_{\mathbb{B}\mathbb{C}}(x, y) \\ &= \begin{cases} \frac{1}{4} i_1 i_2 |x - y| & \text{if } x \neq y \\ 0 & \text{if } x = y \end{cases} \end{aligned} \quad (37)$$

then, f satisfies (15).

So all the hypotheses of Theorem 10 are fulfilled. It is therefore possible to apply Theorem 10 to get the fact that f has a unique fixed point on X (Here, 0 is the unique fixed point of f).

The following fixed point theorem established in metric space is immediate from Theorem 10.

Theorem 12. Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping on X . Suppose that there exists a MT-function $\psi : [0, +\infty) \longrightarrow [0, 1]$ such that

$$d(fx, fy) \leq \psi(d(x, y))d(x, y), \text{ for all } x, y \in X. \quad (38)$$

Then, f has a unique fixed point on X .

Since any nondecreasing function or any nonincreasing function $\psi : [0, +\infty) \longrightarrow [0, 1]$ is an MT-function, by applying Theorem 10, we have the following results.

Corollary 13. Let $(X, d_{\mathbb{B}\mathbb{C}})$ be a complete bicomplex-valued metric space and let $f : X \longrightarrow X$ be a mapping on X . Suppose that there exists a nondecreasing (or nonincreasing) function $\psi : [0, +\infty) \longrightarrow [0, 1]$ such that

$$d_{\mathbb{B}\mathbb{C}}(fx, fy) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(x, y)\|) d_{\mathbb{B}\mathbb{C}}(x, y), \text{ for all } x, y \in X, \quad (39)$$

Then, f has a unique fixed point on X .

Corollary 14. Let (X, d) be a complete metric space and let $f : X \longrightarrow X$ be a mapping on X . Suppose that there exists a nondecreasing (nonincreasing) function $\psi : [0, +\infty) \longrightarrow [0, 1]$ such

that

$$d(fx, fy) \leq \psi(d(x, y))d(x, y), \text{ for all } x, y \in X. \quad (40)$$

Then, f has a unique fixed point on X .

Corollary 15 (bicomplex-valued metric version of Banach contraction principle). Let $(X, d_{\mathbb{B}\mathbb{C}})$ be a bicomplex-valued complete metric space and let $f : X \longrightarrow X$ be a contraction mapping on X , that is,

$$d_{\mathbb{B}\mathbb{C}}(fx, fy) \leq_{i_2} k d_{\mathbb{B}\mathbb{C}}(x, y), \quad (41)$$

for all $x, y \in X$, where $k \in [0, 1)$. Then, f has a unique fixed point on X .

The aim of the next work is to generalize our results by finding a common fixed point for k functions $f_i, i = 1, 2, \dots, k$ defined on a bicomplex-valued complete metric space.

Theorem 16. Let $(X, d_{\mathbb{B}\mathbb{C}})$ be a complete bicomplex-valued metric space and $f_i : X \longrightarrow X$ be k mappings on X . Suppose that there exists a MT-function $\psi : [0, +\infty) \longrightarrow [0, 1]$ such that

$$d_{\mathbb{B}\mathbb{C}}(f_i x, f_{i+1} y) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(x, y)\|) d_{\mathbb{B}\mathbb{C}}(x, y), \quad (42)$$

for all $x, y \in X$ and $i = 1, 2, \dots, k$, then, there exists a unique common fixed point $u \in X$ such that $u = f_1 u = f_2 u = \dots = f_k u$

Proof. For any $x_0 \in X$ Putting $x_1 = f_{i+1} x_0$ and $x_2 = f_i x_1$, then let $x_3 = f_{i+1} x_2$ and $x_4 = f_i x_3$. Inductively, choose a sequence $(x_n)_{n \in \mathbb{N}}$ in X so that $x_{2n+2} = f_i x_{2n+1}$ and $x_{2n+1} = f_{i+1} x_{2n}$ for all $n \geq 0$.

It follows from the property of the function ψ that if n is odd and for $x_{n+1} \neq x_n$, then

$$\begin{aligned} d_{\mathbb{B}\mathbb{C}}(x_{n+1}, x_n) &= d_{\mathbb{B}\mathbb{C}}(f_i x_n, f_{i+1} x_{n-1}) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(x_n, x_{n-1})\|) \\ &\quad \cdot d_{\mathbb{B}\mathbb{C}}(x_n, x_{n-1}). \end{aligned} \quad (43)$$

Similarly, if n is even, we also obtain that

$$\begin{aligned} d_{\mathbb{B}\mathbb{C}}(x_{n+1}, x_n) &= d_{\mathbb{B}\mathbb{C}}(f_{i+1} x_n, f_i x_{n-1}) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(x_{n-1}, x_n)\|) \\ &\quad \cdot d_{\mathbb{B}\mathbb{C}}(x_{n-1}, x_n), \end{aligned} \quad (44)$$

which implies that, for all $n \in \mathbb{N}$

$$\|d_{\mathbb{B}\mathbb{C}}(x_{n+1}, x_n)\| < \|d_{\mathbb{B}\mathbb{C}}(x_n, x_{n-1})\| \quad (45)$$

Arguing like in the proof of Theorem 16, we prove that (x_n) is a Cauchy sequence in the complete bicomplex-valued metric space. Then, there exists $u \in X$ such that

$$\lim_{n \rightarrow +\infty} d_{\mathbb{B}\mathbb{C}}(x_n, u) = 0. \quad (46)$$

Next, we prove that $f_i u = f_{i+1} u = u$, $i = 1, 2, \dots, k$. Assume that $f_i u \neq u$. For each $n \in \mathbb{N}$, by (42), we have

$$d_{\mathbb{B}\mathbb{C}}(x_{2n+1}, f_i u) = d_{\mathbb{B}\mathbb{C}}(f_{i+1} x_{2n}, f_i u) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(u, x_{2n})\|) \cdot d_{\mathbb{B}\mathbb{C}}(u, x_{2n}), \quad (47)$$

then

$$\|d_{\mathbb{B}\mathbb{C}}(x_{2n+1}, f_i u)\| \leq \psi(\|d_{\mathbb{B}\mathbb{C}}(u, x_{2n})\|) \cdot \|d_{\mathbb{B}\mathbb{C}}(u, x_{2n})\| < \|d_{\mathbb{B}\mathbb{C}}(u, x_{2n})\|, \quad (48)$$

then letting $n \rightarrow +\infty$ and by Lemma 6 and Lemma 7, we have

$$\|d_{\mathbb{B}\mathbb{C}}(u, f_i u)\| \leq \|d_{\mathbb{B}\mathbb{C}}(u, u)\| = 0. \quad (49)$$

We obtain a contradiction. Hence, $f_i u = u$. As

$$d_{\mathbb{B}\mathbb{C}}(u, f_{i+1} u) = d_{\mathbb{B}\mathbb{C}}(f_i u, f_{i+1} u) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(u, u)\|) d_{\mathbb{B}\mathbb{C}}(u, u), \quad (50)$$

then

$$\|d_{\mathbb{B}\mathbb{C}}(u, f_{i+1} u)\| < \|d_{\mathbb{B}\mathbb{C}}(u, u)\| = 0. \quad (51)$$

Therefore, $\|d_{\mathbb{B}\mathbb{C}}(u, f_{i+1} u)\| = 0$, i.e., $u = f_{i+1} u = f_i u$. If there exists another point $v \in X$ such that $v = f_i v = f_{i+1} v$, then using an argument similar to the above, we get

$$d_{\mathbb{B}\mathbb{C}}(u, v) = d_{\mathbb{B}\mathbb{C}}(f_i u, f_{i+1} v) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(u, v)\|) d_{\mathbb{B}\mathbb{C}}(u, v), \quad (52)$$

i.e.,

$$\|d_{\mathbb{B}\mathbb{C}}(u, v)\| < \|d_{\mathbb{B}\mathbb{C}}(u, v)\|. \quad (53)$$

We obtain a contradiction, hence $u = v$.

By the same way, if there exists a unique point $w \in X$ such that $w = f_{i+1} w = f_{i+2} w$. We will show that $u = w$. Suppose that $u \neq w$, we have

$$d_{\mathbb{B}\mathbb{C}}(u, w) = d_{\mathbb{B}\mathbb{C}}(f_i u, f_{i+1} w) \leq_{i_2} \psi(\|d_{\mathbb{B}\mathbb{C}}(u, w)\|) d_{\mathbb{B}\mathbb{C}}(u, w), \quad (54)$$

then

$$\|d_{\mathbb{B}\mathbb{C}}(u, w)\| < \|d_{\mathbb{B}\mathbb{C}}(u, w)\|, \quad (55)$$

which is a contradiction. Then, $u = w$ so $u = f_i u = f_{i+1} u = f_{i+2} u$.

The proof is completed.

4. Application

In this section, we give existence theorems for the Volterra-type integral equation. For this purpose, let $X = C([a, b])$, $\mathbb{R}$ be the space of all continuous realvalued functions on $[a, b]$. Note that X is a complete bicomplex-valued metric space by considering $d_{\mathbb{B}\mathbb{C}}(x, y) = i_1 i_2 \max_{t \in [a, b]} \|x(t) - y(t)\|_\infty$, then $\|d_{\mathbb{B}\mathbb{C}}(x, y)\| = \max_{t \in [a, b]} \|x(t) - y(t)\|_\infty$.

Consider the Volterra-type integral equation as

$$x(t) = \int_a^b K(t, s, x(s)) ds + g(t), \quad (56)$$

where $x(t)$ is the unknown solution, $t \in [a, b] \subset \mathbb{R}$ and $x, g \in X$.

The kernel K of the integral equation is defined on $[a, b] \times [a, b] \times \mathbb{R}$.

Theorem 17. Assume that the following conditions are satisfied:

- (i) $K : [a, b] \times [a, b] \times \mathbb{R} \rightarrow \mathbb{R}$ is continuous function
- (ii) there exists $\lambda \in [0, 1)$ such that

$$|K(t, s, y_1) - K(t, s, y_2)| \leq \frac{\lambda}{b-a} |y_1 - y_2| \quad (57)$$

for all $t, s \in [a, b]$ and $y_1, y_2 \in \mathbb{R}$.

Then, the nonlinear Volterra Equation (56) has a unique solution.

Proof. Define a mapping $f : X \rightarrow X$ as follows:

$$fx(t) = \int_a^b K(t, s, x(s)) ds + g(t). \quad (58)$$

Let $x, y \in X$. By the simple calculation and applying assumption (i₂), for each $t \in [a, b]$, we can write

$$|fx(t) - fy(t)| \leq \int_a^b |K(t, s, x(s)) - K(t, s, y(s))| \cdot |ds| \leq \frac{\lambda}{b-a} \int_a^b |x(s) - y(s)| ds. \quad (59)$$

Then,

$$d_{\mathbb{B}\mathbb{C}}(fx, fy) = i_1 i_2 \max_{t \in [a, b]} \|fx(t) - fy(t)\|_\infty \leq \lambda i_1 i_2 \max_{t \in [a, b]} \|x(t) - y(t)\|_\infty = \lambda d_{\mathbb{B}\mathbb{C}}(x, y). \quad (60)$$

It follows that f satisfies all the conditions of Theorem 10 where $w(t) = \lambda$ for all $t \in [a, b]$. Hence, f has a unique fixed point. This yields that there exists a unique solution of the nonlinear Volterra integral equation.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors contributed equally and significantly to writing this article. All the authors read and approved the final manuscript.

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